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ON THE INTERPOLATION IN LINEAR SPACES

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Abstract: In the papers [3], [4], we will study a way of extending the model of interpolation from the real functions with simple nodes to the case of functions defined between linear spaces, especially between linear normed spaces, presenting a model of construction of the abstract interpolation polynomials and of the divided differences based on the properties of multilinear mappings. The aim of the present paper is the study of the conduct of the abstract interpolation polynomial, in the case when the interpolation function is an abstract polynomial.

1. Introduction

In the papers [2], [4] we have defined the abstract interpolation polynomial attached to the function $f : E \to Y$, where $E \subseteq X$, X is a linear space and Y is an algebra with a special structure. So a connection between the ideas of Păvăloiu, I. [6], [7] and of Prenter, M. [8] was realized.

In order to emphasize on some properties of these interpolation polynomials we will recall the elements of the construction from the aforementioned paper.

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Let us consider the real or complex linear spaces X and Y; we note by $\mathcal{L}(X,Y)$ the set of linear mappings from X to Y. For $n \geq 2$, $\mathcal{L}_n(X,Y)$ represents the set of $n-$ linear mappings from X^n to Y. We have $\mathcal{L}_n(X,Y) = \mathcal{L}(X, \mathcal{L}_{n-1}(X,Y))$,with $\mathcal{L}_1(X,Y) = \mathcal{L}(X,Y)$. Particularly $\mathcal{L}_2(Y,Y)$ represents the set of the bilinear mappings from $Y \times Y$ to Y.

Let be θ_X and θ_Y the null elements of the space X and Y respectively. We will note by Θ_n the null element of the space $\mathcal{L}_n(X,Y)$. For $n = 1$ we will use the notation Θ.

For $U \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}_2(Y,Y)$ we introduce the sequence $(A_n)_{n\in\mathbb{N}}$ where for any $n \in \mathbb{N}$, $A_n \in \mathcal{L}_n(X,Y)$ with $A_1(u) = U(u)$ for $u \in X$ and

$$A_n(u_1, \ldots, u_n) = B(A_{n-1}(u_1, \ldots, u_{n-1}), U(u_n)) \tag{1}$$

for $(u_1, \ldots, u_n) \in X^n$ and $n \in \mathbb{N}$, $n \geq 2$.

We now suppose the next properties:

I) the mapping $B \in \mathcal{L}(Y,Y)$ determines in Y a commutative algebra, namely for any $u, v \in Y$ we have $B(u,v) = B(v,u)$, and for any $u, v, w \in Y$ we have $B(B(u,v), w) = B(u, B(v,w))$;

II) there exists $Y_0 \subseteq U(X) \subseteq Y$ so that (Y_0, B) is an abelian group and the mapping $U : U^{-1}(Y_0) \to Y_0$ is a bijective mapping.

Let now be the set $D \subseteq X$ and a sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$. For $k, n \in \mathbb{N}$ and for any $i \in \{k, k+1, \ldots, k+n\}$ we consider the mapping $w'_{k,n}(x_i) \in \mathcal{L}(X,Y)$ defined through

$$w'_{k,n}(x_i)h = A_{n+1}(x_i - x_k, \ldots, x_i - x_{i-1}, x_i - x_{i+1} \ldots, x_i - x_{k+n}, h). \tag{2}$$

In the papers [2], [4] we have shown that if for certain values $k, n \in \mathbb{N}$ and for any $i, j \in \{k, k+1, \ldots, k+n\}$ with $i \neq j$ we have $x_i - x_j \in U^{-1}(x_0)$, then for $i \in \{k, k+1, \ldots, k+n\}$ the restrictions at $U^{-1}(Y_0)$ of the mappings defined by (2) and denoted by $\left[w'_{k,n}(x_i)\right]_0 : U^{-1}(Y_0) \to Y_0$ are bijective, thus there exist the mappings $\left[w'_{k,n}(x_i)\right]_0^{-1} : Y_0 \to U^{-1}(Y_0)$.

Considering the set $\operatorname{sp}(Y_0)$ which represents the linear cover of the set Y_0, the mapping mentioned before will be prolonged through linearity at $\operatorname{sp}(Y_0)$, obtaining the mapping $\left[w'_{k,n}(x_i)\right]_*^{-1} \in \mathcal{L}(\operatorname{sp}(Y_0), X)$, the restriction to Y_0 being $\left[w'_{k,n}(x_i)\right]_0^{-1}$ itself.

Let us consider $n \in \mathbb{N}$, $D \subseteq X$ and the elements $x_0, x_1, \ldots, x_n \in D$, supposing that they satisfy the aforementioned hypothesis for the spaces X, Y and for the mappings $U \in \mathcal{L}(X,Y)$, $B \in \mathcal{L}_2(Y,Y)$.

We suppose that for any $i, j \in \{0, 1, \ldots, n\}$ with $i \neq j$, $x_i - x_j \in \in U^{-1}(x_0)$ and so the mapping $\left[w'_{0,n}(x_i)\right]_*^{-1} \in \mathcal{L}(\mathrm{sp}(Y_0), X)$ exists.

Let now be a function $f: X \to Y$ so that $f(x_1), f(x_2), \ldots, f(x_n) \in \in \mathrm{sp}(Y_0)$.

In this way we can introduce the mapping $\mathbf{L}(x_0, x_1, \ldots, x_n; f) : : X \to Y$ defined by

$$\mathbf{L}(x_0, x_1, \ldots, x_n; f)(x) =$$

$$= \sum_{i=0}^{n} A_{n+1}(x - x_0, \ldots \underset{i}{|} \ldots, x - x_n, \left[w'_{0,n}(x_i)\right]_*^{-1} f(x_i)), \tag{3}$$

where

$$(x - x_0, \ldots \underset{i}{|} \ldots, x - x_n) = (x - x_0, \ldots, x - x_{i-1}, x - x_{i+1} \ldots, x - x_n),$$

having

$$\mathbf{L}(x_0, x_1, \ldots, x_n; f)(x_i) = f(x_i)$$

for any $i = \overline{0, n}$.

At the same time there exists $D_0 \in Y$ and $D_k \in \mathcal{L}_k(X, Y)$ for any $k = \overline{1, n}$ such that

$$\mathbf{L}(x_0, x_1, \ldots, x_n; f)(x) = D_n x^n + D_{n-1} x^{n-1} + \cdots + D_1 x + D_0.$$

Due to the aforementioned reasons the non-linear mapping defined by the equality (3) will be called *(U-B) abstract interpolation polynomial* of the function $f : X \to Y$ corresponding to the nodes $x_0, x_1, \ldots \ldots, x_n$.

In the expression (3) of the abstract interpolation polynomial a very important element is the coefficient of the term in x^n, namely the mapping $D_n \in \mathcal{L}_n(X, Y)$. This mapping will be denoted by $[x_0, x_1, \ldots, x_n; f]$, and will be defined through

$$[x_0, x_1, \ldots, x_n; f]\, h_1 \ldots h_n = \sum_{i=1}^{n} A_{n+1}(h_1, \ldots, h_n; \left[w'_{0,n}(x_i)\right]_*^{-1} f(x_i)). \tag{4}$$

This mapping is called *generalized divided difference of order* n of the function $f : D \to Y$ on the nodes $x_0, x_1, \ldots, x_n$.

The main results of the paper [4] are expressed through the following

Theorem 1.1. *With the given facts and with the aforementioned hypotheses we have for any $n \in \mathbb{N}$ and any $x \in X$ the following equalities:*

(a) $[x_0, x_1, \ldots, x_n; f](x_n - x_0) = [x_1, \ldots, x_n; f] - [x_0, \ldots, x_{n-1}; f]$,
(the equality taking place between the elements of the space $\mathcal{L}_{n-1}(X, Y)$*);*

(b)
$$\mathbf{L}(x_0, x_1, \ldots, x_n; f)(x) =$$
$$= f(x_0) + \sum_{i=1}^{n} [x_0, x_1, \ldots, x_i; f](x - x_0)(x - x_1) \ldots (x - x_{i-1});$$

(the Newton's form (using the abstract divided differences) of the interpolation polynomial);

(c)
$$f(x) = \mathbf{L}(x_0, x_1, \ldots, x_n; f)(x) +$$
$$+ [x_0, x_1, \ldots, x_n, x; f](x - x_0)(x - x_1) \ldots (x - x_n).$$

For the proof one can consult [2], [4].

As compared with other models, the one presented here offers more exact definitions of the intervening mappings, especially of the abstract divided differences. With the help of these differences we can express the rest of a function's approximation through the abstract interpolation polynomial. This expression is necessary if one has in mind the use of the model in the approximation of the functionals' values or of the equations' solutions in different spaces.

In order to establish different approximation formulas it is important to study the conduct of the abstract interpolation polynomial and of the divided difference in the case of their use for some special types of functions. Such a study is the aim of the present paper.

For this purpose it is necessary to introduce the mappings that will be defined hereafter.

We consider the sequence $(x_n)_{n \in \mathbb{N}} \subseteq D$ and the numbers $k, n \in \mathbb{N}$. Let there be afterwards $p \in \mathbb{N}$, $p \leq n + 1$ and $i_1, i_2, \ldots, i_p \in \mathbb{N}$ with the verification of the inequalities $k \leq i_1 < i_2 < \cdots < i_p \leq k + n$.

For $x \in X$ we introduce the mappings $W_{k,n}^{[i_1, i_2, \ldots, i_p]}(x) \in \mathcal{L}(X, Y)$ defined by

$$W_{k,n}^{[i_1, i_2, \ldots, i_p]}(x) h = A_{n-p+2}(t_1, \ldots, t_{n-p+1}, h) \tag{5}$$

where
$$\{t_1, \ldots, t_{n-p+1}\} = \{x - x_k, \ldots, x - x_{k+n}\} \setminus \{x - x_{i_1}, \ldots, x - x_{i_p}\}$$
keeping the succession order from the initial set.

It is evident that we have $W_{k,n}^{[i_1, i_2, \ldots, i_p]}(x_s) = \Theta$ for any $s \in \{k, k+1, \ldots, k+n\} \setminus \{i_1, i_2, \ldots, i_p\}$, where Θ represents the null mapping of the space $\mathcal{L}(X,Y)$, as well as $W_{k,n}^{[i]}(x_i) = w'_{k,n}(x_i)$ for any $i \in \{k, k+1, \ldots, k+n\}$.

It is also easy to notice that the restriction to the set $U^{-1}(Y_0)$ is bijective, so there exists the mapping

$$\left[W_{k,n}^{[i_1,i_2,\ldots,i_p]}(x)\right]_0^{-1} : Y_0 \to U^{-1}(Y_0),$$

representing the inverse of the mapping defined by (5).

2. Some properties of the abstract interpolation polynomial

Definition 2.1. Let be $U \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}_2(Y,Y)$ such that B determines on Y a commutative algebra and let the mappings sequence $(A_n)_{n\in\mathbb{N}}$ be introduced by (1).

(a) The mapping $M_n : X \to Y$, $M_n(x) = A_n x^n$ is called *(U-B) monomial with degree n.*

(b) A mapping $P : X \to Y$ for which there exist the elements $a_0, a_1, \ldots \ldots, a_n \in Y$ so that

$$P(x) = a_0 + \sum_{k=1}^{n} B(a_k, M_k(x)),$$

where for any $k = \overline{1,n}$ the mapping $M_k : X \to Y$ represents the monomial with the degree k, is called *(U-B) polynomial of degree n.*

Let us consider $n, k \in \mathbb{N}$; $k \geq n$ and $u_i^{(k)} = \left[w'_{0,n}(x_i)\right]^{-1} M_k(x_i) \in$ $\in X$, $i = \overline{0,n}$.

Lemma 2.2. *For any $k, n \in \mathbb{N}$, $k \geq n$, $p \leq n+1$ and $i_1, i_2, \ldots, i_p \in \mathbb{N}$ with $0 \leq i_1 < i_2 < \cdots < i_p \leq n$ we have*

$$(6) \qquad \sum_{j=1}^{p} w_{0,n}^{[i_1,\ldots,i_p]}(x_{i_j})\, u_{i_j}^{(k)} = \sum_{\alpha_1+\cdots+\alpha_p=k-p+1} A_{k-p+1} x_{i_1}^{\alpha_1} \ldots x_{i_p}^{\alpha_p}.$$

Proof. We will use induction according to p. Because for any $i \in$ $\in \{0, 1, \ldots, n\}$ we have $u_i^{(k)} = \left[w'_{0,n}(x_i)\right]^{-1} M_k(x_i)$, so

$$w'_{0,n}(x_i)\, u_i^{(k)} = M_k(x_i) = A_k x_i^k,$$

therefore the equality (6) is true for $p = 1$.

We suppose that this equality is true for $p = s$ and we follow how it is established for $p = s+1$.

From the induction hypothesis we deduce that

$$\sum_{j=1}^{s-1} W_{0,n}^{[i_1,\dots,i_{s-1},i_s]}\left(x_{i_j}\right) u_{i_j}^{(k)} + W_{0,n}^{[i_1,\dots,i_{s-1},i_s]}\left(x_{i_s}\right) u_{i_s}^{(k)} =$$

$$= \sum_{\alpha_1+\dots+\alpha_s=k-s+1} A_{k-s+1} x_{i_1}^{\alpha_1} \dots x_{i_s}^{\alpha_s}.$$

Adding now the index i_{s+1} to $i_1, \dots, i_s \in \mathbb{N}$ so that $0 \le i_1 < \dots < < i_s < i_{s+1} \le n$, replacing i_s by i_{s+1} and α_s by α_{s+1} we have

$$\sum_{j=1}^{s-1} W_{0,n}^{[i_1,\dots,i_{s-1},i_{s+1}]}\left(x_{i_j}\right) u_{i_j}^{(k)} + W_{0,n}^{[i_1,\dots,i_{s-1},i_{s+1}]}\left(x_{i_{s+1}}\right) u_{i_{s+1}}^{(k)} =$$

$$= \sum_{\alpha_1+\dots+\alpha_s=k-s+1} A_{k-s+1} x_{i_1}^{\alpha_1} \dots x_{i_{s-1}}^{\alpha_{s-1}} x_{i_{s+1}}^{\alpha_{s+1}}.$$

It follows from the last two equalities that

$$(7) \qquad \sum_{j=1}^{s-1} \left[W_{0,n}^{[i_1,\dots,i_{s-1},i_s]}\left(x_{i_j}\right) - W_{0,n}^{[i_1,\dots,i_{s-1},i_{s+1}]}\left(x_{i_j}\right)\right] u_{i_j}^{(k)} +$$

$$+ W_{0,n}^{[i_1,\dots,i_{s-1},i_s]}\left(x_{i_s}\right) u_{i_s}^{(k)} - W_{0,n}^{[i_1,\dots,i_{s-1},i_{s+1}]}\left(x_{i_{s+1}}\right) u_{i_{s+1}}^{(k)} =$$

$$= \sum_{\alpha_1+\dots+\alpha_s=k-s+1} \left[A_{k-s+1} x_{i_1}^{\alpha_1} \dots x_{i_s}^{\alpha_s} - A_{k-s+1} x_{i_1}^{\alpha_1} \dots x_{i_{s+1}}^{\alpha_{s+1}}\right].$$

The first member of this equality can be written in the form

$$(8) \qquad B\left(U\left(x_{i_s} - x_{i_{s+1}}\right), \sum_{j=1}^{s+1} W_{0,n}^{[i_1,i_2,\dots,i_s,i_{s+1}]}\left(x_{i_j}\right) u_{i_j}^{(k)}\right).$$

In what the second member of (7) is concerned, for any $\alpha_1, \dots, \alpha_s \in \in \mathbb{N}$ with $\alpha_1 + \dots + \alpha_s = k - s + 1$ the expression

$$A_{k-s+1} x_{i_1}^{\alpha_1} \dots x_{i_{s-1}}^{\alpha_{s-1}} x_{i_s}^{\alpha_s} - A_{k-s+1} x_{i_1}^{\alpha_1} \dots x_{i_{s-1}}^{\alpha_{s-1}} x_{i_{s+1}}^{\alpha_s}$$

is equal with

$$B\left(U\left(x_{i_s} - x_{i_{s+1}}\right), \sum_{r=1}^{\alpha_s} A_{k-s} x_{i_1}^{\alpha_1} \dots x_{i_{s-1}}^{\alpha_{s-1}} x_{i_s}^{\alpha_s - r} x_{i_{s+1}}^{r-1}\right).$$

Thus the expression of the second member of the equality (7) will be written

(9)

$$B\Big(U\left(x_{i_s}-x_{i_{s+1}}\right), \sum_{\alpha_1+\cdots+\alpha_s=k-s+1}\ \sum_{r=1}^{\alpha_s} A_{k-s}x_{i_1}^{\alpha_1}\dots x_{i_{s-1}}^{\alpha_{s-1}}x_{i_s}^{\alpha_s-r}x_{i_{s+1}}^{r-1}\Big).$$

But $x_{i_s}-x_{i_{s+1}}\in U^{-1}(Y_0)$, therefore $U\left(x_{i_s}-x_{i_{s+1}}\right)\in Y_0$, while (Y_0,B) is an abelian group. Thus from (7), (8) and (9) we deduce that

$$\sum_{j=1}^{s+1}W_{0,n}^{[i_1,i_2,\dots,i_s,i_{s+1}]}\left(x_{i_j}\right)u_{i_j}^{(k)}= \tag{10}$$

$$=\sum_{\alpha_1+\cdots+\alpha_s=k-s+1}\ \sum_{r=1}^{\alpha_s} A_{k-s}x_{i_1}^{\alpha_1}\dots x_{i_{s-1}}^{\alpha_{s-1}}x_{i_s}^{\alpha_s-r}x_{i_{s+1}}^{r-1}.$$

Let there be the new indexes $\beta_1,\beta_2,\dots,\beta_s,\beta_{s+1}$ introduced through $\beta_1=\alpha_1,\ \dots\dots,\beta_{s-1}=\alpha_{s-1},\ \beta_s=\alpha_{s-r},\ \beta_{s+1}=\alpha_{r-1}$, so the relation (10) will be written

$$\sum_{j=1}^{s+1}W_{0,n}^{[i_1,\dots,i_{s+1}]}\left(x_{i_j}\right)u_{i_j}^{(k)}=\sum_{\beta_1+\cdots+\beta_s+\beta_{s+1}=k-s} A_{k-s}x_{i_1}^{\beta_1}\dots x_{i_s}^{\beta_s}x_{i_{s+1}}^{\beta_{s+1}}, \tag{11}$$

which indicates that the equality (6) is true for $p=s+1$.

Therefore this equality is true for any $p\in\mathbb{N}$. ◊

Remark 2.3. One can see that for $p=1$, denoting $i_1=i$, the only value of j is 1, and in the second member the only possibility is $\alpha_1=k$, therefore the equality (6) becomes $[W_{0,n}^{[i]}(x_i)]u_i^{(k)}=A_kx_i^k$.

Let us consider now the case $p=n+1$. Because $0\le i_1<i_2<$ $<\cdots<i_{n+1}\le n$ the only possibility is $i_j=j-1$ for any $j=\overline{1,n+1}$, therefore the sum of the first member is $\sum\limits_{j=1}^{n+1}W_{0,n}^{[0,1,\dots,n]}\left(x_{j-1}\right)u_{j-1}^{(k)}$.

Evidently however $w_{0,n}^{[0,1,\dots,n]}\left(x_{j-1}\right)h=A_1(h)=U(h)$, therefore the equality (6) becomes

$$\sum_{j=1}^{n}U\left(u_j^{(k)}\right)=\sum_{\alpha_0+\alpha_1+\cdots+\alpha_n=k-n} A_{k-n}x_0^{\alpha_0}x_1^{\alpha_1}\dots x_n^{\alpha_n}, \tag{12}$$

for the summing indexes we have adapted this notation for symmetry reasons.

We have the following

Theorem 2.4. *From the hypotheses and from the above mentioned conditions, we have*

$$(13)\qquad [x_0,x_1,...,x_n;M_k]=\begin{cases}\Theta_n & \text{for } k<n,\\ A_n & \text{for } k=n,\\ \sum\limits_{\alpha_0+\cdots+\alpha_n=k-n} A_k x_0^{\alpha_0}x_1^{\alpha_1}...x_n^{\alpha_n} & \text{for } k>n,\end{cases}$$

Proof. From the definition of the divided difference it results that

$$[x_0,x_1,\ldots,x_n;M_k]\,h_1\ldots h_n =$$
$$= A_{n+1}\Big(h_1,\ldots,h_n,\sum_{i=0}^{n}\left[w'_{0,n}(x_i)\right]_*^{-1} M_k(x_i)\Big).$$

Let us consider first the case $k\geq n$.

Because $u_i^{(k)}=\left[w'_{0,n}(x_i)\right]_*^{-1} M_k(x_i)$ for any $i=\overline{0,n}$, so

$$[x_0,x_1,\ldots,x_n;M_k]\,h_1\ldots h_n =$$
$$= A_{n+1}\Big(h_1,\ldots,h_n,\sum_{i=0}^{n}u_i^{(k)}\Big) = B\Big(U\Big(\sum_{i=0}^{n}u_i^{(k)}\Big),A_n(h_1,\ldots,h_n)\Big) =$$
$$= B\Big(\sum_{\alpha_0+\alpha_1+\cdots+\alpha_n=k-n} A_{k-n}x_0^{\alpha_0}x_1^{\alpha_1}\ldots x_n^{\alpha_n},A_n(h_1,\ldots,h_n)\Big) =$$
$$= \sum_{\alpha_0+\alpha_1+\cdots+\alpha_n=k-n} A_k x_0^{\alpha_0}x_1^{\alpha_1}\ldots x_n^{\alpha_n}h_1\ldots h_n.$$

Because $h_1,\ldots,h_n\in X$ are arbitrary, we deduce that

$$[x_0,x_1,\ldots,x_n;M_k] = \sum_{\alpha_0+\alpha_1+\cdots+\alpha_n=k-n} A_k x_0^{\alpha_0}x_1^{\alpha_1}\ldots x_n^{\alpha_n}.$$

In the special case $k=n$ the only possibility for the choice of the summing indexes is $\alpha_0=\cdots=\alpha_n=0$, therefore

$$[x_0,x_1,\ldots,x_n;M_n]=A_n.$$

Let us consider now the case $k<n$.

If we note $p=k+1$ we deduce that $p\in\{1,\ldots,n\}$. For this p, due to the relation (6), we have

$$(14)\qquad \sum_{j=1}^{p}W_{0,n}^{[i_1,\ldots,i_p]}(x_{i_j})\,u_{i_j}^{(k)}=K\in Y$$

for any $i_1,\ldots,i_p\in\mathbb{N}$ with $0\leq i_1<\ldots\cdots<i_p\leq n$.

We are in the framework of the relation (6) if we consider $\mathcal{L}_0(X,Y) = Y$. Therefore $A_0 \in \mathcal{L}_0(X,Y)$, thus $A_0 = K \in Y$.

If we introduce a new index i_{p+1} with $0 \le i_1 < \cdots < i_p < i_{p+1} \le$ $\le n$ the relation (14) will be true as well in the case when the indexes are changed in $i_1, \ldots, i_{p-1}, i_{p+1}$, so

$$\sum_{j=1}^{p-1} W_{0,n}^{[i_1,\ldots,i_{p-1},i_{p+1}]}\left(x_{i_j}\right) u_{i_j}^{(k)} + W_{0,n}^{[i_1,\ldots,i_{p-1},i_{p+1}]}\left(x_{i_{p+1}}\right) u_{i_{p+1}}^{(k)} = K,$$

from which

$$\sum_{j=1}^{p-1}\left[W_{0,n}^{[i_1,\ldots,i_{p-1},i_p]}\left(x_{i_j}\right) - W_{0,n}^{[i_1,\ldots,i_{p-1},i_{p+1}]}\left(x_{i_j}\right)\right] u_{i_j}^{(k)} +$$
$$+W_{0,n}^{[i_1,\ldots,i_{p-1},i_p]}\left(x_{i_p}\right) u_{i_p}^{(k)} - W_{0,n}^{[i_1,\ldots,i_{p-1},i_{p+1}]}\left(x_{i_{p+1}}\right) u_{i_{p+1}}^{(k)} = \theta_Y,$$

and so $B\left(U\left(x_{i_p} - x_{i_{p+1}}\right), \sum_{j=1}^{p+1} W_{0,n}^{[i_1,i_2,\ldots i_{p+1}]}\left(x_{i_j}\right) u_{i_j}^{(k)}\right) = \theta_Y$.

From $x_{i_p} - x_{i_{p+1}} \in U^{-1}(Y_0)$ we have $U(x_{i_p} - x_{i_{p+1}}) \in Y_0$ and similarly to the proof of Lemma 2.2, we deduce that $\sum_{j=1}^{p+1} W_{0,n}^{[i_1,i_2,\ldots i_{p+1}]}\left(x_{i_j}\right) u_{i_j}^{(k)} =$ $= \theta_Y$. In the special case $q = n+1$ we have $0 \le i_1 < i_2 < \cdots < i_{n+1} \le$ $\le n$ and we obtain $i_j = j-1$ for any $j = \overline{1, n+1}$, thus the previous equality will be written $\sum_{j=0}^{n} W_{0,n}^{[0,1,\ldots,n]} u_j^{(k)} = \theta_Y$, so $\sum_{j=0}^{n} U\left(u_j^{(k)}\right) = \theta_Y$ and because of the linearity of U we have $\sum_{j=0}^{n} u_j^{(k)} = \theta_X$. In this way for any $h_1, \ldots, h_n \in X$ we have

$$[x_0, x_1, \ldots, x_n; M_k]\, h_1 \ldots h_n = A_{n+1}\left(h_1, \ldots, h_n; \sum_{j=0}^{n} u_j^{(k)}\right) =$$
$$= A_{n+1}\left(h_1, \ldots, h_n; \theta_X\right) = \theta_Y,$$

so $[x_0, x_1, \ldots, x_n; M_k] = \Theta_n$. ◊

Let us establish now the following

Theorem 2.5. *Let us consider the previously introduced elements, a set $D \subseteq X$, the points $x_0, x_1, \ldots, x_n \in D$, the function $f : D \to Y$ so that $f(x_0), f(x_1), \ldots, f(x_n) \in \mathrm{sp}(Y_0)$. Let be $a \in \mathrm{sp}(Y_0)$ and the mapping $g : D \to Y$, $g(x) = B(a, f(x))$. Then*

(15) $[x_0, x_1, \ldots, x_n; g]\, h_1 \ldots h_n = B\left(a, [x_0, x_1, \ldots, x_n; f]\, h_1 \ldots h_n\right).$

Proof. From the definition of the divided difference it results that

(16) $$[x_0, x_1, ..., x_n; g]h_1...h_n = \sum_{i=0}^{n} A_{n+1}(h_1, ..., h_n, [w'_{0,n}(x_i)]^{-1} g(x_i)).$$

For any $i \in \{0, 1, \ldots, n\}$ we have evidently

(17) $$\left[w'_{0,n}(x_i)\right]^{-1} g(x_i) = \left[w'_{0,n}(x_i)\right]^{-1} B\left(a, g(x_i)\right).$$

For any $i, j = \overline{0,n};\ i \neq j$ we have $x_i - x_j \in U^{-1}(Y_0)$. We deduce that for $q = A_n\left(x_i - x_1, \ldots, x_i - x_{i-1}, x_i - x_{i+1}, \ldots, x_i - x_{n+1}\right) \in Y_0$ there exists $q' \in Y_0$ such that $B(q, q') = u_0$ (u_0 being the neutral element of the group (Y_0, B)). As well for any $t \in Y_0$ we have $B(t, u_0) = t$, that is $B(t, B(q, q')) = t$ or $B(q, B(q', t)) = t$, namely

$$B(A_n(x_i - x_1, \ldots, x_i - x_{i-1}, x_i - x_{i+1}, ..., x_i - x_{n+1}), B(q', t)) = t,$$

this is

$$A_{n+1}(x_i - x_1, \ldots, x_i - x_{i-1}, x_i - x_{i+1}, \ldots, x_i - x_{n+1}, U^{-1} B(q', t)) = t,$$

or $B(q', t) = U\left[w'_{0,n}(x_i)\right]^{-1} t$.

From this relation we notice that for $b, z \in Y_0$ we have

(18) $$U\left[w'_{0,n}(x_i)\right]^{-1} B(b, z) = B\left(\left[w'_{0,n}(x_i)\right]^{-1} z, b\right).$$

The relation (18) will be extended, using the linearity of the mappings U, $\left[w'_{0,n}(x_i)\right]^{-1}$ and B, as well to the case when the elements $b, z \in Y_0$ are replaced respectively by $a, y \in \mathrm{sp}(Y_0)$.

Now, for $i \in \{0, 1, \ldots, n\}$ we choose $y = f(x_i)$, and we obtain

(19) $$U\left[w'_{0,n}(x_i)\right]^{-1} B(a, f(x_i)) = B\left(U\left[w'_{0,n}(x_i)\right]^{-1} f(x_i), a\right).$$

From the relations (17) and (19) we obtain for any $i \in \{0, 1, \ldots, n\}$ the equalities

$$A_{n+1}(h_1, \ldots, h_n, \left[w'_{0,n}(x_i)\right]^{-1} g(x_i)) =$$
$$= B(A_n(h_1, \ldots, h_n), U\left[w'_{0,n}(x_i)\right]^{-1} B(a, f(x_i))) =$$
$$= B(A_n(h_1, \ldots, h_n), B(U\left[w'_{0,n}(x_i)\right]^{-1} f(x_i), a)) =$$

$$= A_{n+2}(U_*^{-1}(a), h_1, \dots, h_n, \left[w'_{0,n}(x_i)\right]^{-1} f(x_i)) =$$
$$= B(a, A_{n+1}(h_1, \dots, h_n, \left[w'_{0,n}(x_i)\right]^{-1} f(x_i))).$$

In this relation $U_*^{-1}$is the prolongation trough linearity of the mapping U^{-1} to $\mathrm{sp}(Y_0)$.

According to the relation (17) we will have

$$[x_0, x_1, \dots, x_n; g]\, h_1 \dots h_n = B\Big(a, \sum_{i=0}^{n} A_{n+1}(h_1, \dots, h_n, [w'_{0,n}(x_i)]^{-1} f(x_i))\Big) =$$
$$= B(a, [x_0, x_1, \dots, x_n; f] h_1 \dots h_n),$$

the theorem being in this way proved. ◊

Corollary 2.6. *If for* $k \in \mathbb{N}$, $M_k : X \to Y$ *is a (U-B) monomial of the degree* k *and we consider the mapping* $g : X \to Y$, $g(x) = B(a, M_k(x))$ *with* $a \in \mathrm{sp}(Y_0)$ *and supposing that all the hypotheses of the previous theorems are fulfilled, then we have the relation*

$$[x_0, x_1, \dots x_n; g]\, h_1 \dots h_n = \tag{20}$$
$$= \begin{cases} \theta_Y, & k < n, \\ B(a, A_n(h_1, \dots, h_n)), & k = n, \\ \displaystyle\sum_{\alpha_0 + \dots + \alpha_n = k-n} B(a, A_k x_0^{\alpha_0} \dots x_n^{\alpha_n} h_1 \dots h_n), & k > n. \end{cases}$$

Proof. The conclusion of this statement is evident if we use Ths. 2.4 and 2.5. ◊

Theorem 2.7. *If* $P : X \to Y$ *is a (U-B) polynomial of degree* k ($k \le n$) *with the coefficients in* $\mathrm{sp}(Y_0)$ *and supposing that all hypotheses of the previous theorems are fulfilled, then for any* $x_0, x_1, \dots, x_n \in X$ *we have*

$$P = \mathbf{L}(x_0, x_1, \dots, x_n; P). \tag{21}$$

Proof. Th. 1.1-(c) indicates that for any $x \in X$,

$$P(x) = \mathbf{L}(x_0, x_1, \dots, x_n; P)(x) + [x_0, x_1, \dots, x_n, x; P](x - x_0) \dots (x - x_n).$$

For $i \in \{0, 1, \dots, n\}$, if we introduce $g_i : X \to Y$ with $g_0(x) = a_0$ and $g_i(x) = B(a_i, M_i(x))$ for $i \ge 1$, we have

$$P(x) = a_0 + \sum_{k=0}^{n} B(a_k, M_k(x)) = \sum_{k=0}^{n} g_k(x),$$

and so

$$[x_0, x_1, \ldots, x_n, x; P] = \sum_{k=0}^{n} [x_0, x_1, \ldots, x_n, x; g_k].$$

In the divided differences from the second member, in the expression of the mappings g_k there appear monomials having a degree at least two units smaller than the number of the nodes, so for any $k = \overline{0, n}$ we have $[x_0, x_1, \ldots, x_n, x; g_k] = 0$, therefore $[x_0, x_1, \ldots, x_n, x; P] = \Theta_{n+1}$, and the theorem is proved. ◊

For certain concrete examples, different from the case of the real function's interpolation, examples in which this construction is realized, see paper [2].

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