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COMPONENT RESTRICTION PROPERTY FOR CLASSES OF MAPPINGS

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Abstract: A class of mappings has the component restriction property if for each mapping $f : X \to Y$ belonging to the class, each $B \subset Y$ and each $A \subset X$ being the union of some components of the preimage of B the restriction $f|A$ also is in the class. The property is studied for classes of open, monotone, confluent and some related mappings.

1. Introduction

Introducing the class of confluent mappings in [2] it was observed that for an arbitrary subset B of the range space the restriction of a confluent mapping to any union of components of the preimage of B is again confluent, see [2], Th. I, p. 213. Later it was shown that the class of semi-confluent mappings has the same property, [5], Th. 3.7, p. 255. In further investigations the property was shown to be important and useful especially in continuum theory. The aim of this paper is to

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study the property from the standpoint of general theory of mappings between topological spaces, for arbitrary classes of mappings.

Let us accept the following definition.

Definition 1.1. A class $\mathfrak{M}$ of mappings between topological spaces is said to have the component restriction property (abbreviated CRP) provided that for each mapping $f : X \to Y$ belonging to $\mathfrak{M}$ and for each subset B of Y, if $A \subset X$ is the union of some components of $f^{-1}(B)$, then the restriction $f|A : A \to f(A)$ belongs to $\mathfrak{M}$.

The subject of finding some conditions under which the property of belonging to a given class $\mathfrak{M}$ of mappings is kept by taking restrictions of mappings to some subspaces of the domain spaces of the mappings is not new. For example, G. T. Whyburn considered restrictions of some mappings to so called inverse sets in [7]; A. V. Arhangel'skii in [1] introduced and studied a concept of an inductively open mapping, i.e., a mapping $f : X \to Y$ for which there exists a subspace X^* of X such that $f(X^*) = f(X)$ and that the restriction $f|X^* : X^* \to f(X)$ is open. For other related concepts and results see e.g. [3].

The paper consists of five sections. After Introduction and Preliminaries some general results are presented in Section 3. In Section 4 we study classes that have CRP: homeomorphisms, monotone, (feebly) confluent, (feebly) semi-confluent, joining and atriodic, as well as their local variants. The last section is devoted to classes of mappings that do not have CRP: open, weakly confluent and pseudo-confluent ones.

2. Preliminaries

All spaces considered in this paper are assumed to be topological Hausdorff. A continuum means a compact connected space. A mapping means a continuous function. We denote by $\mathbb{N}$ the set of all positive integers, and by $\mathbb{R}$ the space of real numbers.

Given a subset A of a space X, we denote by $\mathrm{cl}_X(A)$, $\mathrm{int}_X(A)$ and $\mathrm{bd}_X(A)$ the closure, the interior and the boundary of A in X, respectively.

Let $\mathfrak{M}$ be an arbitrary class of mappings. A mapping $f : X \to Y$ between spaces X and Y is said to be:

— locally $\mathfrak{M}$ (abbreviated Loc($\mathfrak{M}$)) provided that for each point $x \in X$ there is a closed neighborhood V of x such that $f(V)$ is a closed neighborhood of $f(x)$ and the partial mapping $f|V : V \to f(V)$ belongs to

$\mathfrak{M}$;
— hereditarily $\mathfrak{M}$ if for each subcontinuum $K \subset X$ the partial mapping $f|K : K \to f(K)$ belongs to $\mathfrak{M}$.

The reader is referred to [6, Table II, p. 28] to see interrelations between the above mentioned and some others related classes of mappings.

A mapping $f : X \to Y$ between spaces X and Y is said to be:
— a homeomorphism if f is one-to-one and the inverse mapping f^{-1} is continuous;
— simple, if each point-inverse consists of one or two points;
— light, if each point-inverse has one-point components (note that if the point-inverses are compact, then this condition is equivalent to the property that they are zero-dimensional, [7, p. 130]);
— open, if f maps each open set in X onto an open set in Y;
— a local homeomorphism if for each point $x \in X$ there exists an open neighborhood U of x such that $f(U)$ is an open neighborhood of $f(x)$ and that the restriction $f|U : U \to f(U)$ is a homeomorphism, [7, p. 199]);
— monotone if each point-inverse is connected;
— an OM-mapping (an MO-mapping) if there exist mappings f_1 and f_2 such that $f = f_2 \circ f_1$, where f_1 is monotone and f_2 is open (where f_1 is open and f_2 is monotone);
— confluent, if for each subcontinuum Q of Y each component of $f^{-1}(Q)$ is mapped under f onto Q; equivalently, if for each subcontinuum Q of Y and for every two components C_1 and C_2 of $f^{-1}(Q)$ the equality $f(C_1) = f(C_2)$ holds (note that each open mapping on a compact space is confluent, [7, Chapter 8, (7.5), p. 148]);
— feebly confluent, if for each subcontinuum Q of Y and for every two components C_1 and C_2 of $f^{-1}(Q)$ either $f(C_1) = f(C_2)$ or $f(C_1) \cap \cap f(C_2) = \emptyset$;
— weakly confluent, if for each subcontinuum Q of Y there is a component of $f^{-1}(Q)$ which is mapped under f onto Q;
— pseudo-confluent, if for each irreducible subcontinuum Q of Y there is a component of $f^{-1}(Q)$ which is mapped under f onto Q;
— semi-confluent, if for each subcontinuum Q of Y and for every two components C_1 and C_2 of $f^{-1}(Q)$ either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset \subset f(C_1)$;
— feebly semi-confluent, if for each subcontinuum Q of Y and for every

two components C_1 and C_2 of $f^{-1}(Q)$ either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset \subset f(C_1)$ or $f(C_1) \cap f(C_2) = \emptyset$;
— joining, if for each subcontinuum Q of Y and for every two components C_1 and C_2 of $f^{-1}(Q)$ the inequality $f(C_1) \cap f(C_2) \neq \emptyset$ holds;
— atriodic if for each subcontinuum Q of Y there are two components C_1 and C_2 of $f^{-1}(Q)$ such that $f(C_1) \cup f(C_2) = Q$ and for each component C of $f^{-1}(Q)$ either $f(C) = Q$ or $f(C) \subset f(C_1)$ or $f(C) \subset f(C_2)$.

3. The component restriction property — general results

Let a mapping $f : X \to Y$ be given. Recall that a subset $A \subset X$ is said to be an *inverse set under* f provided that $A = f^{-1}(f(A))$, see [7, p. 137]. The following proposition can easily be verified.

Proposition 3.1. *If a class $\mathfrak{M}$ of mappings has* CRP, *then for each mapping $f : X \to Y$ belonging to $\mathfrak{M}$ and for each inverse set $A \subset X$ under f the restriction $f|A : A \to f(A)$ belongs to $\mathfrak{M}$.*

Let $\mathfrak{M}$ and $\mathfrak{N}$ be two classes of mappings each of which contains the class of homeomorphisms. We define (see [6, p. 15])

$$\mathfrak{M}\mathfrak{N} = \{g \circ f : g \in \mathfrak{M} \text{ and } f \in \mathfrak{N}\}.$$

The reader is referred to [6, Chapter 4, Section A, p. 15] for properties of the above concept. The following is obvious.

Proposition 3.2. *If classes $\mathfrak{M}$ and $\mathfrak{N}$ have* CRP, *then the class $\mathfrak{M}\mathfrak{N}$ has* CRP.

The next statement is evident just by using definitions.

Proposition 3.3. *The following classes of mappings have* CRP*: homeomorphisms, simple mappings, light mappings and monotone ones.*

A class $\mathfrak{M}$ of mappings is said to have the composition factor property provided that for every two mappings $f : X \to Y$ and $g : Y \to Z$ if their composition $g \circ f$ belongs to $\mathfrak{M}$, then the mapping g is in $\mathfrak{M}$.

As a direct consequence of the Whyburn factorization theorem (see [7, Th. 4.1, p. 141]) and of Prop. 3.3 we obtain the following result.

Theorem 3.4. *Let a class $\mathfrak{M}$ of mappings between compact spaces has* CRP. *Then for each mapping $f \in \mathfrak{M}$ there exists a unique factorization $f = f_2 \circ f_1$ into two mappings having* CRP *such that f_1 is monotone and f_2 is light. Moreover, if $\mathfrak{M}$ contains the class of monotone mappings, then $f_1 \in \mathfrak{M}$, and if $\mathfrak{M}$ has the composition factor property, then $f_2 \in \in \mathfrak{M}$.*

Theorem 3.5. *If a class* $\mathfrak{M}$ *of mappings has* CRP, *then the class* $\mathrm{Loc}(\mathfrak{M})$ *also has* CRP.

Proof. Let a mapping $f : X \to Y$ belong to $\mathrm{Loc}(\mathfrak{M})$. This means that for each point $x \in X$ there exists a subset $V \subset X$ such that

(3.5.1) $x \in \mathrm{int}_X(V) \subset V = \mathrm{cl}_X(V)$;
(3.5.2) $f(x) \in \mathrm{int}_Y(f(V)) \subset f(V) = \mathrm{cl}_Y(f(V))$;
(3.5.3) the restriction $f|V : V \to f(V)$ is in $\mathfrak{M}$.

To see that the class $\mathrm{Loc}(\mathfrak{M})$ has CRP take a subset $B \subset Y$, and let A be the union of some components of $f^{-1}(B)$. Put $g = f|A : A \to f(A)$. We have to show that $g \in \mathrm{Loc}(\mathfrak{M})$, i.e., that for each point $x \in A$ there exists a subset V' of A such that

(3.5.4) $x \in \mathrm{int}_A(V') \subset V' = \mathrm{cl}_A(V')$;
(3.5.5) $g(x) \in \mathrm{int}_{f(A)}(g(V')) \subset g(V') = \mathrm{cl}_{f(A)}(g(V'))$;
(3.5.6) the restriction $g|V' : V' \to g(V')$ is in $\mathfrak{M}$.

To this aim for each point $x \in A$ define $V' = A \cap V$. Then, by (3.5.1),

$$x \in \mathrm{int}_A(V') = A \cap \mathrm{int}_X(V) \subset A \cap V = V' = A \cap \mathrm{cl}_X(V) = \mathrm{cl}_A(V').$$

Thus V' is a closed neighborhood of x in A, i.e., (3.5.4) holds. Further, by (3.5.2) we infer that $g(x) = f(x)$ is an interior point of the set $g(V') = g(A \cap V) = f(A \cap V) = \mathrm{cl}_{f(A)}(f(A \cap V)) = \mathrm{cl}_{f(A)}(g(V'))$, whence (3.5.5) follows. Finally, since $\mathfrak{M}$ has CRP, we conclude by Def. 1.1 and (3.5.3) (elementary details are left to the reader) that the mapping $g|V' = (f|A)|V'$ is in $\mathfrak{M}$, i.e., that (3.5.6) is true. ◊

4. Confluent and related mappings — positive results

The following assertion is known, see [2, Th. I, p. 213] and [5, Th. 3.7, p. 255].

Theorem 4.1. *The classes of confluent mappings and of semi-confluent ones have* CRP.

Proposition 4.2. *The classes of feebly confluent, of feebly semi-confluent, of joining, and of atriodic mappings have* CRP.

Proof. Denote by $\mathfrak{M}$ the class of either feebly confluent or feebly semi-confluent or of joining mappings. Let a mapping $f : X \to Y$ be in $\mathfrak{M}$. Take a set $B \subset Y$, the union $A \subset X$ of some components of $f^{-1}(B)$, and let $g = f|A : A \to f(A)$. For a subcontinuum Q of $f(A) \subset Y$ let C_1 and C_2 be components of $g^{-1}(Q)$. Since

$$g^{-1}(Q) = A \cap f^{-1}(Q), \tag{4.2.1}$$

for each $i \in \{1, 2\}$ the component C_i lies in a component C_i' of $f^{-1}(Q)$. It follows from $C_i \subset A$ that

$$C_i = A \cap C_i \subset A \cap C_i', \tag{4.2.2}$$

and from $Q \subset f(A) \subset B$ we infer that

$$C_i' \subset f^{-1}(B). \tag{4.2.3}$$

According to the assumptions regarding A, conditions (4.2.2) and (4.2.3) give $C_i' \subset A$, whence $C_i' \subset g^{-1}(Q)$ by (4.2.1). Thus $C_i' = C_i$ and $g(C_i) = f(C_i)$ for each $i \in \{1, 2\}$. Consequently, $g \in \mathfrak{M}$, as needed.

If $\mathfrak{M}$ is the class of atriodic mappings, the proof is quite similar to the above one. Indeed, besides some two components C_1 and C_2 of $g^{-1}(Q)$ satisfying $g(C_1) \cup g(C_2) = Q$, we have to consider an arbitrary component C of $g^{-1}(Q)$ and to show that either $g(C) = Q$, or $g(C) \subset \subset g(C_i)$ for some $i \in \{1, 2\}$. The details are left to the reader. ◊

Recall that the class of local homeomorphisms coincides with the class Loc($\mathfrak{H}$), where $\mathfrak{H}$ stands for the class of homeomorphisms (see [6, Th. 4.23, p. 18]) and that a similar coincidence holds for the class of locally confluent mappings, [6, Th. 4.24, p. 19]. Thus as an immediate consequence of Prop. 3.3 and Ths. 3.5, 4.1 and 4.2 we have the following result.

Corollary 4.3. *The classes of: local homeomorphisms, locally monotone, locally confluent, locally semi-confluent, locally feebly confluent, locally feebly semi-confluent, locally joining and locally atriodic mappings have* CRP.

5. Open, weakly confluent and related mappings — negative results

In the light of Cor. 4.3 and of a result of Whyburn in [7, Th. 7.2, p. 147] (saying that the restriction of an open mapping to an inverse set is open) a natural question arises if the class of open mappings has CRP. The next theorem shows that the mentioned result of Whyburn cannot be extended to CRP, i.e., that the above question has a negative answer. To show this we recall two concepts used to characterize open mappings between arcs.

Let a positive integer k be given, and let $m \in \{0, 1, \dots, k\}$. Define a surjection $g_k : [0,1] \to [0,1]$ by the following conditions:

(a) if m is even, then $g_k(\frac{m}{k}) = 0$, and if m is odd, then $g_k(\frac{m}{k}) = 1$;

(b) for each m, the restriction $g_k|[\frac{m}{k}, \frac{m+1}{k}] : [\frac{m}{k}, \frac{m+1}{k}] \to [0,1]$ is defined as linear.

Note that g_1 is the identity, g_2 is the well known tent mapping, and for each $k \in \mathbb{N}$ the mapping g_k is open.

Two surjective mappings $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ between topological spaces are said to be *equivalent* provided that there are homeomorphisms $h_X : X_1 \to X_2$ and $h_Y : Y_1 \to Y_2$ such that $f_2 \circ h_X = h_Y \circ f_1$.

The following characterizations of open mappings between arcs (or, equivalently, between closed intervals) and of OM-mappings between topological spaces are known, see [7, (1.3), p. 184] and [4, Cor. 3.1, p. 104, and (2.2), p. 102], respectively.

Proposition 5.1. *A surjective mapping $f : X \to Y$ between arcs X and Y is open if and only if f is equivalent to $g_k : [0,1] \to [0,1]$ for some $k \in \mathbb{N}$.*

Proposition 5.2. *A mapping $f : X \to Y$ between spaces X and Y is an OM-mapping if and only if $\lim y_n = y$ implies that $\operatorname{Ls} f^{-1}(y_n)$ intersects each component of $f^{-1}(y)$.*

Theorem 5.3. *For each open mapping $f : X \to Y$ (that is not a homeomorphism) between arcs X and Y there exist closed subsets $A \subset X$ and $B \subset Y$ such that A is the union of some components of $f^{-1}(B)$ and that the restriction $f|A : A \to f(A) = B$ is neither an OM-mapping nor an MO-mapping (thus in particular it is not open).*

Proof. According to Prop. 5.1 it is enough to show the result for $X = Y = [0,1]$ and for $f = g_k$, where $k \geq 2$. To this aim for a fixed $k \geq 2$ put

$$B = \{0\} \cup \bigcup \{[\tfrac{1}{2^{2n-1}}, \tfrac{1}{2^{2n}}] : n \in \mathbb{N}\}.$$

Thus B is a closed subset of $Y = [0,1]$. Define

$$A = \{0\} \cup ([\tfrac{1}{k}, \tfrac{2}{k}] \cap f^{-1}(B)),$$

and note that A is a closed subset of $X = [0,1]$. To see that $g_k|A : A \to g_k(A) = B$ is not an OM-mapping apply Prop. 5.2 with $y_n = \frac{1}{2^{2n}}$. Then $y = \lim y_n = 0$ and $(g_k|A)^{-1}(y_n)$ is a singleton $\{x_n\}$ in $[\frac{1}{k}, \frac{2}{k}]$ such that $\mathrm{Ls}\{x_n\} = \{\lim x_n\} = \{\frac{2}{k}\}$. On the other hand, $(g_k|A)^{-1}(y) = \{0, \frac{2}{k}\}$ and the singleton $\{0\}$ is a component of $(g_k|A)^{-1}(y)$ that is disjoint with $\mathrm{Ls}\,(g_k|A)^{-1}(y_n)$. So, $g_k|A$ is not an OM-mapping. It is also neither an MO-mapping nor even a locally MO-mapping since all MO-mappings are locally MO-mappings, which in turn are OM-mappings, see [4, Cor. 3.2, p. 104]. The proof is complete. ◊

The previous theorem leads to the following corollary.

Corollary 5.4. *The classes of open mappings,* OM-*mappings,* MO-*mappings and locally* MO-*mappings do not have* CRP.

Note that we need not to consider the class of locally OM-mappings because this class coincide with the class of OM-mappings, [6, (4.29), p. 20].

Consider now the class of weakly confluent mappings. We start with the following example, where $\mathbb{S}^1$ denotes the unit circle in the complex plane.

Example 5.5. There exist a weakly confluent mapping $f : [0,1] \to \mathbb{S}^1$, an arc $B \subset \mathbb{S}^1$ and the union A of some components of $f^{-1}(B)$ such that the restriction $f|A : A \to f(A) = B$ is not pseudo-confluent (thus not weakly confluent).

Proof. Define $f(x) = \exp(4i\pi x)$ for $x \in [0,1]$. Let $B = \{z \in \mathbb{S}^1 : \arg z \in [-\pi/4, \pi/4]\}$. Then $f^{-1}(B)$ has three components: $[0, \frac{1}{16}]$, $[\frac{9}{16}, \frac{11}{16}]$ and $[\frac{15}{16}, 1]$. Taking $A = [0, \frac{1}{16}] \cup [\frac{15}{16}, 1]$ we see that $f(A) = B$ and no component of $(f|A)^{-1}(B)$ is mapped onto the whole B under $f|A$. ◊

Note however, that if $C = [\frac{9}{16}, \frac{11}{16}]$, then $f|C : C \to f(C) = B$ is a homeomorphism, thus (in particular) it is weakly confluent. In the light of the above, the following questions seem to be interesting.

Questions 5.6. Let a mapping $f : X \to Y$ between compact spaces be weakly confluent, and let a subset $B \subset Y$ be given. Does there exist a component C of $f^{-1}(B)$ such that $f(C) = B$ and $f|C : C \to B$ is weakly confluent? If not, is the implication true under an additional assumption that the range space Y is locally connected?

As a consequence of Ex. 5.5 we get the following.

Corollary 5.7. *The classes of weakly confluent and of pseudo-confluent mappings do not have* CRP.

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