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ON DECOMPOSITIONS OF D.G. SEMINEAR-RINGS

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Abstract: The existence of the product of d.g. seminear-rings in the category of all d.g. seminear-rings is proved. Then the d.g. subdirect product of a family of d.g. seminear-rings is defined, and some relations with the underlying generating semigroups are discussed. Furthermore, a well-known result of Birkhoff about subdirect decomposition of an algebra is generalized for the case of seminear-rings. Namely, we prove that every d.g. seminear-ring is a d.g. subdirect product of d.g. subdirectly irreducible d.g. seminear-rings.

1. Introduction

A lot of work concerning d.g. near-rings have been done, in particular, decomposition of d.g. near-rings. It seems that some concepts can be established and a lot of results can be extended to the case of d.g. seminear-rings. In this paper we first prove the existence of the product of d.g. seminear-rings in the category of all d.g. seminear-rings. Then we define the d.g. subdirect product of a family of d.g. seminear-rings in the category of all d.g. seminear-rings and prove some related results. In fact, we prove that every d.g. seminear-ring is a d.g. subdirect product of d.g. subdirectly irreducible d.g. seminear-rings. We note that

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some fundamental ideas are defined in a way analogous to the case of d.g. near-rings. However, some concepts are not because semigroups are involved rather than groups. So, in order to start, we need some basic definitions and elementary results.

A set R with two binary operations $+$ and $.$ is called a (left) seminear-ring if $(R,+)$ and $(R,.)$ are semigroups and the left distributive law $a(b+c) = ab + ac$ for all $a, b, c \in R$ is satisfied. An element $d \in R$ is called distributive if $(a+b)d = ad + bd$ for all $a, b \in R$. A natural example of a seminear-ring is the set $M(S)$ of all mappings on a semigroup $(S,+)$ with the operations of pointwise addition and multiplication given by composition of maps. A seminear-ring R is called a distributively generated (d.g.) seminear-ring if R contains a multiplicative subsemigroup $(S,.)$ of distributive elements which generates $(R,+)$. Note that S need not be the semigroup of all distributive elements and such a d.g. seminear-ring is denoted by (R,S). If we consider the above seminear-ring $M(S)$ then the set $\mathrm{End}(S)$, of all endomorphisms of S, is a distributive subsemigroup of $M(S)$, and generates a d.g. seminear-ring denoted by $(\mathrm{E}(S), \mathrm{End}(S))$. A mapping $\theta : (R,S) \longrightarrow (T,U)$ is called a seminear-ring homomorphism if θ is both a semigroup homomorphism from $(R,+)$ to $(T,+)$ and also from $(R,.)$ to $(T,.)$; and such a homomorphism is called a d.g. seminear-ring homomorphism if, in addition, it satisfies that $S\theta \subseteq U$. It is known [3] that a semigroup homomorphism $\theta : (R,+) \longrightarrow (T,+)$ is a d.g. seminear-ring homomorphism from (R,S) to (T,U) if and only if θ is a semigroup homomorphism from $(S,.)$ to $(U,.)$. Unless otherwise stated, we will be using the term homomorphism to mean a d.g. seminear-ring homomorphism. In [3], the free d.g. seminear-ring $(\mathrm{Frs}(S), S)$ on a semigroup S was constructed, where $(\mathrm{Frs}(S),+)$ is the free semigroup on S. Moreover, every d.g. seminear-ring is a homomorphic image of a free d.g. seminear-ring.

Let $\{A_\lambda; \lambda \in \Lambda\}$ be a family of objects in a category Ω. A product for the family is a family of morphisms $\{\alpha_\lambda : A \longrightarrow A_\lambda; \lambda \in \Lambda\}$ with the property that for any family $\{f_\lambda : B \longrightarrow A_\lambda; \lambda \in \Lambda\}$ there is a unique morphism $\phi : B \longrightarrow A$ such that $\phi\alpha_\lambda = f_\lambda$ for each $\lambda \in \Lambda$.

In order to consider some results on subdirect product of d.g. seminear-rings we first need to prove the existence of the product of d.g. seminear-rings in the category of all d.g. seminear-rings. This is the aim of the following section.

2. Product of d.g. seminear-rings

Let Ω be the category of all d.g. seminear-rings. Let $\{(R_\lambda, S_\lambda) : \lambda \in \Lambda\}$ be a family of d.g. seminear-rings in Ω. Let $P = \prod_{\lambda\in\Lambda} R_\lambda$, then P is a seminear-ring which is not necessarily a d.g. seminear-ring. Now let $S = \prod_{\lambda\in\Lambda} S_\lambda$, then it can be seen that S is a distributive subsemigroup of P. Thus S generates a sub d.g. seminear-ring (R, S) of P. Now we prove the following.

Theorem 2.1. *The d.g. seminear-ring (R, S) is the product in Ω of the family $\{(R_\lambda, S_\lambda) : \lambda \in \Lambda\}$ of d.g. seminear-rings in Ω.*

Proof. Consider the seminear-ring $P = \prod_{\lambda\in\Lambda} R_\lambda$ and also the subsemigroup $S = \prod_{\lambda\in\Lambda} S_\lambda$ of P. Let $p_\lambda : P \longrightarrow R_\lambda$ be the projection map for each $\lambda \in \Lambda$. Then it can be seen that p_λ maps S onto $S_\lambda \subseteq R_\lambda$, for each $\lambda \in \Lambda$. It follows that R is mapped onto R_λ. Thus, for each $\lambda \in \Lambda, p_\lambda|_R$ is a d.g. seminear-ring homomorphism. Let $q_\lambda = p_\lambda|_R, \lambda \in \Lambda$. Then $q_\lambda : (R, S) \longrightarrow (R_\lambda, S_\lambda)$ is a d.g. seminear-ring epimorphism for each $\lambda \in \Lambda$. Let (T, U) be a d.g. seminear-ring in Ω together with a family $\{\psi_\lambda : (T, U) \longrightarrow (R_\lambda, S_\lambda)\}_{\lambda\in\Lambda}$ of d.g. seminear-ring homomorphisms. We can consider $\psi_\lambda : T \longrightarrow R_\lambda, \lambda \in \Lambda$ as seminear-ring homomorphisms. By the property of products there exists a unique seminear-ring homomorphism $\phi : T \longrightarrow P$ such that $\phi p_\lambda = \psi_\lambda$ for each $\lambda \in \Lambda$. For $t \in T$, we have $t\phi = t\psi_\lambda, \lambda \in \Lambda$. Now since ψ_λ is a d.g. seminear-ring homomorphism, then ψ_λ maps U into S_λ in R_λ. Therefore ϕ maps $U \subseteq T$ into $S \subseteq P$ and so T is mapped into R. Hence ϕ is a d.g. seminear-ring homomorphism from (T, U) into (R, S). Moreover, $\phi q_\lambda = \psi_\lambda, \lambda \in \Lambda$ as d.g. seminear-ring homomorphisms. Finally, the uniqueness of ϕ as a d.g. seminear-ring homomorphism follows from the uniqueness of ϕ as a semigroup homomorphism. This completes the proof. $\Diamond$

3. Subdirect decompositions

Now the product of d.g. seminear-rings exists, so given a family $\{(T_\lambda, U_\lambda) : \lambda \in \Lambda\}$ of d.g. seminear-rings in the category of all d.g. seminear-rings, then the product of this family is $\{p_\lambda : (T, U) \to \to (T_\lambda, U_\lambda); \lambda \in \Lambda\}$, where $\{p_\lambda|_U : U \longrightarrow U_\lambda; \lambda \in \Lambda\}$ is the product of the set $\{U_\lambda; \lambda \in \Lambda\}$ of semigroups.

Definition 3.1. A d.g. seminear-ring (R, S) is called a *d.g. subdirect product* of the set $\{(T_\lambda, U_\lambda); \lambda \in \Lambda\}$ of d.g. seminear-rings if there exists a monomorphism $\eta : (R,S) \longrightarrow (T,U)$ such that ηp_λ is an epimorphism for each $\lambda \in \Lambda$. In this case we write $\{(R,S) \xrightarrow{\eta} (T,U) \xrightarrow{p_\lambda} (T_\lambda, U_\lambda);$ $\lambda \in \Lambda\}$.

The above definition will lead to the following result.

Theorem 3.2. *Let* (R,S) *be a d.g. seminear-ring having a family of congruences* $\{\rho_\lambda; \lambda \in \Lambda\}$ *such that* $\bigcap\{\rho_\lambda; \lambda \in \Lambda\}$ *is trivial. Then* (R,S) *is a subdirect product of* $\{(R,S)/\rho_\lambda; \lambda \in \Lambda\}$.

Proof. Let $(M_\lambda, S_\lambda) = (R,S)/\rho_\lambda$, where S_λ is the image of S in M_λ. Then the product $\{(T,U) \xrightarrow{p_\lambda} (M_\lambda, S_\lambda); \lambda \in \Lambda\}$, is defined by considering T as the subseminear-ring of $\prod_{\lambda\in\Lambda} M_\lambda$ generated by $U = \prod_{\lambda\in\Lambda} S_\lambda$. Now let $\eta : (R,S) \longrightarrow \prod_{\lambda\in\Lambda} M_\lambda$, defined by $r\eta = r\rho_\lambda$, $\lambda \in \Lambda$. Then $(R,S)\eta \subseteq T$. Now the result follows using standard methods. $\Diamond$

Remark 3.3. Note that if (R,S) is a subdirect product of d.g. seminear-rings $\{(T_\lambda, U_\lambda); \lambda \in \Lambda\}$; given by $\{(R,S) \xrightarrow{\eta} (T,U) \xrightarrow{p_\lambda} (T_\lambda, U_\lambda); \lambda \in \Lambda\}$, then this induces a subdirect decomposition of S: $\{S \xrightarrow{\eta} U \xrightarrow{p_\lambda} U_\lambda;$ $\lambda \in \Lambda\}$, by considering $\eta|_S, p_\lambda|_U$, for $\lambda \in \Lambda$.

Theorem 3.4. *Let* S *be a semigroup with a subdirect decomposition*

$$\{S \xrightarrow{\eta} U \xrightarrow{p_\lambda} U_\lambda; \lambda \in \Lambda\}. \tag{3.1}$$

Then there exists a d.g. seminear-ring (R,S) *which has a d.g. subdirect decomposition*

$$\{(R,S) \xrightarrow{\alpha} (T,U) \xrightarrow{q_\lambda} (Frs(U_\lambda), U_\lambda); \lambda \in \Lambda\},$$

giving rise to (3.1).

Proof. First consider $\eta : S \longrightarrow U$. Then η can be extended uniquely to a d.g. seminear-ring homomorphism $\bar{\eta} : (Frs(S), S) \longrightarrow (Frs(U), U)$. Indeed, $\bar{\eta}$ is monomorphism since it maps a set of generators to a set of generators. Similarly, p_λ can be extended to a d.g. seminear-ring monomorphism $\overline{p}_\lambda : (Frs(U), U) \longrightarrow (Frs(U_\lambda), U_\lambda)$, for each $\lambda \in \Lambda$. Thus we now have

$$(Frs(S), S) \xrightarrow{\bar{\eta}} (Frs(U), U) \xrightarrow{\overline{p}\lambda} (Frs(U_\lambda), U_\lambda), \text{ for each } \lambda \in \Lambda.$$

Let $\{q_\lambda : (T,U) \longrightarrow (Frs(U_\lambda), U_\lambda); \lambda \in \Lambda\}$ be the product of $\{(Frs(U_\lambda), U_\lambda); \lambda \in \Lambda\}$ in the category of all d.g. seminear-rings. Then there exists a unique homomorphism $\phi : (Frs(U), U) \longrightarrow (T,U)$ such that $\phi q_\lambda = \overline{p}_\lambda$, for each $\lambda \in \Lambda$. Now consider the following diagram

$$\begin{array}{ccccc} (Frs(S),S) & \xrightarrow{\bar{\eta}} & (Frs(U),U) & \xrightarrow{\bar{p}_\lambda} & (Frs(U_\lambda),U_\lambda) \\ \pi\downarrow & & \phi\downarrow & \nearrow q_\lambda & \\ (R,S) & \xrightarrow{\quad\alpha\quad} & (T,U) & & \end{array}$$

where π is the natural homomorphism with $\mathrm{Ker}\pi = \mathrm{Ker}\bar{\eta}\phi$. Hence there exists a unique homomorphism $\alpha : (R,S) \longrightarrow (T,U)$ such that the above diagram commutes. Now, $\pi\alpha = \bar{\eta}\phi$, $\mathrm{Ker}\,\pi = \mathrm{Ker}\,\bar{\eta}\phi$ and π is an epimorphism. Hence α is a monomorphism. This completes the proof. ◊

We close with the following result which extends Birkhoff's result to the case of d.g. seminear-rings.

Theorem 3.5. *Every d.g. seminear-ring is a d.g. subdirect product of d.g. subdirectly irreducible seminear-rings.*

Proof. Let (R,S) be a d.g. seminear-ring. Consider R as a seminear-ring. Applying Birkhoff's decomposition to R, being a seminear-ring, we get a subdirect decomposition

$$\{R \xrightarrow{\eta} N \xrightarrow{p_\lambda} T_\lambda; \lambda \in \Lambda\}, \tag{3.2}$$

where each T_λ is a subdirectly irreducible seminear-ring. For each $\lambda \in \Lambda$, let $U_\lambda = S\eta p_\lambda$. Since ηp_λ is an epimorphism for each $\lambda \in \Lambda$, then U_λ is a distributive subsemigroup of T_λ for each $\lambda \in \Lambda$ and $(T_\lambda, +) = sg\,\langle U_\lambda \rangle$. Hence $\{(T_\lambda, U_\lambda); \lambda \in \Lambda\}$ is a set of d.g. seminear-rings with d.g. seminear-ring homomorphisms ηp_λ. Let $\{q_\lambda : (T,U) \to (T_\lambda, U_\lambda); \lambda \in \Lambda\}$ be the product of $\{(T_\lambda, U_\lambda); \lambda \in \Lambda\}$. As seen in section 2, T is the subseminear-ring of N generated by U and $q_\lambda = p_\lambda|_T$. Moreover, $S\eta \subseteq U$ and $R\eta \subseteq T$. Thus $\{(R,S) \xrightarrow{\eta} (T,U) \xrightarrow{q_\lambda} (T_\lambda, U_\lambda); \lambda \in \Lambda\}$ is a d.g. subdirect decomposition of (R,S). But each T_λ is a subdirctly irreducible as a seminear-ring, and so d.g. subdirectly irreducible. This completes the proof. ◊

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