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LIOUVILLE'S THEOREM IN A PSEUDO-CONFORMAL SPACE

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Abstract: An extension of a famous theorem of Benz–Liouville is given providing also a new proof for Benz result.

1. Introduction

Liouville's Theorem on conformal mappings in space is usually proved in textbooks under the assumption, made tacitly or explicitly, that the mapping considered is of class C^3. In 1958 however, Ph. Hartman gave a proof which is valid for mappings of class C^1 (cf. [5]). Another such proof which does not rely on any differentiability assumptions at all, has since been given by Yuri G. Rešetnyak [7].

The problem of generalizing Liouville's Theorem to spaces where the underlying quadratic form is no longer assumed positive definite has also been dealt with by several authors. Thus Johannes Haantjes in 1937 (cf. [3]) considered a pseudo-Euclidean space E with non-definite form and what he called conformal representations of the space E on itself. But in fact the mappings he arrived at are mappings of E together with certain ideal elements which he did not bother to determine since he could handle his mappings by means of analytic formulae. In his monograph [2] Walter Benz considers E together with a non-degenerate form of arbitrary signature. He defines a group of generalized spherical transformations acting on E together with certain well-specified ideal elements depending on the form, and then proves in complete analogy to Liouville's classical Theorem that mappings conformal with respect to the underlying form and defined in a connected region of E are the parts of a generalized spherical transformation restricted to that region. The assumptions in Benz's work are the same as in the classical proofs, namely, that the mapping is of class C^3. The assumptions required in [3] seem weaker at first sight but we shall see in Section 4 that this is only apparently so.

It is the purpose of this note to show that the arguments of Ph. Hartman hold almost without change also in the more general situation of Benz–Liouville. An exception has to be made only for signatures such as $(\varepsilon_1 = 1, \varepsilon_2 = -1, ..., \varepsilon_d = -1)$ or $(\varepsilon_1 = 1, ..., \varepsilon_{d-1} = 1, \varepsilon_d = -1)$ where either the value $\varepsilon_1 = 1$ or the value $\varepsilon_d = -1$ occurs only once. For these exceptional signatures it remains an open question whether the hypothesis that the considered mapping should be of class C^3 can be relaxed. We shall thus prove

Theorem 1.1. *(Benz–Liouville) Let G be an open and connected region of $\mathbb{R}^d$, where $d \geq 3$, and let*

$$(u^1, u^2, ..., u^d) \to (v^1(u^1, ..., u^d), ..., v^d(u^1, ..., u^d)) \tag{1}$$

be a mapping of class C^1 of G into $\mathbb{R}^d$ whose Jacobian matrix satisfies identically in G the relations

$$\sum_{k=1}^{d} v_i^k v_j^k \varepsilon_k = \varepsilon_i \varepsilon \gamma^2 \delta_{ij} \tag{2}$$

where $\gamma = \gamma(u^1, u^2, ..., u^d) > 0$ and $\varepsilon = \pm 1$. If the ε_i do not belong to one of the exceptional signatures then there exists a spherical transformation of signature $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_d)$ which in the region G coincides with the given mapping.

The proof will follow very closely the ideas of Ph. Hartman [5]. But the arguments given in [5] are often only rough indications where the reader has to provide much of the details by himself. Yet, quod licet Iovi non licet bovi, and therefore it seemed appropriate to be a little more explicit here.

The paper is organized as follows. Section 2 contains the definitions required in the Benz–Liouville Theorem. Sections 3–7 contain auxiliary results from tensor algebra, tensor analysis, and functional analysis. Section 8 contains the proof of the theorem.

2. Generalized spherical geometry

We consider an affine space of dimension $d \geq 3$ over the real numbers. In the underlying vector space V let there be given a non-degenerate symmetric bilinear form $\langle u, v\rangle$. We shall say u and v are orthogonal to each other and write $u \perp v$ in this case when $\langle u, v\rangle = 0$. We are interested in the group S of all mappings leaving invariant the relation $\perp$. These mappings will be called pseudo-Euclidean similarities. Let γ be one of them. As the vectors orthogonal to a fixed non-zero vector form a hyperplane it is easy to see that γ transforms hyperplanes into hyperplanes. From this it follows that γ preserves lines (cf. [2], p. 135). As the field of real numbers does not admit any non-trivial automorphisms such a mapping must have the form $\gamma : x \to x^\sigma + b$ where σ is a linear transformation of the underlying vector space and b a fixed vector. The condition that the relation $\perp$ remains invariant implies that $\langle x, y\rangle = 0$ is equivalent with $\langle x^\sigma, y^\sigma\rangle = 0$.

From this equivalence we may conclude that there exists a constant numerical factor $\mu \neq 0$ such that $\langle x^\sigma, y^\sigma\rangle = \mu\langle x, y\rangle$ for all x, y from V. We may set $\mu = \lambda\varepsilon$ where $\lambda > 0$ and $\varepsilon^2 = 1$.

According to Sylvester's Theorem we can choose a basis $v_1, v_2, \ldots, v_d$ in V such that $\langle v_i, v_j\rangle = \varepsilon_i\delta_{ij}$. Let us set $v_i^\sigma = \alpha_i^1 v_1 + \alpha_i^2 v_2 + \ldots + \alpha_i^d v_d$. Then $A = (\alpha_i^j)$ is the matrix of the linear transformation σ with respect to the basis $v_1, v_2, \ldots, v_d$. It follows that

$$\begin{aligned}\lambda\varepsilon\langle v_i, v_j\rangle &= \langle v_i^\sigma, v_j^\sigma\rangle = \\ &= \langle \alpha_i^1 v_1 + \alpha_i^2 v_2 + \ldots + \alpha_i^d v_d, \alpha_j^1 v_1 + \alpha_j^2 v_2 + \ldots + \alpha_j^d v_d\rangle = \\ &= \alpha_i^1\alpha_j^1\langle v_1, v_1\rangle + \alpha_i^2\alpha_j^2\langle v_2, v_2\rangle + \ldots + \alpha_i^d\alpha_j^d\langle v_d, v_d\rangle = \\ &= \alpha_i^1\alpha_j^1\varepsilon_1 + \alpha_i^2\alpha_j^2\varepsilon_2 + \ldots + \alpha_i^d\alpha_j^d\varepsilon_d.\end{aligned}$$

The rows of the matrix of σ thus satisfy the conditions

$$\text{(R)} \qquad \alpha_i^1\alpha_j^1\varepsilon_1 + \alpha_i^2\alpha_j^2\varepsilon_2 + \ldots + \alpha_i^d\alpha_j^d\varepsilon_d = \lambda\varepsilon\varepsilon_i\delta_{ij}$$

where $\lambda > 0$ and $\varepsilon^2 = 1$. Conversely, if the matrix $A = (\alpha_i^j)$ satisfies relations (R) then it belongs to a linear transformation such that $\langle x^\sigma, y^\sigma\rangle = \lambda\varepsilon\langle x, y\rangle$. Let E denote the diagonal matrix having the elements $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d$ along the main diagonal and zeroes everywhere else. The relations (R) are equivalent to the matrix equation $AEA^\top = \mu E$ where $\mu = \lambda\varepsilon$. The inverse matrix satisfies $A^{-1}E(A^{-1})^\top = \mu^{-1}E$.

Since $(A^{-1})^\top = (A^\top)^{-1}$ and $E^{-1} = E$ it follows that $A^\top EA = \mu E$ and therefore the matrix equations $AEA^\top = \mu E$ and $A^\top EA = \mu E$ are equivalent to each other. The last equation may be expressed as a set of relations satisfied by the columns

$$\text{(C)} \qquad \alpha_1^i \alpha_1^j \varepsilon_1 + \alpha_2^i \alpha_2^j \varepsilon_2 + \ldots + \alpha_d^i \alpha_d^j \varepsilon_d = \lambda\varepsilon\varepsilon_i \delta_{ij}$$

Our findings may be summed up as

Lemma 2.1. *For an arbitrary $d \times d$ matrix A the following assertions are equivalent:*

i) *A belongs to an affine transformation of the group S,*
ii) *relations (R) hold, i.e. $AEA^\top = \mu E$ for $\mu \neq 0$,*
iii) *relations (C) hold, i.e. $A^\top EA = \mu E$ for $\mu \neq 0$.* ◊

The hyperplanes of the affine space can be represented by an equation of the form $H(u, \alpha) : \langle u, x \rangle = \alpha$. Here naturally $u \neq 0$, for otherwise $H(u, \alpha)$ would be the entire space or empty. $H(u, \alpha)$ and $H(u_1, \alpha_1)$ represent the same hyperplane if and only if for a certain $\beta \neq 0$ we have $u_1 = \beta u$ and $\alpha_1 = \beta\alpha$. Let $\gamma : x \rightarrow x^\sigma + b$ be an affine transformation which leaves $\perp$ invariant. If $x \in H(u, \alpha)$, it follows $x^\sigma + b \in H(u^\sigma, \lambda\varepsilon\alpha + \langle u^\sigma, b \rangle)$. Therefore $H(u^\sigma, \lambda\varepsilon\alpha + \langle u^\sigma, b \rangle)$ represents the image of the hyperplane $H(u, \alpha)$.

A hyperplane $H(u, \alpha)$ is called isotropic when $\langle u, u \rangle = 0$. It follows that isotropic hyperplanes are mapped by γ to isotropic ones and non-isotropic hyperplanes are mapped to non-isotropic ones. Let Λ denote the set of isotropic hyperplanes. The group S thus operates on the set $V \cup \Lambda$. Moreover, the group S considered as a permutation group on $V \cup \Lambda$ admits of a unique transitive extension. That means the following:

Theorem 2.1. *Let the symbol ∞ denote an element not belonging to $V \cup \Lambda$. Then there is a unique permutation group Γ operating transitively on the set $V \cup \Lambda \cup \{\infty\}$ such that the stabilizer Γ_∞ of the element ∞ coincides with the group S of pseudo-Euclidean similarities.*

This is proved in [2]. The group Γ is called the group of spherical transformations of signature $s = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d)$. The set $V \cup \Lambda \cup \{\infty\}$ may be called the pseudo-conformal space $M_n(s)$ of the given signature in analogy with the ordinary conformal space (or Möbius space) obtained from V by adjoining only one point at infinity.

3. Tensor algebra

Let V be a vector space of dimension d over a field K of characteristic $\neq 2$. Let $V^* = \mathrm{Hom}(V, K)$ denote the dual vector space of all linear forms on V. A p times contravariant and q times covariant tensor over V is an element t of the tensor product

$$\left(\bigotimes^{p} V\right) \bigotimes \left(\bigotimes^{q} V^*\right).$$

The pair of numbers (p, q) is called the type of the tensor t.

In what follows we shall frequently use Einstein's summation convention: when in a term of a formula a letter occurs as an upper as well as a lower index, then summation is implied over this index from 1 to $d = \dim V$.

Let C^{im} be a non-singular matrix. The denotation is meant to indicate that we refer to the components of a tensor of type $(2,0)$. Such a tensor corresponds to a nonsingular linear transformation $\sigma : V^* \to V$ according to the rule $\sigma(v^m) = C^{mi} v_i$ where v^m and v_i are base vectors from a pair of dual bases of V^* and V respectively.

Theorem 3.1. *When* $\det C^{ij} \neq 0$, *then the tensor equations*

$$C^{im} v^l_{ik} = u^{ml}_k, v^l_{ik} = v^l_{ki}, u^{ml}_k = -u^{lm}_k$$

for $i, k, l, m = 1, 2, ..., d$ *can only be solved in the trivial way such that* $v^l_{ik} = 0$ *for all* i, k, l.

Proof. We consider the tensor $w^{mls} = C^{im} v^l_{ik} C^{ks}$. We have $w^{mls} = = u^{ml}_k C^{ks} = C^{im} u^{sl}_i = w^{slm}$ and moreover $w^{mls} = -w^{lms}$ because of $u^{ml}_k = -u^{lm}_k$. Now it follows that $w^{mls} = w^{slm} = -w^{lsm}$ and on the other hand $w^{mls} = -w^{lms} = -w^{sml} = w^{msl} = w^{lsm}$. Thus we get $w^{mls} = -w^{mls}$ which is only possible if all the components w^{mls} vanish. Because of $\det C^{ij} \neq 0$ this implies that all the components u^{ml}_k and v^l_{ik} vanish too. ◊

Corollary 3.1. *When* $\det C^{ik} \neq 0$, *the homogeneous equations*

$$C^{il} v^m_{ik} + C^{im} v^l_{ik} = 0, v^l_{ik} = v^l_{ki}$$

admit only the trivial solution $v^j_{ik} = 0$ *for all* $i, j, k = 1, 2, ..., d$.

Proof. We set $C^{il} v^m_{ik} = u^{lm}_k$. The given equations imply that $u^{lm}_k = = -u^{ml}_k$ and hence the assertion follows from Th. 3.1. ◊

4. Metric tensors

We consider pseudo metrics given by a tensor g_{ij} according to the formula

$$ds^2 = g_{ij}(u)du^i du^j.$$

With respect to the tensor g_{ij} we make the assumptions that it is symmetric and has nowhere vanishing determinant $\det(g_{ij}) \neq 0$. We do not require the assumption that the quadratic form $g_{ij}du^i du^j$ is positive definite.

By a transformation $v = v(u)$ of the coordinates we get $du^i = (\partial u^i/\partial v^k)dv^k$ and thus

$$ds^2 = g_{ij}\frac{\partial u^i}{\partial v^k}\frac{\partial u^j}{\partial v^m}dv^k dv^m = h_{km}dv^k dv^m.$$

This means that the g_{ij} are transformed into the h_{km} according to the rule

$$h_{km}(v) = g_{ij}(u)\frac{\partial u^i}{\partial v^k}\frac{\partial u^j}{\partial v^m}.$$

The following is an adaptation of Th. II of P. Hartman [5] to our purposes. The only change we made, is that while the original theorem deals with Riemannian metrics here we also consider the more general pseudo metrics as explained above.

Theorem 4.1. *Let $u = (u^1, ..., u^d)$ and $v = (v^1, ..., v^d)$. Let $(g_{ik}(u))$, $(h_{ik}(v))$ be non-singular symmetric matrices of class C^σ defined in the neighbourhood of $u = 0$ and $v = 0$ respectively. Let $v = v(u)$ be a mapping of class C^1 defined in the neighbourhood of $u = 0$ which satisfies $v(0) = 0$ and transforms the pseudo metric*

$$ds^2 = g_{ik}du^i du^k \quad \text{into} \quad ds^2 = h_{ik}dv^i dv^k.$$

Then $v = v(u)$ is necessarily of class $C^{1+\sigma}$.

Proof. From the transformation rule for pseudo metric tensors we have

$$h_{km}(v) = g_{ij}(u)\frac{\partial u^i}{\partial v^k}\frac{\partial u^j}{\partial v^m}.$$

By passing to the inverse matrices h^{km} and g^{ij} we obtain

$$\text{(P)} \qquad g^{ij}(u)v_i^k(u)v_j^m(u) - h^{km}(v) = 0$$

where $v_i^k = \partial v^k(u)/\partial u^i$ and $v_j^m = \partial v^m(u)/\partial u^j$. The functions of $(u, v, v_1^1, ..., v_d^d)$ on the left-hand side in the above system of $\frac{1}{2}d(d +$

$+1)$ partial differential equations are by hypothesis of class C^σ in their $d+d+d^2$ arguments.

By formal differentiation with respect to u^ν for $\nu = 1, ..., d$ we obtain a linear system of equations for the partial derivatives of second order of v. It has the form

$$(g^{ij}(u)v_j^m)v_{i,\nu}^k + (g^{ij}(u)v_i^k)v_{j,\nu}^m = ...$$

where the right-hand side is irrelevant for our purposes, since we are only interested in the determinant of the system. Setting $C^{im} = g^{ij}v_j^m$ because of $g^{ij} = g^{ji}$ we obtain further $C^{jk} = g^{ij}v_i^k$ and thus the equations can also be written in the form:

$$C^{im}v_{i,\nu}^k + C^{jk}v_{j,\nu}^m = ...$$

for $k < m$, and

$$2C^{im}v_{i,\nu}^m = ...$$

for $k = m$. Now from Cor. it follows immediately that the determinant of this system of equations does not vanish. Thus the assumptions of Th. I of [5] are satisfied for the system (P) and this implies the assertion. $\Diamond$

Note that in applications of Th. 4.1 the hypothesis $v(0) = 0$ is not essential. For if $v(0) = c \neq 0$ and the remaining hypotheses are satisfied mutatis mutandis we may set $\bar{v}(u) = v(u) - c$ and $h_{ik}(\bar{v}) = h_{ik}(v)$. Then $\partial u^i/\partial \bar{v}^k = \partial u^i/\partial v^k$ and all the hypotheses are satisfied for $\bar{v}(u)$.

With the aid of Th. 4.1 we are also in a better position to analyze the requirements in [3]. The mapping considered there has to be in class C^1 so that it is possible to define conformality. But in addition it is assumed in [3] that the mapping transforms a metric tensor g_{ik} with vanishing curvature into another such tensor h_{ik}. These tensors must be twice differentiable to express curvature. Therefore by Th. 4.1 the mapping itself also must be differentiable of a higher order.

5. Strong L^2-derivatives

The proof of Ph. Hartman is based on a certain generalization of the ordinary notion of a partial derivative, the so-called strong L^2-derivatives. For convenience of the reader we quote the basic definitions required to introduce this notion from the book [1] of Shmuel Agmon.

We also quote from this book a few facts on mollifiers. These are used to derive some auxiliary results to be used in later sections.

Let Ω be an open set in the n-dimensional Euclidean space E_n. The functions that are quadratically integrable on Ω, i.e. the functions f, for which $\int_\Omega f^2 dx$ exists form the Hilbert space $L^2(\Omega)$ with the scalar product

$$\langle f, g\rangle = \int_\Omega fg dx = \frac{1}{2}\Big[\int_\Omega (f+g)^2 dx - \int_\Omega f^2 dx - \int_\Omega g^2 dx\Big].$$

The L^2-norm of a function $f \in L^2(\Omega)$ is $\left(\int_\Omega f^2 dx\right)^{\frac{1}{2}}$ and is denoted by $\|f\|_{L^2(\Omega)}$. The scalar product $\langle f, g\rangle$ is continuous in the following sense: if $f_n \to f$ in $L^2(\Omega)$ (i.e. if $\|f_n - f\|_{L^2(\Omega)} \to 0$) and if $g \in L^2(\Omega)$ then $\langle f_n, g\rangle \to \langle f, g\rangle$ as $n \to \infty$. In particular, if Ω is bounded, it follows that $\int_\Omega f_n \to \int_\Omega f$, because we may take for g the characteristic function χ_Ω of Ω.

As usual let $C^m(\Omega)$ denote the set of functions on Ω that are at least m-times continuously differentiable. Further let $C^\infty(\Omega)$ denote the intersection $\bigcap_{m=0}^\infty C^m(\Omega)$ and $C_0^\infty(\Omega)$ the subset of $C^\infty(\Omega)$ consisting of all functions of compact support contained in Ω. The functions from $C_0^\infty(\Omega)$ are called test functions. For $x \in E_n$ and composed exponents $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ we use as shorthand $x^\alpha = x_1^{\alpha_1}...x_n^{\alpha_n}$ and similarly for differential operators

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2}...D_n^{\alpha_n} = \frac{\partial^{\alpha_1}}{\partial {x_1}^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial {x_2}^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial {x_n}^{\alpha_n}}.$$

We shall also use $(i), (i,j)$ etc. as shorthand for the vectors α having $\alpha_i = 1$ or $\alpha_i = 1, \alpha_j = 1$ respectively, and all the other components equal to zero.

Definition 5.1. For $u \in C^m(\Omega)$ set $\|u\|_{m,\Omega} = [\int_\Omega \sum_{|\alpha|\le m} |D^\alpha u|^2 dx]^{\frac{1}{2}}$ where $|\alpha| = \alpha_1 + ... + \alpha_n$.

Note that since $D^0 u = u$ it follows that $\|u\|_{0,\Omega}$ coincides with the L^2-norm.

Definition 5.2. Let $C^{m*}(\Omega)$ denote the subset of $C^m(\Omega)$ consisting of all functions u such that $\|u\|_{m,\Omega} < \infty$.

Let $H_m(\Omega)$ denote the completion of $C^{m*}(\Omega)$ with respect to the norm $\| * \|_{m,\Omega}$.

Definition 5.3. A function $u \in L^2(\Omega)$ has strong L^2-derivatives of order up to m, if in $C^{m*}(\Omega)$ there exists a sequence $\{u_k\}$ such that $\{D^\alpha u_k\}$ is a Cauchy-sequence in $L^2(\Omega)$ for all $|\alpha| \le m$ and $u_k \to u$ with respect to the norm in $L^2(\Omega)$.

Suppose u has strong L^2-derivatives of order up to m and for $|\alpha| \leq m$ let u^α denote the function such that $D^\alpha u_k \to u^\alpha$. If $\phi \in C_0^\infty(\Omega)$ is an arbitrary test function then integrating by parts and using the fact that $\phi \equiv 0$ in a neighbourhood of $\partial\Omega$ we get the relation

$$\int_\Omega \phi D^\alpha u_k dx = (-1)^{|\alpha|} \int_\Omega u_k D^\alpha \phi dx.$$

Here it is possible to pass to the limit as $k \to \infty$ under the intgral sign so that

$$\int_\Omega \phi u^\alpha dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha \phi dx, \qquad |\alpha| \leq m.$$

This motivates the following definition:

Definition 5.4. A locally integrable function u on Ω is said to have the weak derivative u^α if u^α is locally integrable on Ω and if for all test functions ϕ we have

$$\int_\Omega \phi u^\alpha dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha \phi dx.$$

As an immediate consequence from these definitions we get: If u has strong L^2-derivatives of order up to m then u also has weak derivatives of order up to m. Weak derivatives are unique in the sense of L^2-functions, i.e. if u^α and v^α are weak derivatives of the function u then we have almost everywhere $u^\alpha = v^\alpha$. For from the definition it follows that $\int_\Omega \phi(u^\alpha - v^\alpha) = 0$ for all test functions ϕ. As the set $C_0^\infty(\Omega)$ of test functions for each compact subset C of Ω is dense in $L^1(C)$ it follows that $u^\alpha - v^\alpha = 0$ almost everywhere.

From the uniqueness of weak derivatives we get as an immediate consequence the uniqueness of strong derivatives in the same sense, i.e. up to equivalence in $L^2(\Omega)$. This justifies the following denotation: if u has strong L^2-derivatives, and if $u_k \in C^{m*}(\Omega)$, $u_k \to u$ in $L^2(\Omega)$, and $D^\alpha u_k \to u^\alpha$ in $L^2(\Omega)$ we set $D^\alpha u = u^\alpha$.

The operators D^α for strong L^2-derivatives of first order commute in a similar way as for ordinary derivatives of C^2-functions. Consider e.g. the vectors (i), (j), and (i,j) so that $D^{(i)} = \partial/\partial x^i$, $D^{(j)} = \partial/\partial x^j$, and $D^{(i,j)} = \partial^2/\partial x^i \partial x^j$ if these operators are used in the ordinary sense. Then if u has strong L^2-derivatives $D^{(i)}u, D^{(j)}$ etc. and these in turn have strong L^2-derivatives $D^{(i)}(D^{(j)}u)$, $D^{(j)}(D^{(i)}u)$ etc. it follows that these latter are weak derivatives, for

$$\int_\Omega D^{(i)}(D^{(j)}u)\varphi dx = -\int_\Omega D^{(j)}uD^{(i)}\varphi dx =$$
$$= \int_\Omega uD^{(j)}(D^{(i)}\varphi)dx = \int_\Omega uD^{(i,j)}\varphi dx.$$

Since weak derivatives are unique we are justified writing $D^{(i)}(D^{(j)}u) = D^{(i,j)}u = D^{(j)}(D^{(i)}u)$.

We shall need some simple facts about strong L^2-derivatives that can be proved using mollifiers. We therefore quote some definitions and basic results about mollifiers from Agmon [1].

By $j(x)$ we mean a function from $C^\infty(E_n)$ with the properties $j(x) \geq 0, j(x) \equiv 0$ if $|x| \geq 1$, and

$$\int_{E_n} j(x)dx = 1.$$

We may for instance take the function

$$j(x) = c\exp\left(-\frac{1}{1-|x|^2}\right) \quad (|x| < 1), \quad j(x) \equiv 0, (|x| \geq 1)$$

where c is chosen so that the integral takes the value 1. Let $j_\epsilon(x) = (1/\epsilon^n)j(x/\epsilon)$. Note that $j_\epsilon(x)$ vanishes for $|x| \geq \epsilon$ and that

$$\int_{E_n} j_\epsilon(x)dx = 1.$$

Definition 5.5. The mollifier J_ϵ is defined by

$$J_\epsilon u(x) = \int_\Omega j_\epsilon(x-y)u(y)dy$$

for arbitrary functions which are locally integrable in Ω.

It is easy to see that $J_\epsilon u(x)$ is defined for all x having distance at least ϵ from the boundary $\partial\Omega$. If u is also integrable on bounded open subsets of Ω we make the assumption $u(x) = 0$ outside Ω. Then $J_\epsilon u(x)$ is defined everywhere in Ω. The properties of mollifiers needed here are summed up in the following theorems the proofs of which may be found in Agmon [1], pp. 5–6.

Theorem 5.1. *If u is integrable on bounded open subsets of Ω then $J_\epsilon u(x) \in C^\infty(\Omega)$.*

Theorem 5.2. *If $u \in L^2(\Omega)$ it follows that $J_\epsilon u \to u$ in $L^2(\Omega)$ as $\epsilon \to 0$. If u is continuous at x then $J_\epsilon u(x) \to u(x)$. The convergence is uniform for any compact subset of continuity points.*

Theorem 5.3. *If $u \in L^2(\Omega)$, then $\|J_\epsilon u\|_{0,\Omega} \leq \|u\|_{0,\Omega}$.*

Theorem 5.4. *Denote by $W_m(\Omega)$ the set of all functions in $L^2(\Omega)$ having weak derivatives in $L^2(\Omega)$ of order up to m. If $u \in W_m(\Omega)$ and $|\alpha| \leq m$ then for $x \in \Omega$ we have $(D^\alpha J_\epsilon u)(x) = (J_\epsilon D^\alpha u)(x)$ provided that x has distance at least ϵ from $\partial\Omega$.*

Let us assume $g \in L^2(\Omega)$ has strong L^2-derivatives of first order. If the i-th coordinate x^i is varied while the remaining coordinates are fixed we get a one-dimensional cross-section of Ω which we denote by Ω_q where q is the point with the fixed coordinates $x^1, ..., x^{i-1}, x^{i+1}, ..., x^n$.

As a consequence of the previous theorems we obtain:

Lemma 5.1. *If $g \in L^2(\Omega)$ is continuous and has strong L^2-derivatives of first order then for almost all points q the function g is absolutely continuous with respect to x^i on each closed interval contained in Ω_q. Moreover, the strong L^2-derivative with respect to x^i coincides a.e. in Ω with the ordinary derivative $\partial g/\partial x^i$.*

Proof. (Compare the proof of Lemma 4.4.5, p. 170 in Nikolskij, [6]). Let us first consider the one-dimensional case. It will suffice to assume $\Omega = (a, b)$. Let $\epsilon_j \to 0$ and $g_j = J_{\epsilon_j} g$. Then

$$g_j(x) = g_j(x_0) + \int_{x_0}^{x} \frac{dg_j}{dt} dt.$$

We have $g_j(x) \to g(x)$ for all $x \in (a, b)$ by Th. 5.2. If $D^{(1)}g$ denotes the L^2-derivative of g then by Th. 5.4 we also have $dg_j/dx \to D^{(1)}g$ in $L^2(I)$ for any closed interval $I \subseteq (a, b)$ since $dg_j/dx = J_{\epsilon_j}(D^{(1)}g)$ on I for almost all j. Hence it follows that

$$g(x) = g(x_0) + \int_{x_0}^{x} D^{(1)} g dt.$$

It follows from the fundamental theorem of differential and integral calculus (cf. [9], p. 342) that g is absolutely continuous on each closed interval contained in (a, b) and $D^{(1)}g$ coincides a.e. with the ordinary derivative dg/dx.

In the general case let I be an arbitrary n-dimensional open interval contained in Ω. Factorize it with respect to the first variable so that $I = (a, b) \times I_1$. If $u_k \to g$, $D^{(1)}u_k \to D^{(1)}g$ in $L^2(\Omega)$ then a fortiori $u_k \to g$, $D^{(1)}u_k \to D^{(1)}g$ in $L^2(I)$. By Fubini's Theorem it follows that there exists a subsequence $g_{k_\nu} = h_\nu$ such that for almost all points $(x^2, ..., x^n) \in I_1$ we have

$$h_\nu \to g \qquad \text{in } L^2(a, b)$$

and at the same time

$$D^{(1)}h_\nu \to D^{(1)}g \qquad \text{in } L^2(a,b).$$

Thus for a certain $(n-1)$-dimensional set M of measure zero the following is true: if $q = (x^2, ..., x^n) \notin M$ then g is absolutely continuous with respect to x^1 in each closed interval contained in I_q and by what has been said above $\partial g/\partial x^1$ exists a.e. in I_q and coincides a.e. with $D^{(1)}g$. This proves the assertions with respect to the first variable. For the remaining variables they follow by analogy. ◊

Lemma 5.2. *If $g \in L^2(\Omega)$ is continuous and has strong L^2-derivatives of first order which are also continuous then g belongs to $C^1(\Omega)$ and the L^2-derivatives are partial derivatives in the usual sense.*

Proof. Write $g(x) = g(x^1, y)$ where $y = (x^2, ..., x^n)$. Let $I = (x_0^1, x^1)\times$ $\times I_1$ denote an n-dimensional interval contained in Ω. Then the integral

$$F(x^1, y) = g(x_0^1, y) + \int_{x_0^1}^{x^1} D^{(1)}g(t, y)dt$$

exists for all $y \in I_1$. We know from the previous lemma that it is equal to $g(x^1, y) = g(x)$ for almost all y. But since $D^{(1)}g(t, y)$ is continuous it is easy to see that $F(x^1, y)$ is continuous in y. Since for each y we can find y_0 arbitrarily close to y such that $F(x^1, y_0) = g(x^1, y_0)$ it follows that $F(x^1, y) = g(x^1, y)$ for all $y \in I_1$. Thus we have proved that

$$g(x^1, y) = g(x_0^1, y) + \int_{x_0^1}^{x^1} D^{(1)}g(t, y)dt$$

for all $y \in I_1$. This implies that $D^{(1)}g = \partial g/\partial x^1$. For the remaining variables the assertion follows by analogy. ◊

Consider a bounded open set T and the set S of all points that have distance at most ρ from T. Here ρ is an arbitrarily chosen positive constant. For a function $g(x^1, ..., x^n)$ denote by $\Delta_i^h g$ the difference $g(x^1, ..., x^{i-1}, x^i + h, x^{i+1}, ..., x^n) - g(x^1, ..., x^n)$.

Lemma 5.3. *Let g be a function defined on S having the properties:* i) $g \in L^2(S)$, ii) *the partial derivatives $\partial g/\partial x^i = g_{x^i}$ exist almost everywhere in T and are quadratically integrable, i.e. $g_{x^i} \in L^2(T)$,* iii) $\frac{1}{h}\Delta_i^h g \to g_{x^i}$ *in $L^2(T)$. Under these hypotheses the functions $g_{x^i} =$ $= \partial g/\partial x^i$ are strong L^2-derivatives of $g|_{T_0}$ where T_0 is any open set contained in T and having positive distance from ∂T.*

Proof. Let $\bar{h}$ denote the vector having h for its i-th component and all other components zero. We then have $J_\epsilon(\Delta_i^h u)(x) = \Delta_i^h(J_\epsilon u)(x)$ if and only if,

$$\int_\Omega j_\epsilon(x+\bar{h}-y)u(y)dy = \int_\Omega j_\epsilon(x-y)u(y+\bar{h})dy.$$

Using the substitution $y-\bar{h}=z$ we see that

$$\int_\Omega j_\epsilon(x+\bar{h}-y)u(y)dy = \int_{\Omega_1} j_\epsilon(x-z)u(z+\bar{h})dz$$

where $\Omega_1 = \{x-\bar{h} \mid x \in \Omega\}$. Since $j_\epsilon(x-z)$ vanishes for $|x-z| \geq \epsilon$ it follows that

$$\int_{\Omega_1} j_\epsilon(x-z)u(z+\bar{h})dz = \int_\Omega j_\epsilon(x-z)u(z+\bar{h})dz,$$

provided that Ω and Ω_1 both contain the open sphere with radius ϵ around x; this is certainly the case if x has distance at least $\epsilon + h$ from $\partial\Omega$. Thus if ϵ and h are small enough we have $J_\epsilon(\Delta_i^h g)(x) = \Delta_i^h(J_\epsilon g)(x)$ and hence obviously also

$$\text{(c1)} \qquad J_\epsilon\left(\frac{1}{h}\Delta_i^h g\right)(x) = \frac{1}{h}\Delta_i^h(J_\epsilon g)(x)$$

for all $x \in T_0$. From this relation we may pass to the limit on both sides as $h \to 0$ and we obtain

$$\text{(c2)} \qquad \left(J_\epsilon\frac{\partial g}{\partial x^i}\right)(x) = \frac{\partial}{\partial x^i}(J_\epsilon g)(x).$$

For if $S(x,\epsilon)$ denotes the open sphere with center x and radius ϵ then $J_\epsilon(\frac{1}{h}\Delta_i^h g)(x)$ is the scalar product of $j_\epsilon(x-y)$ and $\frac{1}{h}\Delta_i^h g(y)$ as functions of y in the Hilbert space $L^2(S(x,\epsilon))$. As $\frac{1}{h}\Delta_i^h g$ tends to $\partial g/\partial x^i$ in $L^2(S(x,\epsilon))$, the left-hand side in (c1) actually tends to the left-hand side in (c2).

As before let $\|\cdot\|_{0,X}$ denote the L^2-norm on X. Let η be an arbitrary positive number. We choose a sequence $\epsilon_\nu \to 0$ and set $g_\nu = J_{\epsilon_\nu} g$. We choose h small enough so that $\|\frac{1}{h}\Delta_i^h g - g_{x^i}\|_{0,T} \leq \frac{1}{3}\eta$ and further $N = N(h)$ such that $\|\frac{1}{h}\Delta_i^h g_\nu - \frac{1}{h}\Delta_i^h g\|_{0,T} \leq \frac{1}{3}\eta$ for $\nu \geq N(h)$. It follows that

$$\left\|\frac{\partial g_\nu}{\partial x^i} - g_{x^i}\right\|_{0,T_0} \leq$$

$$\left\|\frac{\partial g_\nu}{\partial x^i} - \frac{1}{h}\Delta_i^h g_\nu\right\|_{0,T_0} + \left\|\frac{1}{h}\Delta_i^h g_\nu - \frac{1}{h}\Delta_i^h g\right\|_{0,T_0} + \left\|\frac{1}{h}\Delta_i^h g - g_{x^i}\right\|_{0,T_0}.$$

The rightmost and middle terms in the right-hand side of this inequality are easily seen to be $\leq \eta/3$ if $\nu \geq N(h)$. For the leftmost term we get

$$\left\|\frac{\partial g_\nu}{\partial x^i} - \frac{1}{h}\Delta_i^h g_\nu\right\|_{0,T_0} = \left\|J_{\epsilon_\nu}\left(\frac{\partial g}{\partial x^i} - \frac{1}{h}\Delta_i^h g\right)\right\|_{0,T_0} \leq$$

$$\leq \left\|J_{\epsilon_\nu}\left(\frac{\partial g}{\partial x^i} - \frac{1}{h}\Delta_i^h g\right)\right\|_{0,T} \leq \left\|\frac{\partial g}{\partial x^i} - \frac{1}{h}\Delta_i^h g\right\|_{0,T}$$

by Th. 5.3 and the commuting relations (c1) and (c2). It follows that

$$\left\|\frac{\partial g_\nu}{\partial x^i} - g_{x^i}\right\|_{0,T_0} \leq \eta$$

if $\nu > N(h)$. Thus the functions $\partial g_\nu/\partial x^i$ converge in $L^2(T_0)$ to g_{x^i} as $\nu \to \infty$. Moreover we have $g_\nu \in C^\infty(T) \subseteq C^1(T)$. As the closure of T_0 is contained in T it follows that $g_\nu|_{T_0} \in C^{1*}(T_0)$ and thus the lemma is proven. $\Diamond$

In the next two lemmas we shall consider composition of C^1-functions with functions having strong L^2-derivatives and show that the composed functions still have strong L^2-derivatives.

Lemma 5.4. *Let A and B be regions of $\mathbb{R}^n$. Let $f \in L^2(A)$ and assume that f has strong L^2-derivatives of first order. Let $y^i = y^i(x^1, ..., x^n)$ be the components of an invertible C^1-mapping of A onto B such that all the components $\partial y^i/\partial x^j$ of the Jacobian matrix are bounded on A and that there exists $d > 0$ such that $|\partial(y^1, ..., y^n)/\partial(x^1, ..., x^n)| \geq d$ on A. Then the transformed function $\bar{f}$ defined by $\bar{f}(y^1(x^1, ..., x^n), ..., y^n(x^1, ..., x^n)) = f(x^1, ..., x^n)$ belongs to $L^2(B)$ and has strong L^2-derivatives with respect to $y^1, ..., y^n$ which may be computed by the chain rule*

$$D^{(i)}\bar{f} = \sum_{j=1}^{n} D^{(j)} f(x(y))\partial x^j/\partial y^i.$$

Proof. We have to show first that $\bar{f} \in L^2(B)$. According to the substitution rule

$$\int_B \bar{f}^2 dy = \int_A \bar{f}(y(x))^2 |J| dx = \int_A f(x)^2 |J| dx$$

where $J = \partial(y^1, ..., y^n)/\partial(x^1, ..., x^n)$. Since all the components of the Jacobian matrix are bounded it follows that the Jacobian J is bounded too. Since J is also continuous it follows that the integral $\int_A f(x)^2|J|dx$ exists. Thus $\bar{f} \in L^2(B)$.

Let now $f_\nu \in C^{1*}(A)$ be a sequence of functions such that $f_\nu \to f$ and $D^{(j)} f_\nu \to D^{(j)} f$ in $L^2(A)$ for $j = 1, ..., n$ as $\nu \to \infty$. Let $\bar{f}_\nu$

denote the transformed functions $\bar{f}_\nu(y(x)) = f_\nu(x)$. They belong to C^1 and their derivatives can be computed by the chain rule $D^{(i)}\bar{f}_\nu = \sum_{j=1}^n D^{(j)} f_\nu(x(y))\partial x^j/\partial y^i$. Since $f_\nu \in C^{1*}(A)$ it follows that $f_\nu \in L^2(A)$ and in the same way as for $\bar{f}$ we see that $\bar{f}_\nu \in L^2(B)$. Using again the substitution rule it also follows from $\int_A (f_\nu - f)^2 dx \to 0$ that $\int_B (\bar{f}_\nu - \bar{f})^2 dy \to 0$, i.e. $\bar{f}_\nu \to \bar{f}$ in $L^2(B)$ as $\nu \to \infty$.

In order to show that $\bar{f}_\nu \in C^{1*}(B)$ we have to consider the integrals

$$\int_B (D^{(i)}\bar{f}_\nu)^2 dy.$$

Using again the substitution rule we see that

$$\int_B (D^{(i)}\bar{f}_\nu)^2 dy = \int_A \left(\sum_{j=1}^n D^{(j)} f_\nu(x)\partial x^j/\partial y^i(y(x)) \right)^2 |J| dx.$$

The integrand on the right-hand side is a sum of terms of the form

$$D^{(j)} f_\nu \partial x^j/\partial y^i(y(x)) D^{(k)} f_\nu \partial x^k/\partial y^i(y(x))|J|.$$

From the hypotheses of the lemma we can conclude that $\partial x^j/\partial y^i$ and $\partial x^k/\partial y^i$ are bounded and hence all the integrals over these terms exist. This implies that $\bar{f}_\nu \in C^{1*}(B)$.

In a similar way we see that $\int_B (D^{(j)}\bar{f})^2 dz$ exists so that $D^{(j)}\bar{f} \in L^2(B)$. We still have to show that

$$D^{(i)}\bar{f}_\nu \to D^{(i)}\bar{f} = \sum_{j=1}^n D^{(j)} f \partial x^j/\partial y^i$$

in $L^2(B)$ as $\nu \to \infty$. We obtain

$$\int_B \left(\sum_{j=1}^n D^{(j)} f_\nu(x(y))\partial x^j/\partial y^i - \sum_{j=1}^n D^{(j)} f(x(y))\partial x^j/\partial y^i \right)^2 dy =$$

$$= \int_A \left(\sum_{j=1}^n D^{(j)} f_\nu(x)\partial x^j/\partial y^i(y(x)) - \sum_{j=1}^n D^{(j)} f(x)\partial x^j/\partial y^i(y(x)) \right)^2 |J| dx.$$

The last integrand consists of a sum of terms of the four possible types

$$D^{(j)} h_1 D^{(k)} h_2 \partial x^j/\partial y^i(y(x))\partial x^k/\partial y^i(y(x)|J|$$

where (i) $h_1 = f_\nu$, $h_2 = f_\nu$, (ii) $h_1 = f_\nu$, $h_2 = f$, (iii) $h_1 = f$, $h_2 = f_\nu$, and (iv) $h_1 = f, h_2 = f$. The integrals over each such term exist and for the terms of type (i)–(iii) they tend to the integrals over the corresponding term of type (iv) as $\nu \to \infty$. Moreover, the integrals over the mixed terms (ii) and (iii) occur with a minus sign and there are always two mixed terms (ii) or (iii) corresponding to one of type (iv) while there is only one of type (i). Therefore the entire integral tends to zero.

Here we have used the argument that if $a_\nu \to a$, $b_\nu \to b$ in $L^2(A)$ and c is continuous and bounded then it follows that $\int_A |a_\nu b_\nu - ab| dx \to 0$ and hence $\int_A |a_\nu b_\nu c - abc| dx \to 0$ as $\nu \to \infty$. $\Diamond$

Lemma 5.5. *Let f be defined on a region $B \subseteq \mathbb{R}^m$ and in class C^1. Let $y^j(x^1, ...x^n)$, $j = 1, ..., m$ denote the components of a continuous mapping of a region A of $\mathbb{R}^n$ into B, and let Γ denote a bounded open subset of A whose closure is also contained in A. Assume that the functions $y^1, ..., y^m$ have strong L^2-derivatives of first order. Then the function $g(x) = f(y(x))|_\Gamma$ belongs to $L^2(\Gamma)$ and has strong L^2-derivatives in $L^2(\Gamma)$ which may be computed by the chain rule*

$$D^{(i)}g = \sum_{j=1}^{m} (\partial f/\partial y^j)(y(x)) D^{(i)} y^j.$$

Proof. Let us first show that $g \in L^2(\Gamma)$. Since g is continuous it follows that it is bounded on the closure of Γ which is compact. Therefore $\int_\Gamma g^2 dx \leq M^2 \lambda(\Gamma)$ where $|g(x)| \leq M$ on Γ.

Let now $y^j_\nu \in C^{1*}(A)$ and $y^j_\nu \to y^j$, $D^{(i)} y^j_\nu \to D^{(i)} y^j$ in $L^2(A)$. Then a fortiori $y^j_\nu \to y^j$ and $D^{(i)} y^j_\nu \to D^{(i)} y^j$ in $L^2(\Gamma)$ and $y^j_\nu|_\Gamma \in$ $\in C^{1*}(\Gamma)$.

Denote by g_ν the function $g_\nu(x) = f(y_\nu(x))$. Note that $g_\nu \in C^1$ and that $D^{(i)} g_\nu = \sum_{j=1}^{n} D^{(j)} f D^{(i)} y^j_\nu$. Also g_ν and $D^{(i)} g_\nu$ are continuous and hence bounded on Γ and therefore $g_\nu \in C^{1*}(\Gamma)$. Let us verify that $D^{(i)} g(x) = \sum_{j=1}^{m} D^{(j)} f(y(x)) D^{(i)} y^j \in L^2(\Gamma)$. Thus consider

$$\int_\Gamma \left(\sum_{j=1}^{m} D^{(j)} f(y(x)) D^{(i)} y^j \right)^2 dx.$$

The integrand consists of a sum of terms of the form

$$D^{(j)} f(y(x)) D^{(i)} y^j D^{(k)} f(y(x)) D^{(i)} y^k.$$

Now $D^{(j)} f(y(x))$ and $D^{(k)} f(y(x))$ are continuous in A, hence bounded in Γ and therefore the integrals over all these terms exist since

$D^{(i)}y^j, D^{(i)}y^k \in L^2(A)$.

Finally, we have to show that $D^{(i)}g_\nu \to D^{(i)}g$ and $g_\nu \to g$ in $L^2(\Gamma)$ as $\nu \to \infty$. Thus consider

$$\int_\Gamma \left(\sum_{j=1}^{m} D^{(j)} f(y_\nu(x)) D^{(i)} y_\nu^j(x) - \sum_{j=1}^{m} D^{(j)} f(y(x)) D^{(i)} y^j(x) \right)^2 dx.$$

Again the integrand consists of a sum of terms of four types

$$D^{(j)} f(h_1(x)) D^{(i)} h_1^j(x) D^{(k)} f(h_2(x)) D^{(i)} h_2^k(x)$$

where (i) $h_1(x) = y_\nu(x), h_2(x) = y_\nu(x)$, (ii) $h_1(x) = y_\nu(x)$, $h_2(x) = = y(x)$, (iii) $h_1(x) = y(x)$, $h_2(x) = y_\nu(x)$, and (iv) $h_1(x) = y(x)$, $h_2(x) = y(x)$. But here the situation is not as simple as in Lemma 5.4.

Let $\epsilon_\nu \to 0$ as $\nu \to \infty$ and set $\bar{y}_\nu^j = J_{\epsilon_\nu}(y^j)$. By Ths. 5.1, 5.2, and 5.4 these functions have all the properties required of the functions y_ν^j namely, $\bar{y}_\nu^j \to y^j$, $D^{(i)}\bar{y}_\nu^j \to D^{(i)}y^j$ in $L^2(\Gamma)$, $\bar{y}_\nu^j \in C^{1*}(\Gamma)$, and finally $f(\bar{y}_\nu) \in C^{1*}(\Gamma)$. Therefore we may assume that $y_\nu^j = \bar{y}_\nu^j$. But then $y_\nu^j(x)$ converges to $y^j(x)$ and this convergence is uniform on Γ. The integral over a term of type (iv) is

$$\int_\Gamma D^{(j)} f(y(x)) D^{(i)} y^j(x) D^{(k)} f(y(x)) D^{(i)} y^k(x) dx.$$

This integral exists because of the continuity of functions $D^{(j)} f(y(x))$, $D^{(k)} f(y(x))$ in A and since $D^{(i)}y^j(x), D^{(i)}y^k(x) \in L^2(\Gamma)$. The integrals over the corresponding terms of the other types exist for similar reasons, and because of the uniform pointwise convergence of the functions y_ν^j towards y^j it follows that they tend towards the integral over the corresponding term of type (iv). Hence the entire integral exists and it tends to zero as $\nu \to \infty$.

That $g_\nu = f(y_\nu) \to g$ in $L^2(\Gamma)$ is also an easy consequence of the uniform pointwise convergence of y_ν to y and thus we are finished. ◊

6. A.e. constant functions

We shall need the following criterion for a function of two variables to be almost everywhere (briefly a.e.) constant.

Theorem 6.1. *Let $y^1 = y^1(x^1, x^2), y^2 = y^2(x^1, x^2)$ scalar functions of class C^1 in a connected and open domain B of the (x^1, x^2)-plane with nowhere vanishing Jacobian $\partial y^1/\partial x^1 \partial y^2/\partial x^2 - \partial y^1/\partial x^2 \partial y^2/\partial x^1 \neq 0$. Let further $\alpha(x^1, x^2)$ denote a function from $L^2(B)$ and assume that*

$$\int_K \alpha(x^1,x^2)dy^1(x^1,x^2) = \int_K \alpha(x^1,x^2)dy^2(x^1,x^2) = 0$$

for all piecewise smooth Jordan curves K, for which the above integrals exist. Then $\alpha(x^1,x^2)$ is constant up to a set of measure zero.

Proof. (This is an adaptation of the proof of Th. 2 in Ph. Hartman [4], p. 328–329.) Let (x_0^1,x_0^2) denote an arbitrary point of the domain B and let $y_0^1 = y^1(x_0^1,x_0^2)$, $y_0^2 = y^2(x_0^1,x_0^2)$. There exists a local inverse $x^1 = x^1(y^1,y^2), x^2 = x^2(y^1,y^2)$ of class C^1 which transforms a neighbourhood

$$D : |y^1 - y_0^1| < d, |y^2 - y_0^2| < d, (d > 0)$$

of (y_0^1,y_0^2) into a neighbourhood of (x_0^1,x_0^2). The function

$$\beta(y^1,y^2) = \alpha(x^1(y^1,y^2),x^2(y^1,y^2))$$

belongs to $L^2(D)$ and it is enough to show that $\beta(y^1,y^2)$ is constant.

If $J : y^1 = y^1(t), y^2 = y^2(t)$ is a smooth or piecewise smooth curve in D, we consider its image $K : x^1 = x^1(y^1(t),y^2(t)), x^2 = x^2(y^1(t),y^2(t))$ and for an arbitrary total differential $dg = (\partial g/\partial y^1)dy^1 + (\partial g/\partial y^2)dy^2$ we have

$$\int_J \beta(y^1,y^2)dg = \int_K \alpha(x^1,x^2)dh,$$

where $h(x^1,x^2) = g(y^1(x^1,x^2),y^2(x^1,x^2))$. In this formula the line integral along J exists if and only if the line integral along K exists. Let now J be a rectangle contained in D of the form $a_0 \leq y^1 \leq a$, $b_0 \leq y^2 \leq b$ and let us choose for g the function $g(y^1,y^2) = y^1$, so that $h = y^1(x^1,x^2)$ and $dh = dy^1(x^1,x^2)$. The integral

$$\int_J \beta(y^1,y^2)dy^1$$

exists for all rectangles J considered, except perhaps, if the numbers a_0, a, b_0, b lie in a certain one dimensional set of measure zero. With these exceptions it follows from one of the two conditions of the theorem that

$$\int_J \beta(y^1,y^2)dy^1 = \int_K \alpha(x^1,x^2)dy^1(x^1,x^2) = 0.$$

Now evaluation of the left-hand side gives

$$\int_{a_0}^{a} \beta(t,b_0)dt - \int_{a_0}^{a} \beta(t,b)dt = 0$$

We set

$$\varphi(a) = \int_{a_0}^{a} \beta(t, b_0)dt, \psi(a) = \int_{a_0}^{a} \beta(t, b)dt.$$

Thus if a_0, a, b_0, b are not in the exceptional set N of measure zero mentioned above then we have $\varphi(a) = \psi(a)$. If only a_0, b_0, b are not in N then we can define $\varphi(a)$ and $\psi(a)$ also for $a \in N$, for the integrals $\int_{a_0}^{a} \beta(t, b_0)dt$ and $\int_{a_0}^{a} \beta(t, b)$ exist, when $b_0 \notin N$ or $b \notin N$ respectively. If we imagine the definition of φ and ψ extended in this manner we obtain absolutely continuous functions $\varphi(a) = \psi(a)$ and we have almost everywhere $\beta(t, b_0) = d\varphi/da, \beta(t, b) = d\psi/da$. Thus $\beta(t, b_0) = \beta(t, b)$ for almost all t when $b_0, b \notin N$. Using the second condition we obtain in a similar way that $\beta(a_0, s) = \beta(a, s)$ for almost all s, when $a_0, a \notin N$.

Let us keep b_0 fixed and vary b. For distinct b_1, b_2 outside N there exist one-dimensional nullsets $N(b_1), N(b_2)$ such that $\beta(t, b_0) = = \beta(t, b_1) = p_1$, if $t \notin N(b_1)$ and $\beta(t, b_0) = \beta(t, b_2) = p_2$, if $t \notin N(b_2)$. Since $t \notin N(b_1) \cup N(b_2)$ can easily be satisfied it follows that $p_1 = p_2 = = p$. We form the union of the sets $\{(t, b)|b \notin N, t \notin N(b)\}$, add the set $\{(t, b)|b \in N\}$ to it, and obtain in this way a two-dimensional nullset N_1 with the property that $p = \beta(t, b_0) = \beta(t, y^2)$, if $(t, y^2) \notin N_1$. Similarly we obtain a two-dimensional nullset N_2 with the property that $q = = \beta(a_0, s) = \beta(y^1, s)$, if $(y^1, s) \notin N_2$. If now $(y^1, y^2) \notin N_1 \cup N_2$ it follows that $q = \beta(a_0, y^2) = \beta(y^1, y^2) = \beta(y^1, b_0) = p$, whence $p = = q$. Thus β as an element of $L^2(D)$ is equal to the constant p and the theorem is proven. ◊

Corollary 6.1. *In* Th. 6.1 *it suffices to assume that for each point* y_0^1, y_0^2 *of the image domain there exists a certain neighbourhood* D *with the following property: for a certain one-dimensional nullset* N *which may depend on* D*, the integrals occurring in the hypothesis vanish for all closed curves* K *which correspond to the boundary of arbitrary rectangles* $a_0 \leq y^1 \leq a$, $b_0 \leq y^2 \leq b$ *contained in* D*, with the possible exception of those for which one of the numbers* a_0, a, b_0, b *belongs to* N.

We shall apply Th. 6.1 to functions α in $n \geq 3$ variables $u^1, u^2, \ldots$ $\ldots, u^n$ defined on a connected open domain U in the space of these n variables. Let $i \neq j$ be fixed. If u^i, u^j are varied and the remaining coordinates have fixed values $u^k = u_0^k, k \neq i, j$, we get a 2-dimensional cross-section of the domain U which we denote by U_q where q is the point with $n-2$ coordinates $u_0^k, k \neq i, j$. We are interested in a situation where α satisfies the criterion of Th. 6.1 for almost all these cross-sections, and consequently, is a.e. constant in U_q for almost all q.

Theorem 6.2. *Let $\alpha \in L^2(U)$ and suppose that for each pair $i \neq j$ of indices there exists an $(n-2)$-dimensional set N_{ij} of measure zero such that α restricted to U_q is constant a.e. in U_q if $q \notin N_{ij}$. Then α is constant a.e. in U.*

Proof. Since every open set is the union of countably many intervals it suffices to prove this theorem for n-dimensional intervals contained in U. Let I be such an interval. Let $i \neq j$ be a fixed pair of indices. Consider the cross-sections $I_q = U_q \cap I$ belonging to i, j. The union of those I_q where α is not a.e. constant is a nullset. We can change α to an arbitrary value on this set.

We will now show that the union of all the exceptional subsets of measure zero of those I_q where α is a.e. constant is also a nullset. We change α in such a way that the changed function $\bar{\alpha}$ is actually constant on I_q for all q. We have to show that the set where $\alpha \neq \bar{\alpha}$ is a nullset. By Fubini's Theorem

$$\int_I \alpha du = \int \left(\int_{I_q} \alpha du^i du^j \right) dq = \int f(q) dq.$$

Now $f(q) = \bar{\alpha}\lambda(I_q)$ and from this it follows that $\bar{\alpha}$ is integrable (i.e. $\bar{\alpha} \in L(I)$). Hence $|\alpha - \bar{\alpha}| \in L(I)$ and

$$\int_I |\alpha - \bar{\alpha}| du = \int \left(\int_{I_q} |\alpha - \bar{\alpha}| du^i du^j \right) dq = 0$$

and therefore the set where $\alpha(u) \neq \bar{\alpha}(u)$ is a nullset.

Note that the changed function still satisfies the hypothesis of the theorem. We may thus assume that α is strictly constant on all 2-dimensional cross-sections of I in which only the coordinates u^1, u^2 are varied. If $n = 3$ it follows from our assumptions that α is strictly constant on the cross-sections in which only x^1 is varied while it is a.e. constant on almost all complementary cross-sections. From this it follows that α is a.e. constant. If $n = 4$ it follows from the hypothesis of the theorem that α is a.e. constant on almost all of the complementary cross-sections in which u^3, u^4 are varied while u^1, u^2 are fixed. Since α is strictly constant on the cross-sections in which only u^1, u^2 are varied this implies again that α is a.e. constant on I. We may now assume by induction that the theorem is true for smaller dimensions and that $n >$ > 4. It follows that the complementary cross-sections in which $u^3, ..., u^n$ are varied while u^1, u^2 are fixed satisfy the hypotheses of the theorem so that α is a.e. constant on all of them. As before this implies that α is a.e. constant on I. $\Diamond$

7. The Dirichlet integral

If $g = g(u)$ is defined in a u-domain R and is of class C^1 then we denote by $I_R(g)$ the Dirichlet integral of g over the domain R, thus

$$I_R(g) = \sum_{i=1}^{d} \int_R |g_i(u)|^2 du^1 ... du^d.$$

If $g(u)$ is continuous in a domain containing the closure of R then for small $|h| > 0$ we shall denote by $I_{Rh}(g)$ the modified Dirichlet integral

$$I_{Rh}(g) = h^{-2} \sum_{i=1}^{d} \int_R |\Delta_i^h g|^2 du^1 ... du^d,$$

where Δ_i^h is the difference

$$\Delta_i^h(g) = g(u^1, ..., u^{i-1}, u^i + h, u^{i+1}, ..., u^d) - g(u^1, ..., u^d).$$

Lemma 7.1. *Let $g = g(u)$ be a continuous function on a domain containing the closure of an open sphere T. Suppose there exists a constant K such that*

$$I_{Th}(g) \leq K$$

for small $|h| \geq 0$. Then $g(u)$ has strong L^2-derivatives of first order on open subsets T_0 with compact closure contained in T. More precisely, the ordinary partial derivatives of first order of g exist almost everywhere in T_0, belong to $L^2(T_0)$, and are in fact equal to the corresponding strong L^2-derivatives.

Proof. The existence a.e. of the partial derivatives and their quadratic integrability follow from a theorem of Sólyi (cf. [8]) which also ensures that the third hypothesis of Lemma 5.3 is satisfied. That these partial derivatives are strong L^2-derivatives is implied by Lemma 5.3. $\Diamond$

Lemma 7.2. *Let $z = z(u) = (z^1(u), ..., z^d(u))$ denote a vector function of class C^1 on a bounded u-domain R. Suppose there exist constants C_1, C_2, C_{ijkm}, such that $|z^i(u)| \leq C_1$ and*

$$(*) \qquad \sum_{i=1}^{d} \sum_{j=1}^{d} |z_j^i|^2 \leq C_{ijkm} \partial(z^i, z^j)/\partial(u^k, u^m) + C_2.$$

Then for each compact subset T of R there exists a constant K depend-

ing only on R, T, C_1, C_2, C_{ijkm} such that the Dirichlet-integrals of z^j satisfy the inequality

$$\sum_{j=1}^{d} I_T(z^j) \leq K.$$

Note that in the right-hand side of $(*)$ a summation over all indices i, j, k, m is implied. For the proof see [5]. ◊

In the proof of the theorem we shall use these two lemmas in the following way. In a region around the origin let us be given a C^1-mapping $v = v(u)$. For the set T we choose a small closed sphere contained in the region. Let us set

$$z = h^{-1}\Delta_i^h v.$$

Then $z = z(u, h)$ is a vector function of class C^1 defined for small $|u|$ and $|h| > 0$. This function is continuous even at $h = 0$ if we set $z(u, 0) = \partial v / \partial u^i$.

Lemma 7.3. *In the situation considered above the sum of the Dirichlet integrals of z^j from the previous lemma can be computed explicitly:*

$$\sum_{j=1}^{d} I_T(z^j) = h^{-2} \sum_{k=1}^{d} \sum_{m=1}^{d} \int_T |\Delta_i^h v_m^k|^2 du^1 ... du^d$$

for each $i = 1, ..., d$.

Proof. This is an immediate consequence of the definition of z, namely that $z^k = h^{-1}\Delta_i^h v^k$. ◊

Thus if the functions z^j satisfy an inequality of type $(*)$ in which the constants are independent of h, the last two lemmas yield the existence of a constant K_i such that

$$\sum_{j=1}^{d} I_T(z^j) = h^{-2} \sum_{k=1}^{d} \sum_{m=1}^{d} \int_T |\Delta_i^h v_m^k|^2 du^1 ... du^d \leq K_i.$$

Now by definition of the modified Dirichlet integral we have

$$I_{Th}(v_m^k) = h^{-2} \sum_{i=1}^{d} \int_T |\Delta_i^h v_m^k|^2 du^1 ... du^d.$$

From the last two formulae it follows that $I_{Th}(v_m^k) \leq K$ where $K = d \cdot \max(K_i)$. From Lemma 7.1 it follows therefore that each v_m^k has strong L^2-derivatives of first order and that inside the sphere T the second derivatives in the ordinary sense $\partial^2 v^k / \partial u^m \partial u^j$ exist almost everywhere and are in fact equal to the corresponding strong L^2-derivatives. Moreover, since the operators for strong L^2-derivatives commute it follows that $\partial^2 v^k/\partial u^m \partial u^j = \partial^2 v^k/\partial u^j \partial u^m$ a.e.

8. Proof of Theorem 1.1

For better readability the proof is divided into five sections.

1. We shall show first that the v^k_m have strong L^2-derivatives. As we have seen in Lemma 2.1 for an arbitrary matrix (v^i_j) the relations (2) of the theorem are equivalent to

$$\sum_{k=1}^{d} v^i_k v^j_k \varepsilon_k = \varepsilon_i \varepsilon \gamma^2 \delta^{ij} \tag{3}$$

Transforming the u–coordinates by a pseudo-Euclidean similarity we may assume that an arbitrarily chosen point u in the region G is replaced by the point $u = 0$ and that

$$v^j_k(0) = \delta^{jk}. \tag{4}$$

Let $1 \leq i \leq d$ and h, Δ^h_i, and z be defined as in Section 7. Then the relations (3) and (4) imply that

$$\varepsilon_k z^j_k + \varepsilon_j z^k_j = (h^{-1}\Delta^h_i \gamma^2)\varepsilon_j \varepsilon \delta^{jk} + \sum_{m=1}^{d}\sum_{n=1}^{d} o(1)|z^m_n|, \tag{5}$$

where $o(1) \to 0$ as $|u| \to 0$ and $0 < |h| \to 0$. To prove the formulae (5) we apply the operator $h^{-1}\Delta^h_i$ to the function $f(v^1_1, v^2_1, ..., v^d_d) = \sum_{\nu=1}^{d} v^j_\nu v^k_\nu \varepsilon_\nu$. By the intermediate value theorem of differential calculus the result can be written as

$$h^{-1}\Delta^h_i f = (\partial f/\partial v^n_m)_\xi z^n_m.$$

The lower index ξ here means that the partial derivatives are to be taken at $v^n_m + \xi \Delta^h_i v^n_m$ for a suitable value ξ such that $0 < \xi < 1$. The expression on the right-hand side may be further transformed into

$$df^m_n z^n_m - \sum_{m=1}^{d}\sum_{n=1}^{d} o(1)|z^n_m|.$$

Here df^m_n denotes the value of $\partial f/\partial v^n_m$ for the arguments $(v^1_1(0), ..., v^d_d(0))$. Now the formulae (3) and (4) show that $df^m_n = 0$ except for the two cases $df^j_k = \varepsilon_j f$ and $df^k_j = \varepsilon_k$. Thus (5) is proven.

If $j = k$ then (5) becomes

(6)
$$z_j^j = \frac{1}{2}(h^{-1}\Delta_i^h \gamma^2)\varepsilon + \sum_{m=1}^{d}\sum_{n=1}^{d} o(1)|z_n^m|.$$

This formula and the corresponding one for z_k^k yield

(7)
$$z_j^j - z_k^k = \sum_{m=1}^{d}\sum_{n=1}^{d} o(1)|z_n^m|.$$

By squaring (7) and (5) for $j \neq k$, multiplying the square of (5) by $\varepsilon_k\varepsilon_j$, and adding it follows that

(8)
$$z_j^j z_k^k - z_k^j z_j^k = \frac{1}{2}(|z_j^j|^2 + |z_k^k|^2 + \varepsilon_j\varepsilon_k|z_k^j|^2 + \varepsilon_j\varepsilon_k|z_j^k|^2) + \sum_{m=1}^{d}\sum_{n=1}^{d} o(1)|z_n^m|^2.$$

Note here that the expressions $(\sum\sum o(1)|z_n^m|)^2$ which arise in the squared relations can be transformed into $\sum\sum o(1)|z_n^m|^2$ because of the inequality

$$\left(\sum_{i=1}^{N} a_i\right)^2 \leq N \cdot \sum_{i=1}^{N} a_i^2.$$

Forming linear combinations from (8) we get

(8*)
$$\sum_{j<k} C_{jk}(|z_j^j|^2 + |z_k^k|^2 + \varepsilon_j\varepsilon_k|z_k^j|^2 + \varepsilon_j\varepsilon_k|z_j^k|^2) + \sum_{m=1}^{d}\sum_{n=1}^{d} o(1)|z_n^m|^2 =$$
$$= 2\sum_{j<k} C_{jk}(z_j^j z_k^k - z_k^j z_j^k).$$

Let us set $C_{jk} = 2d$ when $\varepsilon_j\varepsilon_k = 1$ and $C_{jk} = -2$ otherwise. In the first sum on the left-hand side of (8*) each term $|z_k^j|^2$ or $|z_j^k|^2$ occurs exactly once and will have coefficient $s = |C_{jk}| \geq 2$. For any fixed j the term $|z_j^j|^2$ will get the coefficient $s = \sum_{l=1}^{j-1} C_{lj} + \sum_{k=j+1}^{d} C_{jk}$. Since we have excluded signatures like $1, -1, ..., -1$ or $1, ..., 1, -1$ there will be at least one l or k such that $C_{lj} = 2d$ or $C_{jk} = 2d$ so that $s \geq 2d - 2(d-1) \geq 2$. Now we can choose constants $r > 0$ and $h_0 > 0$ such that all factors $o(1)$ in (8*) become smaller than 1 in absolute value when $|u| < r$ and $|h| < h_0$. It follows that in the left-hand side of (8*) each term $|z_n^m|^2$

occurs with coefficient at least 1. Thus we get

$$\sum_{m=1}^{d}\sum_{n=1}^{d}|z_n^m|^2 \leq \sum_{j<k} 2C_{jk}(z_j^j z_k^k - z_k^j z_j^k)$$

which is an inequality of type $(*)$ as in Lemma 7.2 with $C_2 = 0$ and $C_{ijkm} = 2C_{ij}$ when $i < j, k = i, m = j$ and $C_{ijkm} = 0$ otherwise.

It follows therefore from the results of Section 7 that in a neighbourhood U of $u = 0$ the function $v = v(u)$ has strong L^2-derivatives of second order; in particular $\partial^2 v^i / \partial u^j \partial u^k$ exists almost everywhere and is equal to $\partial^2 v^i / \partial u^k \partial u^j$. It follows from (3) and $\gamma > 0$ that γ has strong L^2-derivatives of first order. ◊

2. In a similar way as in [2], p. 142, we are now going to show that almost everywhere in U we have

$$v_{ij} = \gamma^{-1}(\gamma_j v_i + \gamma_i v_j), \tag{9}$$

if $i \neq j$ and

$$v_{ii} = \gamma^{-1}(\gamma_i v_i - \sum_{k \neq i} \varepsilon_i \varepsilon_k \gamma_k v_k). \tag{10}$$

For arbitrary vectors $a = (a^1, ..., a^d)$ and $b = (b^1, ..., b^d)$ let us use the scalar product notation $a \cdot b$ or ab for short, for the expression $\sum a^k b^k \varepsilon_k$. To prove the formulae (9) and (10) we start with the relations (2) from the hypothesis of Th. 1.1 which we may rewrite as $v_i v_j = \varepsilon_i \varepsilon \gamma^2 \delta_{ij}$. From this we get $v_i v_{jk} + v_{ik} v_j = 0$ for $i \neq j$. If i, j, k are distinct indices it follows hence

$$v_k v_{ij} = -v_i v_{kj} = -v_i v_{jk} = v_j v_{ki} = -v_k v_{ij},$$

i.e. $v_k v_{ij} = 0$. Thus v_{ij} is a linear combination of v_i and v_j, say $v_{ij} = \alpha v_i + \beta v_j$. By scalar multiplication with v_i it follows on the one hand that $v_i v_{ij} = \alpha v_i v_i$. On the other hand it follows from $v_i v_i = \varepsilon_i \varepsilon \gamma^2$ by differentiating that $v_i v_{ij} = \varepsilon_i \varepsilon \gamma \gamma_i$. This yields $\gamma_j = \alpha\gamma$. In a similar way we get $\gamma_i = \beta\gamma$ and thus $\gamma v_{ij} = \gamma_j v_i + \gamma_i v_j$.

To prove (10) note that v_{ii} is a linear combination of the independent vectors v_j, say $v_{ii} = \alpha^j v_j$. To determine the coefficients we use the relation $v_i v_i = \varepsilon_i \varepsilon \gamma^2$ which implies $v_{ii} v_i = \varepsilon_i \varepsilon \gamma \gamma_i$, and for $i \neq j$ the relations $v_{ii} v_j = \varepsilon_i \varepsilon \gamma \gamma_j$ which follow from (9).

Denote by $y_i = y_i(u)$ the vector

$$y_i = \lambda v_i, \tag{11}$$

where $\lambda = \gamma^{-1}$. (Note that the subscript i in y_i does not denote partial differentiation.) The vectors (11) have strong L^2-derivatives and from (9) and (10) we get

$$\partial y_i/\partial u^j = -\lambda_i v_j, \tag{12}$$

when $i \neq j$ and

$$\partial y_i/\partial u^i = \sum_{k\neq i} \lambda_k v_k \tag{13}$$

almost everywhere. ◊

3. We shall show that λ_i essentially depends only on the i-th coordinate u^i.

Let the i-th coordinate have a fixed value $u^i = c$ and denote by U_c the cross-section of U with $u^i = c$. More precisely, we shall show that for all possible values of c, except perhaps for a one-dimensional set of measure zero, the function λ_i is constant almost everywhere on U_c.

To see this we first observe that the subset $N \subseteq U$ where not all of the relations (12) are satisfied is a nullset. Now let N_1 be the set of all c such that (12) is not satisfied a.e. in U_c. By Fubini's Theorem N_1 is a one-dimensional nullset. Also the set N_2 of all c such that $y_i \notin L^2(U_c)$ or the relations (12) do not represent the L^2-derivative of y_i is a nullset. We will show that λ_i is constant a.e. in U_c provided that $c \notin N_1 \cup N_2$. Let us fix two indices j, k which are distinct from i. We consider 2-dimensional cross-sections $U_{c,q}$ of U_c where u^i, u^j may be changed freely, u^m are fixed for $m \neq i, j, k$, and q is the point with coordinates u^m. The q for which $y_i \notin L^2(U_{c,q})$ or the relations (12) do not represent the L^2-derivatives of y_i in $L^2(U_{c,q})$ form a $(d-3)$-dimensional nullset M and for $q \notin M$ we will apply the criterion of Th. 6.1, i.e.

$$\int_J \lambda_i dv = 0, \tag{14}$$

where J denotes any one out of a certain class of piecewise smooth Jordan curves in $U_{c,q}$ along which this line integral exists. We imagine λ_i changed provisionally in such a way that $\lambda_i = \infty$ in all places where the relation (12) does not hold for one of the two indices j or k. Let now J be a piecewise smooth Jordan curve along which λ_i has value ∞ only in a nullset. Note that this is a necessary condition for the line integral (14) to exist. We have to show that the line integral vanishes under certain additional conditions. To this end consider the line integral along the curve starting from a fixed point P_0 and going up to a variable point P as a vector function of the parameter t, thus

$$\int_{P_0}^{P} \lambda_i dv = \int_{t_0}^{t} \lambda_i \left(v_j \frac{du^j}{dt} + v_k \frac{du^k}{dt} \right) dt = F(t).$$

According to our assumptions along the curve relation (12) holds almost everywhere and thus we get

$$F'(t) = \lambda_i \left(v_j \frac{du^j}{dt} + v_k \frac{du^k}{dt} \right) = -dy_i/dt$$

for almost all t. Note however, that this alone does not imply that $F(t) = -y_i(t)$.

The matrix $(\partial v^\nu / \partial u^\mu), \mu = j, k$ has rank two. Therefore we can choose indices ν_1, ν_2 so that the Jacobian $\partial(v^{\nu_1}, v^{\nu_2})/\partial(u^j, u^k)$ does not vanish. Thus we shall show that locally for $x^1 = u^j, x^2 = u^k, y^1 = = v^{\nu_1}, y^2 = v^{\nu_2}$ and $\alpha = \lambda_i$ the hypotheses of Cor. 6.1 are satisfied.

For let V denote the region in the (v^{ν_1}, v^{ν_2})-plane corresponding to $U_{c,q}$. We may consider the vector y_i locally as a function of v^{ν_1}, v^{ν_2} in $V_1 \subseteq V$ by setting $\bar{y}_i(v^{\nu_1}, v^{\nu_2}) = y_i(u^j, u^k)$ and we may assume that $\bar{y}_i$ belongs to $L^2(V_1)$ and has strong L^2-derivatives. By Lemma 5.1 it follows that $\bar{y}_i$ is absolutely continuous with respect to v^{ν_1}, v^{ν_2} on closed intervals of variation contained in V_1 for almost all values of v^{ν_2} and v^{ν_1} respectively. This means that if J corresponds to a rectangle K in V_1 such that $a \leq v^{\nu_1} \leq b$, $\alpha \leq v_2^\nu \leq \beta$ we may assume that $\bar{y}_i$ is absolutely continuous along the sides of the rectangle and that the partial derivatives are equal a.e. to the given L^2-derivatives of $\bar{y}_i$ except when a, b, α, β belong to a certain one-dimensional set of measure zero. It follows that

$$0 = \int_K -d\bar{y}_i = \int_J -dy_i = \int_J \lambda_i dv$$

and thus the criterion of Cor. 6.1 is satisfied. (To evaluate the first integral use the parameter v^{ν_1} along the segments where v^{ν_2} is constant and vice versa. Then apply the fundamental theorem of differential and integral calculus (cf. [9], p. 342).) It follows that λ_i as a function of u^j, u^k is a.e. constant. And this is not influenced by our provisional change which we can now undo. It also follows from the discussion above that the hypotheses of Th. 6.2 are satisfied and thus λ_i is constant a.e. in U_c provided that $c \notin N_1 \cup N_2$. $\Diamond$

4. By changing λ_i on a d-dimensional set of measure zero we can

achieve that λ_i is constant in the strict sense for all sections U_c.

First the union of all those cross-sections where λ_i is not constant up to a nullset is itself a nullset. We may change λ_i on this set to an arbitrary constant.

Secondly let us change λ_i on each section U_c to the constant value assumed a.e. on U_c and let us denote the new function provisionally by $\bar{\lambda}_i$. We shall show that $\bar{\lambda}_i$ is measurable and that consequently $\bar{\lambda}_i$ and λ_i differ only on a set of measure zero. Since U is a union of countably many d-dimensional intervals, it suffices to prove this for d-dimensional intervals. Let I be such an interval. Then

$$\int_I \lambda_i du = \int_{c_0}^{c_1} \left(\int_{I_{u^i}} \lambda_i d\bar{u} \right) du^i = \int_{c_0}^{c_1} f(u^i) du^i$$

where I_{u^i} denotes the cross-section of I corresponding to the actual value of the coordinate u^i. Since λ_i is a.e. constant on I_{u^i} it follows that $f(u^i) = \bar{\lambda}_i(u^i) \cdot q$ where q is the constant content of I_{u^i}. Thus from the existence of the integrals above it follows that $\bar{\lambda}_i$ is measurable on I. By Fubini's Theorem this implies that $\bar{\lambda}_i \in L(I)$. Let $g(u) = \lambda_i(u) - \bar{\lambda}_i(u)$. It follows that $g \in L(I)$, hence $|g| \in L(I)$ and

$$\int_I |g(u)| du = \int_{c_0}^{c_1} \left(\int_{I_{u^i}} |\lambda_i - \bar{\lambda}_i| d\bar{u} \right) du^i = 0.$$

Thus the set N of all $u \in I$ where $\lambda_i(u) \neq \bar{\lambda}_i(u)$ is a nullset. $\Diamond$

Since we have thus shown that λ_i and $\bar{\lambda}_i$ only differ in a set of measure zero there will be no harm if in the sequel we drop the notation $\bar{\lambda}_i$ and use λ_i as before.

5. If $i, j (j \neq i)$ are fixed (12) and (13) imply

$$\int_J \left(\lambda_i v_j du^j - \left(\sum_{k \neq i} \lambda_k v_k \right) du^i \right) = 0, \tag{15}$$

if J is the boundary of a rectangle of the form $a \leq u^i \leq b$, $\alpha \leq u^j \leq \beta$, $u^k = \text{const}$ for $k \neq i, j$, provided that a, b, α, β do not belong to a certain 1-dimensional nullset and that the point q formed by the coordinates $u^k, k \neq i, j$ does not belong to a certain $(n-2)$-dimensional nullset. To see this note that by Fubini's Theorem the L^2-derivatives of the vector function y_i are integrable in almost all planar sections U_q where $u^k = \text{const}$, $k \neq i, j$, and in any such section we may assume that the relations (12) and (13) are satisfied a.e. and that they represent the L^2-

derivative of y_i. Consequently, we may again apply Lemma 5.1 in order to show that the four integrals along straight line segments occurring in (15) exist for almost all a, b, α, β and can be evaluated using the fundamental theorem of differential and integral calculus. $\diamondsuit$

Since λ_k depends only on u^k relation (15) yields

$$\lambda_i(b)s(b) - \lambda_i(a)s(a) - \sum_{k\neq i} \lambda_k(u^k)[t_k(\beta) - t_k(\alpha)] = 0, \tag{16}$$

where

$$s(u^i) = s(u^i, \alpha, \beta) = v(u^1, ..., u^{j-1}, \beta, u^{j+1}, ..., u^d)-$$

$$-v(u^1, ..., u^{j-1}, \alpha, u^{j+1}, ..., u^d), \tag{17}$$

and

$$t_k(u^j) = t_k(u^j, a, b) = \int_a^b v_k(u)du^i. \tag{18}$$

We may transform (16) into

$$s(b)[\lambda_i(b)-\lambda_i(a)] = -\lambda_i(a)[s(b)-s(a)]+\sum_{k\neq i} \lambda_k(u^k)[t_k(\beta)-t_k(\alpha)] \tag{19}$$

and consider a, α, β, u^k for $k \neq i$ as fixed and b as variable. Then $s(b)$ is of class C^1. Likewise $t_k(u^j, a, b)$ as functions of b are of class C^1 (cf. (18)). Because of (19) also $s(b)[\lambda_i(b) - \lambda_i(a)]$ is of class C^1. The vector $s(b)$ does not vanish as long as $\alpha \neq \beta$ because of the injectivity of the transformation $v = v(u)$.

It follows that $\lambda_i(u^i) = \partial\lambda/\partial u^i$ can be extended to a continuous function, and thus λ has continuous strong L^2-derivatives with respect to $u^i, i = 1, ..., d$. By Lemma 5.2 it follows that λ and hence also γ are of class C^1. Let $g_{ij} = \varepsilon_i\varepsilon\gamma^2\delta_{ij}$ and $h_{km} = \varepsilon_k\delta_{km}$. The relations (2) show that g_{ij} is transformed by $v = v(u)$ into h_{km}. By Th. 4.1 the mapping $v = v(u)$ is of class C^2.

We divide (19) by $b - a$ and let b converge to a. If follows that λ_i has a derivative λ_{ii} with respect to u^i and that

$$s(a)\lambda_{ii}(a) = -\lambda_i(a)\{v_i\} + \sum_{k\neq i} \lambda_k(u^k)\{v_k\},$$

where $\{g\}$ denotes the difference of the values of g at $u^i = a, u^j = \beta$ and $u^i = a, u^j = \alpha$. Consequently λ has continuous second order derivatives

λ_{ii} and $\lambda_{ij} = 0$ for $i \neq j$. Therefore $v = v(u)$ is of class C^3 by Th. 4.1. The proof can now be finished as in Benz [2], pp. 143–149. $\Diamond$

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