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ON THE STRUCTURE OF NON-SINGULAR ITERATION GROUPS ON THE CIRCLE

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Abstract: The aim of this paper is to investigate the structure of non-singular iteration groups on the unit circle $\mathbb{S}^1$, that is, families $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1, v \in V\}$ of homeomorphisms such that

$$F^{v_1} \circ F^{v_2} = F^{v_1+v_2}, \qquad v_1,\ v_2 \in V,$$

and at least one $F^v \in \mathcal{F}$ has no periodic point (V is a linear space over $\mathbb{Q}$ with $\dim V \geq 1$). Our main result shows that iteration groups under study are direct sums of some special subgroups.

1. Introduction

Denote by $\mathbb{S}^1$ the unit circle and let V be a linear space over $\mathbb{Q}$ such that $\dim V \geq 1$.

Recall that a family $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1, \quad v \in V\}$ of homeomorphisms for which

$$F^{v_1} \circ F^{v_2} = F^{v_1+v_2}, \qquad v_1,\ v_2 \in V$$

is called an *iteration group* or a *flow* (on $\mathbb{S}^1$). An iteration group is said

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to be *disjoint* if every its element either is the identity mapping or has no fixed point.

Some special cases of such iteration groups under the assumption that $V = \mathbb{R}$ have been investigated in [2] and [3]. A complete description of disjoint iteration groups $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ can be found in [6] and [kc6].

An iteration group $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ is said to be *non-singular* if at least one its element has no periodic point, otherwise $\mathcal{F}$ is called a *singular* iteration group.

The aim of this paper is to investigate the structure of non-singular iteration groups which do not need to be disjoint. We shall show that every such group is a direct sum of two subgroups and one of these subgroups is a special disjoint iteration group. In order to do this we use some ideas from [11].

2. Preliminaries

We begin by recalling the basic definitions and introducing some notation.

Throughout the paper $\mathbb{N}$ stands for the set of all positive integers and the set of all cluster points of the set $A \subset \mathbb{S}^1$ will be denoted by A^{d}.

A set $A \subset \mathbb{S}^1$ is said to be an *open arc* if there are distinct $v,\ z \in \mathbb{S}^1$ for which

$$A = \overrightarrow{(v,\ z)} := \{e^{2\pi it},\ t \in (t_v,\ t_z)\},$$

where $t_v, t_z \in \mathbb{R}$ are such that $e^{2\pi it_v} = v$, $e^{2\pi it_z} = z$ and $0 < t_z - t_v < 1$.

It is well-known (see for instance [1], [2] and [12]) that for every continuous mapping $F : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ there is a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$, which is unique up to translation by an integer, and a unique integer k such that

$$F(e^{2\pi ix}) = e^{2\pi if(x)}, \qquad x \in \mathbb{R}$$

and

$$f(x+1) = f(x) + k, \qquad x \in \mathbb{R}.$$

The integer k is called the *degree* of F, and is denoted by $\deg F$. If $F : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is a homeomorphism, then so is f. Furthermore, $|\deg F| = 1$. We say that a homeomorphism $F : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ *preserves orientation* if $\deg F = 1$, which is clearly equivalent to the fact that f is increasing.

Moreover, F preserves orientation if and only if for any $v,\ w,\ z \in \mathbb{S}^1$ such that $w \in \overrightarrow{(v,\ z)}$ we have $F(w) \in \overrightarrow{(F(v),\ F(z))}$ (see [5]). Recall also that every element of an iteration group $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ preserves orientation (see [6]).

For every orientation-preserving homeomorphism F the number $\alpha(F) \in [0,\ 1)$ defined by

$$\alpha(F) := \lim_{n \to \infty} \frac{f^n(x)}{n} (\mathrm{mod}\, 1), \qquad x \in \mathbb{R}$$

is called the *rotation number* of F. This number always exists and does not depend on x and f. Furthermore, $\alpha(F)$ is rational if and only if F has a periodic point. If $\alpha(F) \notin \mathbb{Q}$, then the non-empty set

$$L_F := \{F^n(z),\ n \in \mathbb{Z}\}^{\mathrm{d}},$$

(the *limit set* of F) does not depend on $z \in \mathbb{S}^1$ (see for instance [9] and [10]).

By the *limit set* of a disjoint iteration group $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ we mean the set

$$L_{\mathcal{F}} := \{F^v(z),\ v \in V\}^{\mathrm{d}}$$

which does not depend on $z \in \mathbb{S}^1$. By the *limit set* of a non-singular iteration group $\mathcal{F}$ we mean the set

$$L_{\mathcal{F}} := L_{F^v},$$

where $F^v \in \mathcal{F}$ is an arbitrary homeomorphism with $\alpha(F^v) \notin \mathbb{Q}$. This set does not depend on the choice of such a homeomorphism.

Although the above definitions are different, in the case when the iteration group is both disjoint and non-singular they determine the very same set.

A non-singular or disjoint iteration group $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ is called:

— *dense*, if $L_{\mathcal{F}} = \mathbb{S}^1$;

— *non-dense*, if $\emptyset \neq L_{\mathcal{F}} \neq \mathbb{S}^1$;

— *discrete*, if $L_{\mathcal{F}} = \emptyset$.

It is worth pointing out that every discrete iteration group is both disjoint and singular, and every dense iteration group is disjoint (see [6]). Therefore we shall investigate only non-dense non-singular iteration groups.

We now repeat the relevant, slightly modified, material from [6].

Lemma 1 (see [6] and also [8]). *If $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ is a dense or non-dense iteration group, then there exists a unique pair*

$(\varphi_{\mathcal{F}},\ c_{\mathcal{F}})$ such that $\varphi_{\mathcal{F}} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is a continuous function of degree 1 with $\varphi_{\mathcal{F}}(1) = 1$ and $c_{\mathcal{F}} : V \longrightarrow \mathbb{S}^1$ satisfying the following system of functional equations

$$\varphi_{\mathcal{F}}(F^v(z)) = c_{\mathcal{F}}(v)\varphi_{\mathcal{F}}(z), \quad z \in \mathbb{S}^1,\ v \in V.$$

Regarding the structure of dense non-singular iteration groups we have the following theorem.

Theorem 1 (see [6]). *If $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ is a dense iteration group, then there exists a unique orientation-preserving homeomorphism $\varphi_{\mathcal{F}} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ having fixed point 1 such that*

$$F^v(z) = \varphi_{\mathcal{F}}^{-1}(e^{2\pi i \alpha(F^v)}\varphi_{\mathcal{F}}(z)), \qquad z \in \mathbb{S}^1,\ v \in V.$$

If $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ is a non-dense iteration group, then its limit set is a non-empty perfect and nowhere dense subset of $\mathbb{S}^1$, and therefore we have the following decomposition

$$\mathbb{S}^1 \setminus L_{\mathcal{F}} = \bigcup_{q \in \mathbb{Q}} I_q,$$

where I_q for $q \in \mathbb{Q}$ are open pairwise disjoint arcs.

Lemma 2 (see [6]). *If $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ is a non-dense iteration group, then:*

(i) *for every $q \in \mathbb{Q}$ the mapping $\varphi_{\mathcal{F}}$ is constant on I_q,*

(ii) *for any distinct $p,\ q \in \mathbb{Q}$, $\varphi_{\mathcal{F}}[I_p] \cap \varphi_{\mathcal{F}}[I_q] = \emptyset$,*

(iii) $\varphi_{\mathcal{F}}[\mathbb{S}^1 \setminus L_{\mathcal{F}}] \cdot \mathrm{Im} c_{\mathcal{F}} = \varphi_{\mathcal{F}}[\mathbb{S}^1 \setminus L_{\mathcal{F}}]$.

According to Lemma 2 we can correctly define

$$\{\Phi_{\mathcal{F}}(q)\} := \varphi_{\mathcal{F}}[I_q], \qquad q \in \mathbb{Q}$$

and

$$T_{\mathcal{F}}(q,\ v) := \Phi_{\mathcal{F}}^{-1}(\Phi_{\mathcal{F}}(q)c_{\mathcal{F}}(v)), \qquad q \in \mathbb{Q},\ v \in V.$$

Lemma 3 (see [6]). *If $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ is a non-dense iteration group, then:*

(i) $T_{\mathcal{F}}(T_{\mathcal{F}}(q,\ v_1),\ v_2) = T_{\mathcal{F}}(q,\ v_1 + v_2)$ *for $q \in \mathbb{Q}$, $v_1,\ v_2 \in V$,*

(ii) $T_{\mathcal{F}}(q,\ 0) = q$ *for $q \in \mathbb{Q}$,*

(iii) $F^v[I_q] = I_{T_{\mathcal{F}}(q,\ v)}$ *for $q \in \mathbb{Q}$, $v \in V$.*

3. Main results

We start with some auxiliary results which are valid without any assumption on the iteration group $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$.

It is easily seen that we have

Remark 1. Let $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ be an iteration group. If $z_0 \in \mathbb{S}^1$ is a fixed point of $F^v \in \mathcal{F}$, then so is $F^w(z_0)$ for $w \in V$.

Lemma 4. *Assume that $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ is an iteration group. Then:*

(i) *if $F^{v_1},\ F^{v_2} \in \mathcal{F}$ have fixed points, then they have a common fixed point,*

(ii) *if $\alpha(F^{v_0}) \in \mathbb{Q}$ for a $v_0 \in V$, then $\alpha(F^{rv_0}) \in \mathbb{Q}$ for $r \in \mathbb{Q}$.*

Proof. (i) Fix $v_1,\ v_2 \in V$ and assume that $z_1 \in \mathbb{S}^1$ is a fixed point of F^{v_1}. If $F^{v_2}(z_1) = z_1$, then z_1 has the desired property. Now, assume that $F^{v_2}(z_1) \neq z_1$ and let $z_2 \in \mathbb{S}^1$ be a fixed point of F^{v_2}. If z_2 is the unique fixed point of F^{v_2}, then from Remark 1 it follows that $F^{v_1}(z_2) = z_2$. Finally, we turn to the case when F^{v_2} has at least two fixed points. Denote by $\overrightarrow{(a_1,\ b_1)}$ the maximal open arc without fixed points of F^{v_2} such that $z_1 \in \overrightarrow{(a_1,\ b_1)}$. Since $F^{v_2}(a_1) = a_1$, $F^{v_2}(b_1) = b_1$ and the homeomorphism F^{v_2} preserves orientation, we have $F^{v_2}(z_1) \in \overrightarrow{(a_1,\ b_1)}$. This together with $F^{v_2}(z_1) \neq z_1$ shows that either $F^{v_2}(z_1) \in \overrightarrow{(a_1,\ z_1)}$ or $F^{v_2}(z_1) \in \overrightarrow{(z_1,\ b_1)}$. Assume, for instance, that $F^{v_2}(z_1) \in \overrightarrow{(a_1,\ z_1)}$. Then $F^{nv_2}(z_1) \in \overrightarrow{(a_1,\ F^{(n-1)v_2}(z_1))}$ for $n \in \mathbb{N}$ and consequently

$$F^{lv_2}(z_1) \in \overrightarrow{(F^{kv_2}(z_1),\ F^{jv_2}(z_1))} \quad \text{for } j,\ l,\ k \in \mathbb{N} \cup \{0\} \text{ with } j < l < k. \tag{1}$$

Suppose that there are subsequences

$$(F^{n_k v_2}(z_1))_{k \in \mathbb{N} \cup \{0\}},\ (F^{m_k v_2}(z_1))_{k \in \mathbb{N} \cup \{0\}}$$

of the sequence $(F^{nv_2}(z_1))_{n \in \mathbb{N} \cup \{0\}}$ for which

$$\lim_{k \to \infty} F^{n_k v_2}(z_1) = g_1 \neq g_2 = \lim_{k \to \infty} F^{m_k v_2}(z_1),$$

where

$$g_1,\ g_2 \in \overrightarrow{(a_1,\ b_1)} \cup \{a_1,\ b_1\}$$

and let O_{g_1}, O_{g_2} be neighbourhoods of g_1 and g_2, respectively, with $O_{g_1} \cap O_{g_2} = \emptyset$. Then, by (1), there exist non-negative integers m_k, n_{k_1}, n_{k_2} such that $n_{k_2} < m_k < n_{k_1}$ and

$$\overrightarrow{(F^{n_{k_1} v_2}(z_1),\ F^{n_{k_2} v_2}(z_1))} \subset O_{g_1}, \quad F^{m_k v_2}(z_1) \in O_{g_2}.$$

Therefore $F^{m_k v_2}(z_1) \notin \overrightarrow{(F^{n_{k_1} v_2}(z_1), F^{n_{k_2} v_2}(z_1))}$, contrary to (1).

We have thus shown that the sequence $(F^{nv_2}(z_1))_{n\in\mathbb{N}\cup\{0\}}$ is convergent. It is obvious that its limit, which will be denoted by g, is a fixed point of F^{v_2}. Moreover,

$$F^{v_1}(g) = \lim_{n\to\infty} F^{v_1}(F^{nv_2}(z_1)) = \lim_{n\to\infty} F^{nv_2}(F^{v_1}(z_1))$$
$$= \lim_{n\to\infty} F^{nv_2}(z_1) = g.$$

(ii) Let $z_0 \in \mathbb{S}^1$ and $n \in \mathbb{Z}\setminus\{0\}$ be such that $F^{nv_0}(z_0) = z_0$. Fix an $r \in \mathbb{Q}$ and take $k \in \mathbb{Z}$, $l \in \mathbb{N}$ for which $r = \frac{k}{l}$. Putting $m := nl \in \mathbb{Z}\setminus\{0\}$ we get

$$F^{mrv_0}(z_0) = (F^{nv_0})^k(z_0) = z_0,$$

which gives $\alpha(F^{rv_0}) \in \mathbb{Q}$. ◊

Corollary 1. *If $\mathcal{F} = \{F^w : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ w \in \mathbb{Q}\}$ is an iteration group, then either $\alpha(F^w) \in \mathbb{Q}$ for $w \in \mathbb{Q}$ or $\alpha(F^w) \notin \mathbb{Q}$ for $w \in \mathbb{Q} \setminus \{0\}$.*

Definition. An iteration group $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ is said to be *strictly disjoint* if the fact that $F^v \in \mathcal{F}$ has a fixed point implies $v = 0$.

Lemma 5. *If $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ is an iteration group, then the following conditions are equivalent:*

(i) *$\mathcal{F}$ is strictly disjoint,*
(ii) *$\alpha(F^v) \notin \mathbb{Q}$ for $v \in V \setminus \{0\}$,*
(iii) *for any $z \in \mathbb{S}^1$ the mapping $V \ni v \longmapsto F^v(z) \in \mathbb{S}^1$ is an injection.*

Proof. It is immediate that (ii) yields (i). Now, assume that (i) holds true and let $\alpha(F^v) \in \mathbb{Q}$ for a $v \in V$. Then $F^{nv} \in \mathcal{F}$ has a fixed point for an $n \in \mathbb{Z} \setminus \{0\}$, which together with (i) gives $nv = 0$, and consequently $v = 0$. To finish the proof it suffices to observe that conditions (i) and (iii) are also equivalent. ◊

Let us observe that every strictly disjoint iteration group $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ is disjoint and, by Lemma 5, non-singular.

Lemma 6. *If $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ is an iteration group, then the set*

$$U_{\mathcal{F}} := \{u \in V :\ \alpha(F^u) \in \mathbb{Q}\}$$

is a linear subspace of V.

Proof. Since $0 \in U_{\mathcal{F}}$, we have $U_{\mathcal{F}} \neq \emptyset$. Fix $u_1,\ u_2 \in U_{\mathcal{F}}$ and let $z_1,\ z_2 \in \mathbb{S}^1$, $n_1,\ n_2 \in \mathbb{Z}\setminus\{0\}$ be such that $F^{n_1u_1}(z_1) = z_1$ and $F^{n_2u_2}(z_2) = z_2$. By Lemma 4(i) there is a $z_0 \in \mathbb{S}^1$ for which $F^{n_1u_1}(z_0) = z_0 = F^{n_2u_2}(z_0)$, and therefore

$$(F^{u_1+u_2})^{n_1 n_2}(z_0) = F^{n_1 n_2 u_1}(F^{n_1 n_2 u_2}(z_0)) = z_0.$$

As $n_1 n_2 \in \mathbb{Z} \setminus \{0\}$, we get $\alpha(F^{u_1+u_2}) \in \mathbb{Q}$, and consequently $u_1 + u_2 \in \in U_{\mathcal{F}}$. To finish the proof it suffices to apply Lemma 4(ii). $\Diamond$

The following fact follows immediately from Lemma 5.

Corollary 2. *Assume that $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ is an iteration group and let W be a complementary subspace to $U_{\mathcal{F}}$ in V. Then*

$$\mathcal{F}_W := \{F^w : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ w \in W\}$$

is a strictly disjoint iteration group if $\dim W \geq 1$, *whereas* $\mathcal{F}_W = \{\mathrm{id}\}$ *if* $W = \{0\}$.

It is easily seen that we also have

Remark 2. Assume that $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ is an iteration group and let W be a complementary subspace to $U_{\mathcal{F}}$ in V. Then $\mathcal{F}$ is non-singular if and only if $\dim W \geq 1$.

From now on we shall make some assumptions on iteration groups under study.

We start with

Lemma 7. *Let $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ be a non-singular (respectively, singular and disjoint) iteration group. If $F^v \in \mathcal{F}$ has a fixed point, then*

$$F^v(z) = z, \qquad z \in L_{\mathcal{F}}.$$

Proof. Let $v \in V$ and $z_0 \in \mathbb{S}^1$ be such that $F^v(z_0) = z_0$. If the iteration group $\mathcal{F}$ is singular and disjoint, then our assertion follows from Remark 1 and the continuity of F^v. Next, assume that $\mathcal{F}$ is non-singular and let $w \in V$ be such that $\alpha(F^w) \notin \mathbb{Q}$. Using the same arguments as before we see that

$$F^v(z) = z, \qquad z \in \{F^{nw}(z_0),\ n \in \mathbb{Z}\}^{\mathrm{d}} = L_{F^w} = L_{\mathcal{F}}. \quad \Diamond$$

Next, let us note that an immediate consequence of Remark 2 and Cor. 2 is

Corollary 3. *Assume that $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ is a non-singular iteration group and let W and $\mathcal{F}_W$ be as in* Cor. 2. *Then*

$$L_{\mathcal{F}} = L_{\mathcal{F}_W}.$$

Lemma 8. *If $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ is a non-dense iteration group and $F^u \in \mathcal{F}$ has a fixed point, then:*

(i) $F^u[I_p] = I_p$ *for* $p \in \mathbb{Q}$,
(ii) $F^{u+v}[I_p] = I_{T_{\mathcal{F}}(p,\ v)}$ *for* $p \in \mathbb{Q},\ v \in V$.

Proof. (i) Fix a $p \in \mathbb{Q}$ and let $a_p,\ b_p \in \mathbb{S}^1$ be such that $I_p = \overrightarrow{(a_p,\ b_p)}$. Since $a_p,\ b_p \in L_{\mathcal{F}}$, from Lemma 7 it follows that $F^u(a_p) = a_p$ and $F^u(b_p) = b_p$, which together with the fact that F^u preserves orientation gives

$$F^u[I_p] = \overrightarrow{(F^u(a_p),\ F^u(b_p))} = I_p.$$

(ii) Fix $p \in \mathbb{Q}$, $v \in V$. Using (i) and Lemma 3(iii) we obtain

$$F^{u+v}[I_p] = F^u[F^v[I_p]] = F^u[I_{T_{\mathcal{F}}(p,\ v)}] = I_{T_{\mathcal{F}}(p,\ v)}. \quad \Diamond$$

Lemma 9. *Assume that $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ is a non-dense non-singular iteration group and let W be a complementary subspace to $U_{\mathcal{F}}$ in V. If $T_{\mathcal{F}}(p,\ w_1) = T_{\mathcal{F}}(p,\ w_2)$ for some $p \in \mathbb{Q}$, $w_1,\ w_2 \in W$, then $w_1 = w_2$.*

Proof. Fix $p \in \mathbb{Q}$, $w_1,\ w_2 \in W$ for which $T_{\mathcal{F}}(p,\ w_1) = T_{\mathcal{F}}(p,\ w_2)$. Then, by Lemma 3(i) and (ii), we have

$$\begin{aligned} T_{\mathcal{F}}(p,\ w_1 - w_2) &= T_{\mathcal{F}}(T_{\mathcal{F}}(p,\ w_1),\ -w_2) = T_{\mathcal{F}}(T_{\mathcal{F}}(p,\ w_2),\ -w_2) \\ &= T_{\mathcal{F}}(p,\ 0) = p \end{aligned}$$

and Lemma 3(iii) now shows that

$$F^{w_1-w_2}[I_p] = I_p. \tag{2}$$

Let $a_p,\ b_p \in \mathbb{S}^1$ be such that $I_p = \overrightarrow{(a_p,\ b_p)}$. Since $F^{w_1-w_2}$ is an orientation-preserving homeomorphism, from (2) it follows that $F^{w_1-w_2}(a_p) = a_p$. Therefore $\alpha(F^{w_1-w_2}) \in \mathbb{Q}$, and consequently $w_1 - w_2 \in$ $\in U_{\mathcal{F}}$. But we also have $w_1 - w_2 \in W$, and $U_{\mathcal{F}} \cap W = \{0\}$ finally yields $w_1 = w_2$. $\Diamond$

The following fact follows immediately from Lemma 9.

Corollary 4. *Assume that $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ is a non-dense non-singular iteration group and let W be a complementary subspace to $U_{\mathcal{F}}$ in V. Then the mapping $c_{\mathcal{F}} \mid_W : W \longrightarrow \mathbb{S}^1$ is an injection.*

Lemma 10. *Assume that $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\}$ is a non-dense non-singular iteration group and let W be a complementary subspace to $U_{\mathcal{F}}$ in V. Then*

$$1 \leq \dim W \leq \aleph_0. \tag{3}$$

Proof. According to Remark 2 it suffices to show that $\dim W \leq \aleph_0$. Let the iteration group $\mathcal{F}_W$ be as in Cor. 2. This group is non-singular and, by Cor. 3, non-dense. Therefore from Lemma 2 and Cor. 4 it follows that $\mathrm{card}\mathrm{Im}\, c_{\mathcal{F}_W} = \aleph_0$ and the mapping $c_{\mathcal{F}_W} : W \longrightarrow \mathbb{S}^1$ is an injection. Consequently,

$$\dim W \leq \mathrm{card} W = \mathrm{card}\mathrm{Im} c_{\mathcal{F}_W} = \aleph_0. \quad \Diamond$$

Theorem 2. *If $\mathcal{F} = \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1, v \in V\}$ is a non-dense non-singular iteration group, then there is a linear subspace W of V satisfying condition* (3) *and a linear subspace U of V such that $V = U \oplus W$ and*

$$\mathcal{F} = \mathcal{F}_U \oplus \mathcal{F}_W,$$

where $\mathcal{F}_W := \{F^w : \mathbb{S}^1 \longrightarrow \mathbb{S}^1, w \in W\}$ is a strictly disjoint non-dense iteration group with $L_{\mathcal{F}_W} = L_{\mathcal{F}}$ and $\mathcal{F}_U := \{F^u : \mathbb{S}^1 \longrightarrow \mathbb{S}^1, u \in U\}$ is a singular iteration group if $\dim U \geq 1$, whereas $\mathcal{F}_U = \{\mathrm{id}\}$ if $U = \{0\}$.

Proof. Put $U := U_{\mathcal{F}}$ and note that, by Lemma 6 U is a linear subspace of V. Let W be a complementary subspace to U in V. Since from Lemma 10 it follows that W satisfies (3), Corollaries 2 and 3 show that $\mathcal{F}_W := \{F^w : \mathbb{S}^1 \longrightarrow \mathbb{S}^1, w \in W\}$ is a strictly disjoint non-dense iteration group for which $L_{\mathcal{F}} = L_{\mathcal{F}_W}$. It is also obvious that $\mathcal{F}_U := = \{F^u : \mathbb{S}^1 \longrightarrow \mathbb{S}^1, u \in U\}$ is a singular iteration group if $\dim U \geq 1$, whereas $\mathcal{F}_U = \{\mathrm{id}\}$ if $U = \{0\}$. Finally, $\mathcal{F}_U$ and $\mathcal{F}_W$ are subgroups of $(\mathcal{F}, \circ)$ with $\mathcal{F}_U \cap \mathcal{F}_W = \{\mathrm{id}\}$ and

$$\begin{aligned}
\mathcal{F}_U \circ \mathcal{F}_W &= \{F_1 \circ F_2,\ F_1 \in \mathcal{F}_U,\ F_2 \in \mathcal{F}_W\} \\
&= \{F^u \circ F^w : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ u \in U,\ w \in W\} \\
&= \{F^{u+w} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ u \in U,\ w \in W\} \\
&= \{F^v : \mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ v \in V\} = \mathcal{F}. \qquad \Diamond
\end{aligned}$$

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