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A NEW KUROSH–AMITSUR RADICAL THEORY FOR PROPER SEMIFIELDS I.

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Abstract: In developing a radical theory for semifields, one has to work with kernels of semifields, certain substructures which are either subsemifields or merely groups. Thus a universal class catering a radical theory must contain, in general, semifields as well as groups. In [12] we were too generous and admitted groups in abundance. Here we take only as many groups as necessary. This enables us to build up a new theory which incorporates the old one as a special case, and which does not make use of groups if the considered universal class contains only idempotent semifields, a concept closely related to lattice-ordered groups (cf. [13]). We also solve a problem posed in [12], and present several conditions for the hereditariness of semisimple classes. The

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claim stated in [12] that every semisimple class is hereditary in the old theory, has not been substantiated, and is still open. In the new theory, however, not every semisimple class is hereditary and so not every radical class has the ADS-property.

In Part I we present the new general theory and in Part II (cf. [14]) we compare the old and new theories and deal with special features of the new theory.

1. Introduction

A proper semifield is defined as a semiring $(A, +, \cdot)$ such that $(A, \cdot)$ is a group (cf. Section 2). In contrast with the case of fields, such a semifield may have various non-trivial homomorphic images. Moreover, similarly to rings and groups, each congruence κ of a semifield $(A, +, \cdot)$ is determined by a kernel K, a particular substructure of $(A, +, \cdot)$ (cf. [6]). Based on this, a Kurosh–Amitsur radical theory dealing with homomorphic images and kernels was developed in [12] for proper semifields.

The kernels of a proper semifield $(A, +, \cdot)$ are normal subgroups of $(A, \cdot)$ with an additional property. Among these kernels, in general, there are those which are additively closed and so subsemifields of $(A, +, \cdot)$, as well as those which are not additively closed and hence merely certain groups. Therefore, a radical theory for arbitrary proper semifields has to deal simultaneously with semifields and groups. More precisely, the frame for such a theory is a suitable universal subclass $\mathfrak{H}$ of $\mathfrak{S}^* \cup \mathfrak{G}$, where $\mathfrak{S}^*$ denotes the class of all proper semifields and $\mathfrak{G}$ the class of all groups. Beyond that certain interrelations between semifields and groups were assumed in [12]. (For instance, the axioms defining a semisimple class $\mathbb{S}$ of $\mathfrak{H}$ given there include the property that $(A, +, \cdot) \in \mathbb{S}$ implies $(A, \cdot) \in \mathbb{S}$.) However, applying this radical theory only to additively idempotent semifields as done in [13], all kernels of such semifields are again semifields. Hence, in this case the use of groups is unnatural and apparently superfluous (but not avoidable as long as semisimple classes have the above mentioned property).

This and several other possible improvements motivated us to present here a more general radical theory for proper semifields. It is based on a weaker interrelation between semifields and groups which becomes vacant when working merely with idempotent semifields or merely with groups. Our new theory contains that given in [12] as a

special case. Nevertheless, the old theory remains of interest inasmuch as there are general results which depend on the stronger assumptions used in that theory. For this reason and for comparing both theories without the fear of confusion we shall use all notations, concepts and terminology introduced in [12] also here in the meaning of that paper. This concerns, in particular, the concepts universal class, radical class, radical operator and semisimple class, and we speak now of universal classes in the new meaning, briefly n-universal classes, and use likewise the terminology n-radical class, n-semisimple class, etc.

For the sake of selfcontainedness, we repeat in the next section concepts and statements on semifields and their kernels which are used throughout the paper. In Section 3 we describe the frame of the new theory, namely n-universal classes $\mathfrak{H}$ of $\mathfrak{S}^* \cup \mathfrak{G}$. Such a class $\mathfrak{H}$ contains is general semifields and groups, but it may consist merely of idempotent semifields or merely of groups. In Section 4 we investigate two kinds of subclasses $\mathbb{R}$ and $\mathbb{S}$ of $\mathfrak{H}$, defined by properties which, interpreted accordingly, characterize radical classes and semisimple classes of rings or of groups. In particular, we obtain a bijective correspondence between these classes. However, these classes are in general not yet n-radical classes and n-semisimple classes. To define the latter (cf. Defs. 5.1 and 7.1) one needs a further condition in both cases, denoted by (Rc) and $(S\gamma)$, respectively. They assure the simultaneous occurrence of semifields and groups in n-radical classes and n-semisimple classes, but only under certain conditions which become void if the considered n-universal class $\mathfrak{H}$ consists only of idempotent semifields (or only of groups).

In the subsequent Sections 5–7 we develop the new radical theory for proper semifields dealing with n-radical classes, n-radical operators and n-semisimple classes, their properties, interrelation and with several other characterizations of these basic concepts. Apart from conditions as (Rc) and $(S\gamma)$ and a corresponding one for n-radical operators, which clearly occur in various statements, the obtained results are very similar to those in the radical theory of rings or groups.

In the first part of Section 8, we prove that one needs exactly property (Rc) to assure the following property for an n-radical class $\mathbb{R}$ of $\mathfrak{H}$: for each $A \in \mathfrak{H}$, all $\mathbb{R}$-kernels of A are contained in a greatest $\mathbb{R}$-kernel of A. Clearly, this property is indispensable for a Kurosh–Amitsur radical theory where congruences are determined by kernels,

and (Rc) for n-radical classes yields $(S\gamma)$ for n-semisimple classes. We give in Th. 8.3 also a set of properties (similar to those used in a torsion theory) which characterize a pair of subclasses of $\mathfrak{H}$ as an n-radical class $\mathbb{R}$ and the corresponding n-semisimple class $\mathbb{S}$. From this we obtain examples of such non-trivial classes which satisfy $\mathbb{R} \cup \mathbb{S} = \mathfrak{H}$, a special feature of this radical theory which occurs neither for rings nor for groups.

Sections 9 and 10 will appear as part II of the present paper in the forthcoming issue of this journal (cf. [14]).

In Section 9 we consider the radical theory for proper semifields developed in [12] as a special case of the new theory presented here. We show that all statements of the latter (apart from a particular exception) yield statements of the former theory just by replacing the new concepts and conditions by the corresponding former ones. Moreover, as a simplification of the old theory we obtain that one condition in the definition of a radical class in [12] is superfluous (cf. Prop. 9.3). We also show by the somewhat complicated Ex. 9.9 that another simplification of that definition is not possible. This solves a problem posed in [12] in the negative (cf. Remark 9.2 and Th. 9.8).

In the last section we deal with the question whether n-semisimple classes and in particular semisimple classes are hereditary. We have to correct our paper [12], where we claimed in Th. 6.5 that every semisimple class $\mathbb{S}$ of a universal class $\mathfrak{H} \subseteq \mathfrak{S}^* \cup \mathfrak{G}$ would be hereditary. In the meantime, however, we discovered a gap in the proof of the auxiliary Prop. 2.11 b) of [12] used in proving Th. 6.5 in that paper. Despite of all attempts, up to now we could not decide as whether these assertions in [12] are true or false. (The same applies to some minor statements of that paper, namely to the Supplement of Th. 5.7, part b) of Cor. 6.3 and Ex. 7.2, which also depend on the questionable proposition.)

Therefore, starting with this question again from the beginning, we present several sufficient conditions for the hereditariness of n-semisimple classes and semisimple classes. The most far-reaching results are that semisimple classes of universal classes are hereditary if all involved semifields have commutative addition, and that the corresponding statement for n-semisimple classes of n-universal classes is false (cf. Ths. 10.11 and 10.12).

Before going on, we remark that our new theory fits to the framework of the general radical theory of [7] as the old theory does. We fur-

ther note that there is a Kurosh–Amitsur radical theory for semirings, more precisely, for additively commutative semirings with an absorbing zero (cf. [8], [4], [5], and the references given there). It deals with semiring-ideals as kernels, regardless of the fact that a semiring has in general more congruences as those described by such a kernel. However, common objects of this theory and that of proper semifields are just semirings which consist of one element, since other proper semifields have no zero.

2. Kernels of proper semifields

The background of our considerations is a rather general concept of semiring, defined as a universal algebra $(A, +, \cdot)$ where $(A, +)$ and $(A, \cdot)$ are arbitrary semigroups connected by ring-like distributivity. Such a semiring $(A, +, \cdot)$ is called a *semifield*, if $(A, \cdot)$ or $(A \setminus \{0\}, \cdot)$ is a group, the latter in the case when $(A, +)$ has a neutral element 0. The former one, called proper semifield in this paper, deserve particular interest, since all other semifields are either fields or skewfields, or are obtained from proper semifields by adjoining one element, an absorbing zero 0 (cf. [3, I.5]).

Definition 2.1. a) A semiring $(A, +, \cdot)$ is called a *proper semifield* if $(A, \cdot)$ is a group. Note that neither $(A, +)$ nor $(A, \cdot)$ are assumed to be commutative and that a proper semifield may consist of a single element. The class of all proper semifields (which clearly may be considered as a variety) is denoted by $\mathfrak{S}^*$.

b) If not specified, we write e for the multiplicative identity of a semifield $A \in \mathfrak{S}^*$, and we call A *idempotent* if $e + e = e$ holds. The latter implies $a + a = a$ for all $a \in A$. The class of all idempotent proper semifields (a subvariety of $\mathfrak{S}^*$) is denoted by $\mathfrak{S}^{\mathrm{idp}}$.

c) A non-idempotent semifield $A \in \mathfrak{S}^*$ consists of at least two elements, and we denote the class of all these semifields by $\mathfrak{S}^\circ$.

The following results of [6] are used throughout this paper. As for arbitrary universal algebras, each homomorphism $\varphi : (A, +, \cdot) \to \to (B, +, \cdot)$ of proper semifields corresponds to a congruence κ of $(A, +, \cdot)$, and the homomorphic image $(\varphi(A), +, \cdot) \in \mathfrak{S}^*$ is isomorphic to the algebra $(A/\kappa, +, \cdot)$ of congruence classes $[a]_\kappa$.

Theorem 2.2. *Let $(A, +, \cdot)$ be a proper semifield. Then each congruence κ of $(A, +, \cdot)$ is determined by the congruence class $K = [e]_\kappa$*

according to

$$a\kappa b \Leftrightarrow aK = bK \Leftrightarrow a^{-1}b \in K, \tag{2.1}$$

where K is a normal subgroup of $(A,\cdot)$ with the property that

$$s + t = e \text{ for } s,t \in A \text{ implies } s + tk \in K \text{ for all } k \in K. \tag{2.2}$$

Conversely, each normal subgroup K of $(A,\cdot)$ which satisfies (2.2) determines by (2.1) a congruence κ of $(A,+,\cdot)$.

Definition 2.3. For each $(A,+,\cdot) \in \mathfrak{S}^*$, a normal subgroup K of $(A,\cdot)$ satisfying (2.2) is called a *kernel* of $(A,+,\cdot)$, and we denote by $\mathfrak{K}(A)$ the set of all kernels of A. If $K \in \mathfrak{K}(A)$ corresponds by Th. 2.2 to the congruence κ of A, we use $(A/K,+,\cdot)$ synonymously with $(A/\kappa,+,\cdot)$.

In this context we note that (2.2) can be replaced by several other properties which characterize a normal subgroup of $(A,\cdot)$ as a kernel of $(A,+,\cdot)$. For this and various examples of kernels of proper semifields we refer to [6]. Here we give only one example which, apart from later use, illustrates that *a kernel K of $A \in \mathfrak{S}^*$ may be a subsemifield of A or merely a (normal) subgroup of $(A,\cdot)$.*

Example 2.4. The direct product $A = \prod_{i\in I} B_i$ of proper semifields B_i, $i \in I$, is again such a semifield. For $(A,+,\cdot) = (B_1,+,\cdot) \times (B_2,+,\cdot)$ we consider the natural projection $\varphi_1 : (A,+,\cdot) \to (B_1,+,\cdot)$ which maps each $(b_1,b_2) \in A$ onto b_1. Then $K_1 = \{(e_1,b_2) \mid b_2 \in B_2\}$ is the corresponding kernel of A, where e_1 denotes the identity of B_1. Now, let B_1 be idempotent. Then $e_1 + e_1 = e_1$ yields that $K_1 = (K_1,+,\cdot)$ is a subsemifield of $(A,+,\cdot)$, clearly isomorphic to $(B_2,+,\cdot)$. However, if B_1 is not idempotent, then K_1 is not additively closed. Hence in this case K_1 is only a subgroup $K_1 = (K_1,\cdot)$ of $(A,\cdot)$ isomorphic to $(B_2,\cdot)$.

Proposition 2.5. *As a consequence of* Th. 2.2, *the set $\mathfrak{K}(A)$ of all kernels of $A \in \mathfrak{S}^*$ is a complete lattice $(\mathfrak{K}(A),\subseteq)$, isomorphic to the lattice of all congruences of A. We further state*

$$K \vee L = KL = \{k \cdot \ell \mid k \in K, \ell \in L\}$$

for the supremum of $K,L \in \mathfrak{K}(A)$, and $K \wedge L = K \cap L$ for their infimum. Moreover, each subset $S \subseteq A$ is contained in a smallest kernel of A which is denoted by $\mathrm{hull}_A(S)$.

Remark 2.6. Let $(A,+,\cdot)$ be a proper, non-idempotent semifield. Then each element $a \in A$ has infinite additive order. Moreover, the proper semifield $(\mathbb{H},+,\cdot)$ of positive rational numbers is an operator domain for the semigroup $(A,+)$, if one defines the action of $\alpha = \frac{m}{n} \in$

$\in \mathbb{H}$ on $a \in A$ by

$$\alpha a = (ne)^{-1}(ma) = (e + \cdots + e)^{-1}(a + \cdots + a). \tag{2.3}$$

This yields that $(\mathbb{H}e, +, \cdot)$ for $\mathbb{H}e = \{\alpha e \mid \alpha \in \mathbb{H}\}$ is the smallest subsemifield contained in $(A, +, \cdot)$, which is isomorphic to $(\mathbb{H}, +, \cdot)$. Furthermore, $(\mathbb{H}, +, \cdot)$ can be considered as operator domain also for an idempotent semifield A where (2.3) yields $\alpha a = a$ for all $\alpha \in \mathbb{H}$ and $a \in A$ and thus $\mathbb{H}e = \{e\}$.

Theorem 2.7. *For each kernel K of a proper semifield A the following statements are equivalent:*

a) *K is a subsemifield of A, i.e. $h + k \in K$ for all $h, k \in K$.*

b) *There are elements $h, k \in K$ satisfying $h + k \in K$.*

c) *$(A/K, +, \cdot)$ is an idempotent semifield.*

Corollary 2.8. a) *A proper semifield A is idempotent iff the smallest kernel of A is the one element semifield $\langle e \rangle = (\{e\}, +, \cdot)$. In this case each kernel $K \in \mathfrak{K}(A)$ is an idempotent semifield.*

b) *A semifield $A \in \mathfrak{S}^*$ is not idempotent iff the smallest kernel of A is the one element group $(e) = (\{e\}, \cdot)$. In this case each kernel $K \in \in \mathfrak{K}(A)$ is either a non-idempotent subsemifield of A or merely a group, and K is a subsemifield iff K contains the subsemifield $(\mathbb{H}e, +, \cdot) \cong \cong (\mathbb{H}, +, \cdot)$.*

c) *For all $A \in \mathfrak{S}^*$, $\mathrm{hull}_A(\mathbb{H}e)$ is the smallest kernel of A which is a subsemifield.*

Example 2.9. a) A semifield $A \in \mathfrak{S}^*$ is called *simple*, if $\mathfrak{K}(A)$ consists of A and $\langle e \rangle$ or (e). Using (2.2), one can verify that the proper semifields of positive rational and of positive real numbers are simple, and the same holds for every proper subsemifield of an algebraic number field (cf. [6, Th. 6.9]).

b) Later we need the following example of a simple idempotent semifield $B \in \mathfrak{S}^*$. Let $(B, \cdot)$ be an infinite cyclic group generated by $b \in B$ and define

$$b^i + b^j = b^{\max(i,j)} \quad \text{for all } b^i, b^j \in B.$$

It is clear that $(B, +, \cdot)$ is an idempotent proper semifield. To show that B is simple, let $K \in \mathfrak{K}(B)$ contain an element $b^i \neq e$. Choosing $i > 0$, we obtain $b^0 + b^{1-i} = b^0$ and $b^0 + b^{1-i}b^i = b^1 \in K$ by (2.2), that is, $K = B$.

Since a kernel $L \in \mathfrak{K}(A)$ need not be a subsemifield of A and a subsemifield U of A need not be a kernel of A, the First Isomorphism

Theorem has two versions:

Theorem 2.10. *Let A be a proper semifield and K a kernel of A.*

a) *If U is a subsemifield of A, then $U \cap K$ is a kernel of U and K a kernel of the subsemifield $UK = \{u \cdot k \mid u \in U, k \in K\}$ of A, and one has the isomorphism*

$$U/(U \cap K) \cong UK/K.$$

b) *If L is a kernel of A, then $L \cap K$ is a kernel of L and K a kernel of $LK \in \mathfrak{K}(A)$. Now, one has in general only the group isomorphism*

$$L/(L \cap K) \cong LK/K,$$

which is a semifield isomorphism exactly in the case when L is also a subsemifield of A.

Remark 2.11. *For $A \in \mathfrak{S}^*$, let $\varphi : A \to B$ be a surjective homomorphism. Then $L \in \mathfrak{K}(A)$ implies $\varphi(L) \in \mathfrak{K}(A)$ and $\overline{L} \in \mathfrak{K}(B)$ yields $\varphi^{-1}(\overline{L}) \in \mathfrak{K}(A)$.* A corresponding statement holds for subsemifields U of A and $\overline{U}$ of B.

Theorem 2.12. *Let A be a proper semifield and L and K kernels of A satisfying $K \subseteq L$. Then L/K is a kernel of A/K and one has the semifield isomorphism*

$$A/L \cong (A/K)/(L/K).$$

We close this section with two propositions. The first one was also proved in [12], whereas the latter is unpublished so far.

Proposition 2.13. *Let A be a proper semifield and $(N, \cdot)$ a normal subgroup of $(A, \cdot)$. Then $K = \mathrm{hull}_A(N)$, the smallest kernel of A containing N, consists of all finite sums $\sum s_i n_i$ of elements $n_i \in N$ and $s_i \in A$ satisfying $\sum s_i = e$.*

Proposition 2.14. *Each non-trivial (cf.* Def. 3.1*) proper semifield A has a non-trivial homomorphic image B with the property that each non-trivial kernel C of B is a subsemifield of B.*

Proof. If A itself has this property, there is nothing to prove. Otherwise, there are non-trivial kernels $K_i \in \mathfrak{K}(A) \cap \mathfrak{G}$. Let $\mathfrak{C}$ be a chain $\ldots \subseteq K_i \subseteq \ldots$ of those kernels. One easily checks that $K = \cup(K_i \mid K_i \in \mathfrak{C})$ is a kernel of A, and we show $K \in \mathfrak{G}$ by way of contradiction. Indeed, $k_1 + k_2 = k_3$ for elements of K yields that these elements are contained in some $K_j \in \mathfrak{C}$. Hence K_j would be a subsemifield of A by Th. 2.7, whereas $K_j \in \mathfrak{G}$ was assumed. So we can apply Zorn's Lemma and obtain that $\mathfrak{K}(A) \cap \mathfrak{G}$ has a maximal element, say again K. This yields that $B = A/K$ is a homomorphic image $\varphi(A)$ of A as

claimed above, since a non-trivial kernel $\overline{L} = \mathfrak{K}(B) \cap \mathfrak{G}$ would yield $L = \varphi^{-1}(\overline{L}) \in \mathfrak{K}(A) \cap \mathfrak{G}$ and $K \subset L$ (cf. Rem. 2.11). Moreover, $K \subset A$ shows that $B = A/K$ is non-trivial. $\Diamond$

3. Universal classes in the new meaning

Recall that $\mathfrak{G}$ denotes the class of all groups and $\mathfrak{S}^*$ that of all proper semifields. We further have introduced $\mathfrak{S}^{\mathrm{idp}}$ for the class of all idempotent proper semifields and $\mathfrak{S}^\diamond$ for the class of all non-idempotent proper semifields. Moreover, we shall deal with a suitable subclass $\mathfrak{H}$ of $\mathfrak{S}^* \cup \mathfrak{G}$ which, in general, contains semifields and groups. Hence $A \in \mathfrak{H}$ means that A is either a semifield $(A, +, \cdot) \in \mathfrak{S}^*$ or a group $(A, \cdot) \in \mathfrak{G}$. As already done in [12], in the following we consider three types of morphisms $\varphi : A \to B$:

1) All semifield homomorphisms $\varphi : (A, +, \cdot) \to (B, +, \cdot)$ for $A, B \in \mathfrak{S}^*$.

2) All group homomorphisms $\varphi : (A, \cdot) \to (B, \cdot)$ for $A, B \in \mathfrak{G}$.

3) All group homomorphisms $\varphi : (A, \cdot) \to (B, +, \cdot)$ for $A \in \mathfrak{G}$ and $B \in \mathfrak{S}^*$, where one does not care about the addition in B.

In particular, we call a morphism *injective*, or *surjective* or *bijective* if the set theoretical mapping of the underlying sets has this property. Moreover, $\varphi : A \to B$ is called an *isomorphism* if $\varphi^{-1} : B \to \to A$ exists and is a morphism, too. Hence an isomorphism has to be of type 1) or 2).

Clearly, we may consider $\mathfrak{S}^*$ as a category with all morphisms of type 1), $\mathfrak{G}$ as a category with all morphisms of type 2), and $\mathfrak{S}^* \cup \mathfrak{G}$ as a category with all morphisms introduced above.

Again, for each morphism $\varphi : A \to B$, its kernel is defined by $K = \varphi^{-1}(\varphi(e))$ for the identity $e \in A$, and we write $\mathfrak{K}(A)$ for the set of all kernels of A. Recall from Cor. 2.8 that $\mathfrak{K}(A) \subseteq \mathfrak{S}^{\mathrm{idp}} \subseteq \mathfrak{S}^*$ holds for each $A \in \mathfrak{S}^{\mathrm{idp}}$, whereas each $A \in \mathfrak{S}^\diamond$ has kernels contained in $\mathfrak{S}^\diamond$ (at least A itself) as well as in $\mathfrak{G}$ (at least $(\{e\}, \cdot)$). For each group $A \in \mathfrak{G}$, the kernels are, of course, its normal subgroups.

Definition 3.1. a) For each non-empty subclass $\mathbb{M}$ of $\mathfrak{S}^* \cup \mathfrak{G}$ we define the following properties:

I) $\mathbb{M}$ *is closed under surjective homomorphisms of type* 1) *and* 2): $A \in \mathbb{M}$ implies $\varphi(A) \in \mathbb{M}$ for all $A \in \mathfrak{S}^* \cup \mathfrak{G}$ and all morphisms $\varphi : A \to B$ of these types.

II) $\mathbb{M}$ is hereditary: $A \in \mathbb{M}$ implies $\mathfrak{K}(A) \subseteq \mathbb{M}$ for all $A \in \mathfrak{S}^* \cup \mathfrak{G}$.

b) A non-empty subclass $\mathfrak{H}$ of $\mathfrak{S}^* \cup \mathfrak{G}$ is called a *universal class in the new meaning*, briefly an *n-universal class of* $\mathfrak{S}^* \cup \mathfrak{G}$, if $\mathfrak{H}$ satisfes I) and II).

As a consequence of I), each n-universal class $\mathfrak{H}$ of $\mathfrak{S}^* \cup \mathfrak{G}$ contains all one-element semifields $\langle e \rangle = (\{e\}, +, \cdot)$ whenever $\mathfrak{H} \cap \mathfrak{S}^* \neq \emptyset$, and all one-element groups $(e) = (\{e\}, \cdot)$ whenever $\mathfrak{H} \cap \mathfrak{G} \neq \emptyset$, called *trivial semifields* and *trivial groups*, or *trivial objects* henceforth. Moreover, depending on $\mathfrak{H}$, we denote by $\mathfrak{T}$ the subclass of trivial objects contained in $\mathfrak{H}$.

Later on we use also the concept of a universal class $\mathfrak{H}$ of $\mathfrak{S}^* \cup \mathfrak{G}$ as defined in [12, Def. 3.1] by $\mathfrak{H} \neq \emptyset$, I), II) and

III) $\quad (A, +, \cdot) \in \mathfrak{H} \Rightarrow (A, \cdot) \in \mathfrak{H}$ for all $(A, +, \cdot) \in \mathfrak{S}^*$.

This additional condition looks natural and has some advantage, but it involves sometimes too many superfluous group as, for instance, in the case of $\mathfrak{H} \cap \mathfrak{S}^* \subseteq \mathfrak{S}^{\mathrm{idp}}$.

The following statement, however, shows that many reasonable n-universal classes are already universal classes.

Proposition 3.2. *Let $\mathfrak{H}$ be an n-universal class in $\mathfrak{S}^* \cup \mathfrak{G}$ which is closed under taking finite direct products. If $\mathfrak{H}$ contains at least one non-idempotent semifield, say $(B, +, \cdot) \in \mathfrak{H} \cap \mathfrak{S}^\diamond$, then $\mathfrak{H}$ satisfies also* III), *and hence it is a universal class in $\mathfrak{S}^* \cup \mathfrak{G}$.*

Proof. Let $(A, +, \cdot)$ be any semifield contained in $\mathfrak{H}$. Then $\mathfrak{H}$ contains also the direct product $(B, +, \cdot) \times (A, +, \cdot)$. Since $B \in \mathfrak{S}^\diamond$, by Ex. 2.4 this direct product has a kernel $K_1 = \{(e_1, a) \mid a \in A\} \cong (A, \cdot)$. Hence, taking into account I) and II), we get $(A, \cdot) \in \mathfrak{H}$. Thus also III) is satisfied. $\Diamond$

4. Subclasses satisfying (Ra) and (Rb) or (Sa) and (Sb)

Definition 4.1. Let $\mathfrak{H}$ be an n-universal class of $\mathfrak{S}^* \cup \mathfrak{G}$.

a) For each $A \in \mathfrak{H}$ we denote by $B \lhd A$ that B is a non-trivial kernel of A, and by $A \rightarrowtriangle B$ a surjective homomorphism $\varphi : A \to B$ of type 1) or 2) such that $\varphi(A) = B \notin \mathfrak{T}$.

b) For a subclass $\mathbb{R}$ of $\mathfrak{H}$ we define the following properties (Ra) and (Rb) which refer to condition

(4.1) $$\forall B(A \to\!\triangleright B)\exists C(C \triangleleft B \wedge C \in \mathbb{R}).$$

(Ra) For all $A \in \mathfrak{H}$, (4.1) implies $A \in \mathbb{R}$.

(Rb) All $A \in \mathbb{R}$ satisfy (4.1).

c) For a subclass $\mathbb{S}$ of $\mathfrak{H}$ we define the following properties (Sa) and (Sb) which refer to condition

(4.2) $$\forall B(B \triangleleft A)\exists C(B \to\!\triangleright C \wedge C \in \mathbb{S}).$$

(Sa) For all $A \in \mathfrak{H}$, (4.2) implies $A \in \mathbb{S}$.

(Sb) All $A \in \mathbb{S}$ satisfy (4.2).

A class $\mathbb{S}$ with property (Sb) is called a *regular class*. Each hereditary class is clearly regular.

Note that there is a kind of duality between b) and c) inasmuch as they are related to each other by interchanging $X \to\!\triangleright Y$ and $Y \triangleleft X$ for all $X, Y \in \mathfrak{H}$. Moreover, these notations and properties correspond in an obvious way to well-known ones used in the radical theories for rings and for groups. The same applies to all statements of this section. They deal with subclasses $\mathbb{R}$ and $\mathbb{S}$ of $\mathfrak{H} = \mathfrak{S}^* \cup \mathfrak{G}$ where $\mathbb{R}$ stands for a subclass satisfying (Ra) and (Rb) and $\mathbb{S}$ for a subclass satisfying (Sa) and (Sb), and are, in fact, independent on the concrete meaning of $B \triangleleft A$ and $A \to\!\triangleright B$. However, whereas these statements already characterize radical classes $\mathbb{R}$ and semisimple classes $\mathbb{S}$ in the radical theories for rings and for groups, one needs further properties to define corresponding concepts for arbitrary proper semifields (cf. Defs. 5.1 and 7.1 and Section 9 for the theory of [12]). In this context we refer also to Section 8, in particular for cases where these further properties are trivially satisfied and thus superfluous.

Using only (Ra), (Rb) and the fact that $A \to\!\triangleright B$ is a transitive relation, one obtains at first

Lemma 4.2. a) *A subclass* $\mathbb{R}$ *of* $\mathfrak{H}$ *which satisfies* (Ra) *and* (Rb) *has the following property:*

($R\eta$) $\mathbb{R}$ *is closed under surjective homomorphisms of type* 1) *and* 2).

b) *Conversely,* ($R\eta$) *implies* (Rb) *for every subclass* $\mathbb{R}$ *of* $\mathfrak{H}$. *Hence* (Ra) *and* (Rb) *are equivalent to* (Ra) *and* ($R\eta$) *for every* $\mathbb{R} \subseteq \mathfrak{H}$.

Definition 4.3. For each subclass $\mathbb{M}$ of $\mathfrak{H}$ we define *dual operators* $\mathcal{U}$ and $\mathcal{S}$ (more precisely: $\mathcal{U}_{\mathfrak{H}}$ and $\mathcal{S}_{\mathfrak{H}}$) by

(4.3) $$\mathcal{U}\mathbb{M} = \{A \in \mathfrak{H} \mid \forall B(A \to\!\triangleright B \Rightarrow B \notin \mathbb{M})\}$$

and

(4.4) $$\mathcal{S}\mathbb{M} = \{A \in \mathfrak{H} \mid \forall B(B \lhd A \Rightarrow B \notin \mathbb{M})\}.$$

Note that $\mathbb{M}_1 \subseteq \mathbb{M}_2$ implies $\mathcal{U}\mathbb{M}_2 \subseteq \mathcal{U}\mathbb{M}_1$ and $\mathcal{S}\mathbb{M}_2 \subseteq \mathcal{S}\mathbb{M}_1$. Moreover,
$\mathbb{M} \cap \mathcal{U}\mathbb{M} = \mathfrak{T}$ and $\mathbb{M} \cap \mathcal{S}\mathbb{M} = \mathfrak{T}$ are satisfied.

Proposition 4.4. a) *If* (Ra) *and* (Rb) *hold for a subclass* $\mathbb{R} \subseteq \mathfrak{H}$, *then* $\mathbb{S} = \mathcal{S}\mathbb{R}$ *satisfies* (Sa) *and* (Sb).

b) *If* (Sa) *and* (Sb) *hold for a subclass* $\mathbb{S} \subseteq \mathfrak{H}$, *then* $\mathbb{R} = \mathcal{U}\mathbb{S}$ *satisfies* (Ra) *and* (Rb).

c) *The operators* $\mathcal{U}$ *and* $\mathcal{S}$ *provide a bijective correspondence between the subclasses of* $\mathfrak{H}$ *considered above, that is,* $\mathcal{U}\mathcal{S}\mathbb{R} = \mathbb{R}$ *and* $\mathcal{S}\mathcal{U}\mathbb{S} = \mathbb{S}$. *Moreover,* $\mathbb{R}$ *and* $\mathbb{S} = \mathcal{S}\mathbb{R}$ *satisfy* $\mathbb{R} \cap \mathbb{S} = \mathfrak{T}$.

Proof. a) Using only (Ra) and (Rb) for $\mathbb{R}$, one obtains from (4.4)

$$A \in \mathcal{S}\mathbb{R} \Leftrightarrow \forall B(B \lhd A)\exists C(B \twoheadrightarrow C \wedge C \in \mathcal{S}\mathbb{R}) \text{ for all } A \in \mathfrak{H},$$

which states (Sa) and (Sb) for $\mathbb{S} = \mathcal{S}\mathbb{R}$.

b) This is the dual statement of a), but also a consequence of the subsequent Lemma 4.5.

c) Applying (4.3) and (4.4) to any subclass $\mathbb{S} \subseteq \mathfrak{H}$, we obtain

$$A \in \mathcal{S}\mathcal{U}\mathbb{S} \Leftrightarrow \forall B(B \lhd A)\exists C(B \twoheadrightarrow C \wedge C \in \mathbb{S}) \text{ for all } A \in \mathfrak{H},$$

that is, $A \in \mathcal{S}\mathcal{U}\mathbb{S} \Leftrightarrow A$ satisfies (4.2). This yields $\mathcal{S}\mathcal{U}\mathbb{S} = \mathbb{S}$ if $\mathbb{S}$ satisfies (Sa) and (Sb), and, for later use,

(4.5) $$\mathbb{S} \subseteq \mathcal{S}\mathcal{U}\mathbb{S} \Leftrightarrow (Sb) \text{ for } \mathbb{S}, \quad \mathbb{S} \supseteq \mathcal{S}\mathcal{U}\mathbb{S} \Leftrightarrow (Sa) \text{ for } \mathbb{S}.$$

Dually, one obtains $A \in \mathcal{U}\mathcal{S}\mathbb{R} \Leftrightarrow A$ satisfies (4.1), hence $\mathcal{U}\mathcal{S}\mathbb{R} = \mathbb{R}$ holds if $\mathbb{R}$ satisfies (Ra) and (Rb), and

(4.6) $$\mathbb{R} \subseteq \mathcal{U}\mathcal{S}\mathbb{R} \Leftrightarrow (Rb) \text{ for } \mathbb{R}, \quad \mathbb{R} \supseteq \mathcal{U}\mathcal{S}\mathbb{R} \Leftrightarrow (Ra) \text{ for } \mathbb{R}. \ \Diamond$$

Lemma 4.5. *Assume only* (Sb) *for a subclass* $\mathbb{S}$ *of* $\mathfrak{H}$. *Then* $\mathbb{R} = \mathcal{U}\mathbb{S}$ *satisfies* (Ra) *and* (Rb). *Moreover,* $\mathbb{R}' \subseteq \mathbb{R} = \mathcal{U}\mathbb{S}$ *holds for every subclass* $\mathbb{R}' \subseteq \mathfrak{H}$ *which satisfies* (Ra), (Rb) *and* $\mathbb{R}' \cap \mathbb{S} = \mathfrak{T}$.

Proof. The transitivity of $A \twoheadrightarrow B$ yields $(R\eta)$ for $\mathbb{R} = \mathcal{U}\mathbb{S}$, in fact, without any assumption on $\mathbb{S}$. Now, assume (Sb) for $\mathbb{S}$. Then $\mathbb{S} \subseteq \mathcal{S}\mathcal{U}\mathbb{S}$ holds by (4.5). The latter implies $\mathcal{U}\mathbb{S} \supseteq \mathcal{U}\mathcal{S}\mathcal{U}\mathbb{S}$ and thus (Ra) for $\mathbb{R} = \mathcal{U}\mathbb{S}$ by (4.6). Hence (Ra) and (Rb) hold for $\mathbb{R} = \mathcal{U}\mathbb{S}$ by Lemma 4.2 b). For the last assertion we need only $(R\eta)$ for $\mathbb{R}'$ to show $\mathbb{R}' \cap \mathbb{S} \subseteq \mathfrak{T} \Rightarrow \mathbb{R}' \subseteq \mathcal{U}\mathbb{S}$. Otherwise there would be some $A \in \mathbb{R}'$ satisfying $A \notin \mathcal{U}\mathbb{S}$, and so $A \twoheadrightarrow B$ for some $B \in \mathbb{S}$. Now, $B \in \mathbb{R}'$ by $(R\eta)$ for $\mathbb{R}'$ and $B \in \mathbb{S}$ contradict $\mathbb{R}' \cap \mathbb{S} \subseteq \mathfrak{T}$. $\Diamond$

Remark 4.6. In the following we always assume tacitly that every subclass of an n-universal class $\mathfrak{H}$ denoted by $\mathbb{R}$ or by $\mathbb{S}$ is abstract (that is, closed under isomorphisms) and contains all trivial objects of $\mathfrak{H}$. In most cases this assumption does not require anything since e.g. (Ra) and (Rb) for $\mathbb{R}$ or (Sa) and (Sb) for $\mathbb{S}$ imply these properties, but we want to avoid any ambiguity on this matter.

5. Radical classes in the new meaning

Definition 5.1. A subclass $\mathbb{R}$ of an n-universal class $\mathfrak{H} \subseteq \mathfrak{S}^* \cup \mathfrak{G}$ is called a *radical class of $\mathfrak{H}$ in the new meaning*, briefly an *n-radical class of $\mathfrak{H}$*, if $\mathbb{R}$ satisfies (Ra) and (Rb) and the following axiom

(Rc) For all $(A,+,\cdot) \in \mathfrak{H}$ if $(A,\cdot) \in \mathbb{R}$ then $(A,+,\cdot) \in \mathbb{R}$.

Remark 5.2. Note that (Rc) demands nothing if one deals only with idempotent semifields in an n-universal class $\mathfrak{H}$ which contains no groups, and also if one deals merely with groups.

Conditions (Ra), (Rb) and (Rc), together with a further one, have already been used in [12] to define a subclass $\mathbb{R}$ of a universal class $\mathfrak{H} \subseteq \subseteq \mathfrak{S}^* \cup \mathfrak{G}$ as a radical class of $\mathfrak{H}$ (cf. Def. 9.1 and Remark 9.2).

Lemma 5.3. *Let $\mathbb{R}$ be an n-universal class of $\mathfrak{H}$ and $A \in \mathfrak{H}$. Then $K \in \mathfrak{K}(A) \cap \mathbb{R}$ and $L \in \mathfrak{K}(A)$ imply $(KL)/L \in \mathbb{R}$.*

Proof. By Th. 2.10 b), $(KL)/L \cong K/(K \cap L)$ holds always with respect to the multiplication, and also with respect to the addition if both sides are semifields. Hence, if both KL and K are either groups or semifields, $(KL)/L \in \mathbb{R}$ follows from $K/(K \cap L) \in \mathbb{R}$, where the latter holds since $\mathbb{R}$ satisfies $(R\eta)$ by Lemma 4.2 a). Otherwise, we have $K \in \mathfrak{G}$ and $KL \in \in \mathfrak{S}^*$ (where L may be a group or a semifield). However, $((KL)/L,\cdot)$ is contained in $\mathbb{R}$, since its isomorphic image $(K/(K \cap L),\cdot)$ is in $\mathbb{R}$ as stated above. Applying (Rc) to $((KL)/L,+,\cdot)$, we obtain $(KL)/L \in \mathbb{R}$ also in this case. $\Diamond$

Lemma 5.4. *Let $\mathbb{R}$ be an n-radical class of $\mathfrak{H}$ and $U \in \mathfrak{H}$. If $U \notin \mathbb{R}$, then there is a kernel $L \neq U$ of U which satisfies $|\mathfrak{K}(U/L) \cap \mathbb{R}| = 1$ and $K \subseteq L$ for all $K \in \mathfrak{K}(U) \cap \mathbb{R}$.*

Proof. If $U \notin \mathbb{R}$, then (Ra) implies the existence of some B satisfying $U \twoheadrightarrow B$ and $|\mathfrak{K}(B) \cap \mathbb{R}| = 1$. Hence $B \cong U/L$ yields $L \neq U$ and $|\mathfrak{K}(U/L) \cap \mathbb{R}| = 1$ for the corresponding kernel $L \in \mathfrak{K}(U)$. Now we assume $K \in \mathfrak{K}(U) \cap \mathbb{R}$. Then $(KL)/L \in \mathbb{R}$ follows from Lemma 5.3, and $KL \in \mathfrak{K}(U)$ implies by Remark 2.11 that $(KL)/L \in \mathfrak{K}(U/L) \cap \mathbb{R} \subseteq \mathfrak{T}$.

This shows $|(KL)/L| = 1$ and thus $K \subseteq L$. $\Diamond$

Proposition 5.5. *Each n-radical class $\mathbb{R}$ of $\mathfrak{H}$ is closed under extensions in $\mathfrak{H}$, which means for all $A, K \in \mathfrak{H}$,*

(Re) $K \in \mathfrak{K}(A) \cap \mathbb{R}$ *and* $A/K \in \mathbb{R}$ *imply* $A \in \mathbb{R}$.

Proof. We show (Re) by way of contradiction, and assume that $A \notin \mathbb{R}$. Then, applying Lemma 5.4 to $A = U$, there would be a kernel $L \neq A$ of A satisfying $|\mathfrak{K}(A/L) \cap \mathbb{R}| = 1$ and $K \subseteq L$ for all $K \in \mathfrak{K}(A) \cap \mathbb{R}$. So we have $A/L \cong (A/K)/(L/K)$, regardless whether A is a group or a semifield (cf. Th. 2.12). Hence $A/K \in \mathbb{R}$ and $(R\eta)$ for $\mathbb{R}$ imply $A/L \in \mathbb{R}$. Now $|\mathfrak{K}(A/L) \cap \mathbb{R}| = 1$ yields $L = A$, contradicting $L \neq A$. $\Diamond$

Lemma 5.6. *Let $\mathbb{R}$ be an n-radical class of $\mathfrak{H}$ and $A \in \mathfrak{H}$. Then $K, L \in \mathfrak{K}(A) \cap \mathbb{R}$ implies $KL \in \mathbb{R}$.*

Proof. From $(KL)/L \in \mathbb{R}$ by Lemma 5.3 and $L \in \mathbb{R}$ we get $KL \in \mathbb{R}$ in view of Prop. 5.5. $\Diamond$

Theorem 5.7. *Let $\mathbb{R}$ be an n-radical class of $\mathfrak{H}$. Then*

$(R\varrho)$ $\bigcup\{K \in \mathfrak{K}(A) \mid K \in \mathbb{R}\} = U \in \mathfrak{K}(A) \cap \mathbb{R}$ *holds for all* $A \in \mathfrak{H}$.

This property implies $\bigvee\{K \in \mathfrak{K}(A) \mid K \in \mathbb{R}\} = S \in \mathbb{R}$ for the supremum in the lattice $(\mathfrak{K}(A), \subseteq)$ and $U = S$. Moreover, $(R\varrho)$ is equivalent to the statement that each $A \in \mathfrak{H}$ has a unique largest $\mathbb{R}$-kernel U which contains any $\mathbb{R}$-kernel of A, called the radical of A determined by $\mathbb{R}$. We write $U = \varrho_{\mathbb{R}} A$ for this $\mathbb{R}$-radical of A and refer to Section 6 for the operator $\varrho_{\mathbb{R}}$ used in this notation.

Proof. At first we show that $(U, \cdot)$ is a normal subgroup of $(A, \cdot)$, and assume that $k, \ell \in U$. Then $k \in K$ and $\ell \in L$ hold for some kernels $K, L \in \mathfrak{K}(A) \cap \mathbb{R}$, and we get $KL \in \mathfrak{K}(A) \cap \mathbb{R}$ by Lemma 5.6. This shows that $k \cdot \ell \in KL \subseteq U$. Hence $(U, \cdot)$ is a subsemigroup of $(A, \cdot)$, which clearly satisfies $k^{-1} \in U$ and $a^{-1}ka \in U$ for each $k \in U$ and all $a \in A$. So we have $U \in \mathfrak{K}(A)$ if $A \in \mathfrak{G}$. To show $U \in \mathfrak{K}(A)$ for $A \in \mathfrak{S}^*$ we assume that $s + t = e$ for $s, t \in A$ and $k \in U$, that is, $k \in K$ for some $K \in \mathfrak{K}(A) \cap \mathbb{R}$. This yields $s + tk \in K$ by Th. 2.2, and thus $s + tk \in U$ for all $k \in U$, which in turn implies $U \in \mathfrak{K}(A)$ by the same theorem.

To prove $U \in \mathbb{R}$, note that each $K \in \mathfrak{K}(A) \cap \mathbb{R}$ satisfies $K \subseteq U \in \in \mathfrak{K}(A)$, which yields $K \subseteq \mathfrak{K}(U)$, and thus $K \in \mathfrak{K}(U) \cap \mathbb{R}$. Now, by way of contradiction, assume that $U \notin \mathbb{R}$ for some $U \in \mathfrak{H}$. Then by Lemma 5.4, there would be a kernel $L \neq U$ of U which contains all $K \in \mathfrak{K}(U) \cap \mathbb{R}$, that is, all $K \in \mathfrak{K}(A) \cap \mathbb{R}$ as just stated. This yields the contradiction $U = \cup\{K \in \mathfrak{K}(A) \mid K \in \mathbb{R}\} \subseteq L \subset U$, and completes the

proof of $(R\varrho)$. The other statements of Th. 5.7 are now obvious. $\Diamond$

Proposition 5.8. *Let $\mathbb{R}$ be a subclass of $\mathfrak{H}$ which satisfies $(R\varrho)$. Then, clearly for each $A \in \mathfrak{H}$,*

(5.1) $\bigvee_{i\in I} K_i \in \mathbb{R}$ *holds for any set of kernels* $K_i \in \mathfrak{K}(A) \cap \mathbb{R}$.

This yields the inductive property for $\mathbb{R}$, which means that $(Ri) K_1 \subseteq \cdots \subseteq K_i \subseteq \ldots$ *for* $K_i \in \mathfrak{K}(A) \cap \mathbb{R}$, $i \in I$, *implies* $\bigcup_{i\in I} K_i \in \mathbb{R}$.

Proof. For each set of kernels $K_i \in \mathfrak{K}(A) \cap \mathbb{R}$, the supremum $B = \bigvee_{i\in I} K_i$ is a kernel of A. This yields $K_i \in \mathfrak{K}(B) \cap \mathbb{R}$ for each $i \in I$, and we obtain

$$B = \bigvee_{i\in I} K_i \subseteq \bigvee \{K \in \mathfrak{K}(B) \cap \mathbb{R}\} \subseteq B.$$

Applying $(R\varrho)$ to $B = \bigvee \{K \in \mathfrak{K}(B) \cap \mathbb{R}\}$, we get $B \in \mathbb{R}$. $\Diamond$

Theorem 5.9. *For any subclass $\mathbb{R}$ of $\mathfrak{H}$, each of the following sets of properties characterizes $\mathbb{R}$ as an n-radical class of $\mathfrak{H}$:*

i) *(Ra), (Rb) and (Rc),*
ii) *(Ra), $(R\eta)$ and (Rc),*
iii) *$(R\varrho)$, (Re), $(R\eta)$ and (Rc),*
iv) *(Ri), (Re), $(R\eta)$ and (Rc).*

Proof. We know i)$\Leftrightarrow$ii) by Lemma 4.2, further i)$\Rightarrow$iii) by Prop. 5.5 and Th. 5.7, and iii)$\Rightarrow$iv) by Prop. 5.8. We complete the proof by iv)$\Rightarrow$ii), and show that (Ri) and (Re) imply (Ra). So we assume $\forall B(A \twoheadrightarrow B)\exists C(C \lhd B \wedge B \in \mathbb{R})$ and, by way of contradiction, $A \notin \mathbb{R}$ for some $A \in \mathfrak{H}$. By (Ri) for $\mathbb{R}$ we can apply Zorn's Lemma and obtain the existence of a maximal kernel M in $\mathfrak{K}(A) \cap \mathbb{R}$. Then $A \notin \mathbb{R}$ implies $M \neq A$, and thus $\varphi : A \twoheadrightarrow B = A/M$. By the other assumption on A, there is some $C \lhd B$ such that $C \in \mathbb{R}$, and $\varphi^{-1}(C) = D$ is a kernel of A by Rem. 2.11 which satisfies $M \subset D$. Now $D/M = C \in \mathbb{R}$ and $M \in \mathbb{R}$ imply $D \in \mathbb{R}$ by (Re), contradicting the maximality of M. $\Diamond$

Remark 5.10. According to this proof, the implications i)$\Leftrightarrow$ii) and iii)$\Rightarrow$iv)$\Rightarrow$ii) remain true if one considers the sets i)–iv) in Th. 5.9 without property (Rc). Concerning the implication i)$\Rightarrow$iii) we note that (Re) as well as $(R\varrho)$ need not be true for a subclass $\mathbb{R}$ of $\mathfrak{H}$ which satisfies only (Ra) and (Rb) (cf. Th. 8.1).

6. Radical operators in the new meaning

In Th. 5.7 we have already introduced the $\mathbb{R}$-radical $\varrho_{\mathbb{R}}A$ of $A \in \mathfrak{H}$ by an operator $\varrho_{\mathbb{R}}$. We now characterize such an operator ϱ by properties not depending on an n-radical class $\mathbb{R}$ of $\mathfrak{H}$, and deal with the bijective correspondence between these concepts.

Definition 6.1. An operator ϱ in an n-universal class $\mathfrak{H} \subseteq \mathfrak{S}^* \cup \mathfrak{S}$ which assigns to each $A \in \mathfrak{H}$ a kernel $\varrho A \in \mathfrak{K}(A)$ is called a *radical operator in the new meaning*, briefly an *n-radical operator*, if it satisfies the following axioms for all $A \in \mathfrak{H}$:

- $(\varrho\alpha)$ $\varphi(\varrho A) \subseteq \varrho B$ holds for all surjective morphisms $\varphi : A \twoheadrightarrow B$ of type 1) and 2).
- (ϱb) $|\varrho(A/\varrho A)| = 1$, or equivalently, $\varrho(A/\varrho A) \in \mathfrak{T}$.
- (ϱc) $\forall B(\varrho B = B \lhd A \Rightarrow B \subseteq \varrho A)$.
- (ϱd) $\varrho\varrho A = \varrho A$.
- $(\varrho\omega)$ For all $(A, +, \cdot) \in \mathfrak{H}$, if $(A, \cdot) \in \mathfrak{H}$ and $\varrho(A, \cdot) = (A, \cdot)$ then $\varrho(A+, \cdot) = (A, +, \cdot)$.

We emphasize that Remark 5.2 applies likewise to $(\varrho\omega)$. Moreover, in [12] a radical operator ϱ in a universal class $\mathfrak{H} \subseteq \mathfrak{S}^* \cup \mathfrak{S}$ was defined by (ϱb), (ϱc), (ϱd) and a further axiom (ϱa). The latter reads as $(\varrho\alpha)$, but it includes also morphisms φ of type 3) (cf. Section 9).

Theorem 6.2. *Let $\mathbb{R}$ be an n-radical class of $\mathfrak{H}$. Then the operator $\varrho_{\mathbb{R}}$ in $\mathfrak{H}$ defined in Th. 5.7 for all $A \in \mathfrak{H}$ by*

$$\varrho_{\mathbb{R}}A = \bigcup\{K \in \mathfrak{K}(A) \mid K \in \mathbb{R}\} = \bigvee\{K \in \mathfrak{K}(A) \mid K \in \mathbb{R}\} \tag{6.1}$$

is an n-radical operator in $\mathfrak{H}$. Conversely, let ϱ be an n-radical operator in $\mathfrak{H}$. Then ϱ defines an n-radical class $\mathbb{R}_\varrho$ of $\mathfrak{H}$ by

$$\mathbb{R}_\varrho = \{A \in \mathfrak{H} \mid \varrho A = A\}. \tag{6.2}$$

Moreover, the correspondence between these concepts given by (6.1) *and* (6.2) *is bijective, that is, $\mathbb{R}_{\varrho_{\mathbb{R}}} = \mathbb{R}$ holds for each $\mathbb{R}$ and $\varrho_{\mathbb{R}_\varrho} = \varrho$ for each ϱ.*

Proof. Let $\mathbb{R}$ be an n-radical class of $\mathfrak{H}$. Then $\varrho_{\mathbb{R}}$ assigns by Th. 5.7 to each $A \in \mathfrak{H}$ a kernel $\varrho_{\mathbb{R}}A \in \mathfrak{K}(A) \cap \mathbb{R}$ which contains all $K \in \mathfrak{K}(A) \cap \cap \mathbb{R}$. This yields immediately that $\varrho_{\mathbb{R}}$ satisfies (ϱc) and (ϱd). Further, $\varphi(\varrho_{\mathbb{R}}A) \in \mathfrak{K}(B) \cap \mathbb{R}$ follows from Remark 2.11 and $(R\eta)$ for $\mathbb{R}$, which yields $\varphi(\varrho_{\mathbb{R}}A) \subseteq \varrho_{\mathbb{R}}B$ and thus $(\varrho\alpha)$.

Next we show that $(R\varrho)$ and (Re) for any subclass $\mathbb{R}$ of $\mathfrak{H}$ imply (ϱb) for $\varrho_{\mathbb{R}}$, and assume by way of contradiction that $\varrho_{\mathbb{R}}(A/\varrho_{\mathbb{R}}A) \notin \mathfrak{T}$ for

some $A \in \mathfrak{H}$. The latter yields the existence of a kernel $B \lhd A/\varrho_{\mathbb{R}}A$ with $B \in \mathbb{R}$ and thus $B \cong C/\varrho_{\mathbb{R}}A$ for a kernel $C \in \mathfrak{K}(A)$ satisfying $\varrho_{\mathbb{R}}A \subset C$ (cf. Remark 2.11). But $C/\varrho_{\mathbb{R}}A \in \mathbb{R}$ and $\varrho_{\mathbb{R}}A \in \mathbb{R}$ imply $C \in \mathbb{R}$ by (Re) for $\mathbb{R}$, and thus $C \subseteq \varrho_{\mathbb{R}}A$ follows, contradicting $\varrho_{\mathbb{R}}A \subset C$.

Finally, $(R\varrho)$ yields $\varrho_{\mathbb{R}}A = A \Leftrightarrow A \in \mathbb{R}$ for each $A \in \mathfrak{H}$. The latter shows that $(\varrho\omega)$ is just a reformulation of (Rc) for $\mathbb{R}$, and also that $\mathbb{R}_{\varrho_{\mathbb{R}}} = \{A \in \mathfrak{H} \mid \varrho_{\mathbb{R}}A = A\}$ according to (6.2) coincides with $\mathbb{R}$.

Now let ϱ be any n-radical operator in $\mathfrak{H}$ and define a subclass $\mathbb{R}_\varrho$ of $\mathfrak{H}$ by (6.2). To show that $\mathbb{R}_\varrho$ is an n-radical class of $\mathfrak{H}$, we use Th. 5.9 and check (Ra), $(R\eta)$ and (Rc). For (Ra), we go by way of contradiction, and assume $\forall B(A \twoheadrightarrow B)\exists C(C \lhd B \wedge C \in \mathbb{R}_\varrho)$ for some $A \in \mathfrak{H}$, but $A \notin \mathbb{R}_\varrho$. The latter yields $\varrho A \subset A$ by (6.2) and thus $A \twoheadrightarrow B = A/\varrho A$. Applying the first assumption on A, there is a kernel $C \lhd B$ satisfying $C \in \mathbb{R}_\varrho$, that is, $\varrho C = C$. Hence (ϱc) for ϱ implies $C \subseteq \varrho B$, and (ϱb) for ϱ yields $\varrho B = \varrho(A/\varrho A) \in \mathfrak{T}$ and thus $C \in \mathfrak{T}$, contradicting $C \lhd B$.

Turning to $(R\eta)$, let $\varphi : A \twoheadrightarrow B$ be a surjective morphism of type 1) or 2), and assume $A \in \mathbb{R}_\varrho$, that is, $\varrho A = A$. Then $B \in \mathbb{R}_\varrho$ follows from $(\varrho\alpha)$ for ϱ by $B = \varphi(A) = \varphi(\varrho A) \subseteq \varrho B$. Finally, again by $A \in \in \mathbb{R}_\varrho \Leftrightarrow \varrho A = A$, one obtains (Rc) for $\mathbb{R}_\varrho$ from $(\varrho\omega)$ for ϱ.

It remains to show that $\varrho_{\mathbb{R}_\varrho}A = \cup\{K \in \mathfrak{K}(A) \mid K \in \mathbb{R}_\varrho\}$ and ϱA coincide for each $A \in \mathfrak{H}$. Since each non-trivial kernel $K \in \mathfrak{K}(A) \cap \mathbb{R}_\varrho$ satisfies $\varrho K = K \lhd A$, we obtain $K \subseteq \varrho A$ by (ϱc) for ϱ and thus $\varrho_{\mathbb{R}_\varrho}A \subseteq$ $\subseteq \varrho A$. For the other inclusion we use (ϱd) for ϱ, by the way, for the first time in this proof: $\varrho(\varrho A) = \varrho A$ implies $\varrho A \in \mathbb{R}_\varrho$, and $\varrho A \in \mathfrak{K}(A) \cap \mathbb{R}_\varrho$ shows $\varrho A \subseteq \varrho_{\mathbb{R}_\varrho}A$. $\Diamond$

We close this section with proving a further equivalence, which yields another characterization for n-radical classes in addition to those in Th. 5.9.

Proposition 6.3. a) *Assume* $(R\varrho)$ *and* $(R\eta)$ *for a subclass* $\mathbb{R}$ *of* $\mathfrak{H}$. *The former yields that each* $A \in \mathfrak{H}$ *has a greatest* $\mathbb{R}$*-kernel, which we denote by* $\varrho_{\mathbb{R}}A$. *Then* (Re) *holds for* $\mathbb{R}$ *if and only if this operator* $\varrho_{\mathbb{R}}$ *satisfies* (ϱb), *that is,* $\varrho_{\mathbb{R}}(A/\varrho_{\mathbb{R}}A) \in \mathfrak{T}$ *for all* $A \in \mathfrak{H}$.

b) *A subclass* $\mathbb{R}$ *of* $\mathfrak{H}$ *is an* n*-radical class if and only if* $\mathbb{R}$ *satisfies* $(R\varrho)$, (ϱb), $(R\eta)$ *and* (Rc).

Proof. a) In the second step of the proof of Th. 6.2 we have shown that $(R\varrho)$ and (Re) for $\mathbb{R}$ imply (ϱb) for $\varrho_{\mathbb{R}}$. To show that $(R\varrho)$, $(R\eta)$ and (ϱb) imply (Re), we assume that $K \in \mathfrak{K}(A) \cap \mathbb{R}$ and $A/K \in \mathbb{R}$ for

some $K, A \in \mathfrak{H}$. Then $K \subseteq \varrho_{\mathbb{R}} A$ holds by $(R\varrho)$, which yields $A/\varrho_{\mathbb{R}} A \cong \cong (A/K)/(\varrho_{\mathbb{R}} A/K)$ by Th. 2.12. Now $A/\varrho_{\mathbb{R}} A \in \mathbb{R}$ follows from $(R\eta)$, and so $A/\varrho_{\mathbb{R}} A \subseteq \varrho_{\mathbb{R}}(A/\varrho_{\mathbb{R}} A) \in \mathfrak{T}$ by $(R\varrho)$ and (ϱb). This shows $A = = \varrho_{\mathbb{R}} A \in \mathbb{R}$ what we were to prove.

b) Since $(R\varrho)$, (ϱb) and $(R\eta) \Leftrightarrow (R\varrho)$, (Re) and $(R\eta)$ holds for each $\mathbb{R} \subseteq \mathfrak{H}$ as just proved, we obtain b) from iii) in Th. 5.9. $\Diamond$

7. Semisimple classes in the new meaning

Now we include semisimple classes into our considerations, the third basic concept for each radical theory.

Definition 7.1. A subclass $\mathbb{S}$ of an n-universal class $\mathfrak{H} \subseteq \mathfrak{S}^* \cup \mathfrak{G}$ is called a *semisimple class of $\mathfrak{H}$ in the new meaning*, briefly an *n-semisimple class of $\mathfrak{H}$*, if $\mathbb{S}$ satisfies (Sa) and (Sb) and the following axiom:

$(S\gamma)$ For all $(A,+,\cdot) \in \mathfrak{H}$, if $(A,+,\cdot) \in \mathbb{S}$ and $|(A,+,\cdot)| \neq 1$, then $(A,\cdot) \notin \mathcal{U}\mathbb{S}$,

that is, $(A,+,\cdot) \in \mathbb{S} \setminus \mathfrak{T}$ implies either $(A,\cdot) \notin \mathfrak{H}$ or the existence of a group B such that $(A,\cdot) \twoheadrightarrow (B,\cdot)$ and $(B,\cdot) \in \mathbb{S}$ hold.

Again we emphasize that Remark 5.2 applies also to $(S\gamma)$, and we refer to Section 9 for the interrelation between $(S\gamma)$ and the axiom (Sc) considered in [12].

Theorem 7.2. a) *Each n-radical class $\mathbb{R}$ of $\mathfrak{H}$ determines an n-semisimple class $\mathbb{S} = \mathcal{S}\mathbb{R}$ of $\mathfrak{H}$.*

b) *Each n-semisimple class $\mathbb{S}$ of $\mathfrak{H}$ determines an n-radical class $\mathbb{R} = \mathcal{U}\mathbb{S}$ of $\mathfrak{H}$.*

c) *For all these classes one has $\mathcal{U}\mathcal{S}\mathbb{R} = \mathbb{R}$ and $\mathcal{S}\mathcal{U}\mathbb{S} = \mathbb{S}$.*

Based on Prop. 4.4, this theorem follows from

Lemma 7.3. *Already (Ra) and (Rc) for a subclass $\mathbb{R}$ of $\mathfrak{H}$ imply $(S\gamma)$ for $\mathbb{S} = \mathcal{S}\mathbb{R}$, and alone $(S\gamma)$ for a subclass $\mathbb{S}$ of $\mathfrak{H}$ yields (Rc) for $\mathbb{R} = = \mathcal{U}\mathbb{S}$.*

Proof. To show $(S\gamma)$ for $\mathbb{S} = \mathcal{S}\mathbb{R}$, we go by way of contradiction, and suppose that $(A,+,\cdot) \in \mathbb{S} \setminus \mathfrak{T}$ but $(A,\cdot) \in \mathcal{U}\mathbb{S}$ for some $(A,+,\cdot) \in \mathfrak{H}$. Since (Ra) for $\mathbb{R}$ implies $\mathcal{U}\mathbb{S} = \mathcal{U}\mathcal{S}\mathbb{R} \subseteq \mathbb{R}$ by (4.6), we obtain $(A,\cdot) \in \mathbb{R}$, and thus $(A,+,\cdot) \in \mathbb{R}$ by (Rc), which contradicts $(A,+,\cdot) \in \mathbb{S}$.

To show (Rc) for $\mathbb{R} = \mathcal{U}\mathbb{S}$, again by way of contradiction, we assume $(A,\cdot) \in \mathbb{R}$ but $(A,+,\cdot) \notin \mathbb{R}$ for a semifield $(A,+,\cdot) \in \mathfrak{H}$. Then $(A,+,\cdot) \notin \mathbb{R} = \mathcal{U}\mathbb{S}$ implies $(A,+,\cdot) \twoheadrightarrow (B,+,\cdot)$ for some $(B,+,\cdot) \in \mathbb{S}$,

which yields $(A,\cdot)\to\triangleright(B,\cdot)$. So we can apply $(S\gamma)$ to $(B,+,\cdot)$ and obtain $(B,\cdot)\notin\mathcal{US}$. This contradicts $(A,\cdot)\in\mathbb{R}=\mathcal{US}$ and $(A,\cdot)\to\triangleright(B,\cdot)$, since $\mathcal{US}$ satisfies $(R\eta)$ for each subclass $\mathbb{S}$ of $\mathfrak{H}$. $\Diamond$

Remark 7.4. Combining the second part of this lemma and Lemma 4.5, we obtain: *if a subclass* $\mathbb{M}\subseteq\mathfrak{H}$ *satisfies* (Sb) *and* $(S\gamma)$, *then* $\mathbb{R}=\mathcal{UM}$ *is an n-radical class of* $\mathfrak{H}$, *the greatest one which satisfies* $\mathbb{R}\cap\mathbb{M}\in\mathfrak{T}$.

As a consequence of Ths. 7.2 and 6.2, each n-semisimple class $\mathbb{S}$ corresponds via $\mathbb{R}=\mathcal{US}$ bijectively to an n-radical operator $\varrho=\varrho_{\mathbb{R}}$. In the following we obtain a description of this operator directly in terms of $\mathbb{S}$ together with an important property of n-semisimple classes. For this end we state that for a subclass $\mathbb{S}$ of $\mathfrak{H}$ and each $A\in\mathfrak{H}$ the intersection $\bigcap\{K\in\mathfrak{K}(A)\mid A/K\in\mathbb{S}\}$ is meaningful and, clearly, a kernel of A. (Note that at least $A/A\in\mathbb{S}$ holds by our convention in Remark 4.6.)

Definition 7.5. A subclass $\mathbb{S}$ of $\mathfrak{H}$ may satisfy the following property

$(S\varrho)$ $\eta_{\mathbb{S}}A=\bigcap\{K\in\mathfrak{K}(A)\mid A/K\in\mathbb{S}\}$ *implies* $A/\eta_{\mathbb{S}}A\in\mathbb{S}$ *for all* $A\in\mathfrak{H}$.

This property states that $\eta_{\mathbb{S}}A$ *is the smallest kernel* K *of* A *which satisfies* $A/K\in\mathbb{S}$, and it yields, in particular,

$$A\in\mathbb{S}\Leftrightarrow\eta_{\mathbb{S}}A\in\mathfrak{T}\text{ for all }A\in\mathfrak{H}. \tag{7.1}$$

Theorem 7.6. *Let* $\mathbb{S}$ *be an n-semisimple class of* $\mathfrak{H}$ *and* $\mathbb{R}=\mathcal{US}$ *the corresponding n-radical class. Then* $\mathbb{S}$ *satisfies* $(S\varrho)$ *and* $\eta_{\mathbb{S}}$ *coincides with the n-radical operator* $\varrho_{\mathbb{R}}$. *Moreover,* $\varrho_{\mathbb{R}}$ *characterizes the n-semisimple class* $\mathbb{S}=\mathcal{SR}$ *by*

$$A\in\mathbb{S}\Leftrightarrow\varrho_{\mathbb{R}}A\in\mathfrak{T}\quad\text{for all }A\in\mathfrak{H}. \tag{7.2}$$

Proof. For $\mathbb{S}$ and $\mathbb{R}=\mathcal{US}$ as assumed, $A\in\mathbb{S}=\mathcal{SR}$ holds if and only if A satisfies $\forall B(B\lhd A\Rightarrow B\notin\mathbb{R})$, that is, if and only if the greatest $\mathbb{R}$-kernel of A is trivial. This shows (7.2) and yields $A/K\in\mathbb{S}\Leftrightarrow$ $\Leftrightarrow\varrho_{\mathbb{R}}(A/K)\in\mathfrak{T}$. Using the latter, we prove that

$$\varrho_{\mathbb{R}}A=\bigcup\{K\in\mathfrak{K}(A)\mid K\in\mathbb{R}\}$$

and

$$\eta_{\mathbb{S}}A=\bigcap\{K\in\mathfrak{K}(A)\mid\varrho_{\mathbb{R}}(A/K)\in\mathfrak{T}\}$$

are equal for each $A\in\mathfrak{H}$. Since $\varrho_{\mathbb{R}}A$ is a kernel of A which satisfies $\varrho_{\mathbb{R}}(A/\varrho_{\mathbb{R}}A)\in\mathfrak{T}$ by (ϱb), $K=\varrho_{\mathbb{R}}A$ ocurs on the right-hand side such that $\eta_{\mathbb{S}}A\subseteq\varrho_{\mathbb{R}}A$ holds. For the converse inclusion we show $\varrho_{\mathbb{R}}A\subseteq K$ for each $K\in\mathfrak{K}(A)$ satisfying $\varrho_{\mathbb{R}}(A/K)\in\mathfrak{T}$.

Indeed, applying $(\varrho\alpha)$ to the natural homomorphism $\varphi : A \to A/K$, we get $\varphi(\varrho_{\mathbb{R}}A) \subseteq \varrho_{\mathbb{R}}(A/K) \in \mathfrak{T}$, and so $\varrho_{\mathbb{R}}A \subseteq K$. This shows $\eta_{\mathbb{S}} = \varrho_{\mathbb{R}}$, which implies $(S\varrho)$ for $\mathbb{S}$ since $A/\varrho_{\mathbb{R}}A \in \mathbb{S}$ for each $A \in \mathfrak{H}$ follows from $\varrho_{\mathbb{R}}(A/\varrho_{\mathbb{R}}A) \in \mathfrak{T}$ and (7.2). $\Diamond$

We note without proof that one needs only (Sa) and (Sb) to show $(S\varrho)$ for a subclass $\mathbb{S}$ of $\mathfrak{H}$, but more assumptions on $\mathbb{S}$ to obtain $\eta_{\mathbb{S}} = \varrho_{\mathcal{U}\mathbb{S}}$ where the latter is, in fact, the main statement of Th. 7.6.

Proposition 7.7. *For any subclass $\mathbb{S}$ of $\mathfrak{H}$, each of the following properties is equivalent to $(S\varrho)$:*

(7.3) *For all $A \in \mathfrak{H}$ and any subset $\{K_i \mid i \in I\}$ of $\{K \in \mathfrak{K}(A) \mid A/K \in \mathbb{S}\}$, $\bigcap_{i\in I} K_i = B$ implies $A/B \in \mathbb{S}$.*

(7.4) *$\bigcap_{i\in I} K_i \in \mathfrak{T}$ implies $A \in \mathbb{S}$ for all $A \in \mathfrak{H}$ and any subset $\{K_i \mid i \in I\}$ as above.*

(7.5) *$\mathbb{S}$ is subdirectly closed in $\mathfrak{H}$, that is, each subdirect product $A \in \mathfrak{H}$ of groups $A_i \in \mathfrak{H} \cap \mathfrak{G}$ or of semifields $A_i \in \mathfrak{H} \cap \mathfrak{S}^*$ satisfies $A \in \mathbb{S}$.*

Proof. The implications $(S\varrho) \Rightarrow$ (7.4) and (7.3) $\Rightarrow (S\varrho)$ are clear. For (7.4) $\Rightarrow$ (7.3) we use $(A/B)/(K_i/B) \cong A/K_i \in \mathbb{S}$ for all $i \in I$ (cf. Th. 2.12), and apply (7.4) to A/B. Then $\bigcap_{i\in I}(K_i/B) = B/B \in \mathfrak{T}$ implies $A/B \in \mathbb{S}$. Finally, (7.4) $\Leftrightarrow$ (7.5) is well known for groups and can be obtained in the same way for subdirect products of proper semifields. $\Diamond$

Note that (7.3) implies that $\mathbb{S}$ *has the coinductive property*, that is,

$$K_1 \supseteq \cdots \supseteq K_i \supseteq \cdots \text{ for } K_i \text{ as in (7.3) implies } A/\bigcap_{i\in I} K_i \in \mathbb{S}.$$

Proposition 7.8. *Each n-semisimple class $\mathbb{S}$ of $\mathfrak{H}$ is closed under extensions, which means for all $A, K \in \mathfrak{H}$:*

(Se) $K \in \mathfrak{K}(A) \cap \mathbb{S}$ and $A/K \in \mathbb{S}$ imply $A \in \mathbb{S}$.

Proof. We show that, for each $B \lhd A$, the assumption of (Se) implies $B \overset{\triangleright}{\to} C$ for some $C \in \mathbb{S}$, which yields $A \in \mathbb{S}$ by (Sa) for $\mathbb{S}$. If $B \subseteq K$ holds, $B \lhd K \in \mathbb{S}$ implies the existence of such a $C \in \mathbb{S}$ by (Sb) for $\mathbb{S}$. Otherwise, for $B \not\subseteq K$, the natural homomorphism $\varphi : A \to A/K$ yields $B \overset{\triangleright}{\to} \varphi(B) \lhd A/K$. Applying (Sb) to $A/K \in \mathbb{S}$, we get $\varphi(B) \overset{\triangleright}{\to} C$ for some $C \in \mathbb{S}$, and thus $B \overset{\triangleright}{\to} C \in \mathbb{S}$. $\Diamond$

Theorem 7.9. *Let $\mathbb{S}$ be a subclass of $\mathfrak{H}$. Then each of the following sets of properties characterizes $\mathbb{S}$ as an n-semisimple class of $\mathfrak{H}$:*

i) (Sa), (Sb) *and* $(S\gamma)$;

ii) $(S\varrho)$, (Sb), $(S\gamma)$ *and* $\eta_{\mathbb{S}}\eta_{\mathbb{S}}A = \eta_{\mathbb{S}}A$ *for all* $A \in \mathfrak{H}$;

iii) $(S\varrho)$, (Sb), $(S\gamma)$ *and* $\varrho_{\mathcal{U}\mathbb{S}}A \in \mathfrak{T} \Rightarrow \eta_{\mathbb{S}}A \in \mathfrak{T}$ *for all* $A \in \mathfrak{H}$;

iv) $(S\varrho)$, (Sb), $(S\gamma)$, (Se) *and* $\eta_{\mathbb{S}}\eta_{\mathbb{S}}A \in \mathfrak{K}(A)$ *for all* $A \in \mathfrak{H}$;

v) $(S\varrho)$, $\mathcal{U}\mathbb{S}$ *is an* n*-radical class of* $\mathfrak{H}$ *and* $\varrho_{\mathcal{U}\mathbb{S}}A \in \mathfrak{T} \Leftrightarrow \eta_{\mathbb{S}}A \in \in \mathfrak{T}$ *for all* $A \in \mathfrak{H}$.

From these characterizations for an n*-semisimple class* $\mathbb{S}$ *of* $\mathfrak{H}$ *several other ones can be obtained as follows:*

a) *Replace* $(S\varrho)$ *by one of the equivalent properties given in* Prop. 7.7, *in particular, by* $\mathbb{S}$ *is subdirectly closed in* $\mathfrak{H}$.

b) *The last condition in* iii) *can be replaced by* $\eta_{\mathbb{S}}A \subseteq \varrho_{\mathcal{U}\mathbb{S}}A$, *and also by the weaker condition* $\eta_{\mathbb{S}}A \in \mathcal{U}\mathbb{S}$, *clearly for all* $A \in \mathfrak{H}$.

c) *The last condition in* v) *can be replaced by* $\varrho_{\mathcal{U}\mathbb{S}}A = \eta_{\mathbb{S}}A$.

d) *The implications* $\eta_{\mathbb{S}}A = \varrho_{\mathcal{U}\mathbb{S}}A \Rightarrow \eta_{\mathbb{S}}\eta_{\mathbb{S}}A = \eta_{\mathbb{S}}A \Rightarrow \eta_{\mathbb{S}}\eta_{\mathbb{S}}A \in \in \mathfrak{K}(A)$ *can be used to replace the last condition in* ii) *or in* iv).

Proof. At first, let $\mathbb{S}$ be an n-semisimple class defined by i). Then $\mathcal{U}\mathbb{S}$ is an n-radical class and $\mathbb{S}$ satisfies (Se), $(S\varrho)$ and $\eta_{\mathbb{S}} = \varrho_{\mathcal{U}\mathbb{S}}$ (cf. Th. 7.2, Prop. 7.8 and Th. 7.6), and the latter equality yields all other properties occurring in this theorem (either trivially or by $\varrho\varrho A = \varrho A \in \mathfrak{K}(A)$, cf. Def. 6.1).

For the converse considerations we note the following: each of the assumptions ii) – iv) on an arbitrary subclass $\mathbb{S}$ of $\mathfrak{H}$ implies that $\mathcal{U}\mathbb{S}$ is an n-radical class, since the latter follows from (Sb) and $(S\gamma)$ by Remark 7.4. Hence the n-radical operator $\varrho_{\mathcal{U}\mathbb{S}}$ can be used in iii) and also for other replacements formulated in b), c) and d). So it remains to show that each of ii)–v) implies i).

iii)$\Rightarrow$i) As just stated, (Sb) and $(S\varrho)$ for $\mathbb{S}$ imply that $\mathbb{R} = \mathcal{U}\mathbb{S}$ is an n-radical class of $\mathfrak{H}$, and hence $\mathcal{S}\mathcal{U}\mathbb{S}$ is an n-semisimple class. Applying Th. 7.6 to $\mathcal{S}\mathcal{U}\mathbb{S}$ and $\mathcal{U}\mathcal{S}\mathcal{U}\mathbb{S} = \mathcal{U}\mathbb{S} = \mathbb{R}$, we obtain $A \in \mathcal{S}\mathcal{U}\mathbb{S} \Leftrightarrow \varrho_{\mathcal{U}\mathbb{S}}A \in \mathfrak{T}$ for all $A \in \mathfrak{H}$ by (7.2). On the other hand, $(S\varrho)$ for $\mathbb{S}$ yields $A \in \mathbb{S} \Leftrightarrow$ $\Leftrightarrow \eta_{\mathbb{S}}A \in \mathfrak{T}$ by (7.1), again for each $A \in \mathfrak{H}$. Hence the assumption $\varrho_{\mathcal{U}\mathbb{S}}A \in \mathfrak{T} \Rightarrow \eta_{\mathbb{S}}A \in \mathfrak{T}$ implies $\mathcal{S}\mathcal{U}\mathbb{S} \subseteq \mathbb{S}$, which in turn yields (Sa) for $\mathbb{S}$ by (4.5), and thus i) for $\mathbb{S}$.

v)$\Rightarrow$i) In the above proof, (Sb) and $(S\gamma)$ have been used only to get that $\mathbb{R} = \mathcal{U}\mathbb{S}$ is an n-radical class. Hence, if $\mathbb{S}$ satisfies v), we obtain $A \in \mathcal{S}\mathcal{U}\mathbb{S} \Leftrightarrow \varrho_{\mathcal{U}\mathbb{S}}A \in \mathfrak{T}$ and $A \in \mathbb{S} \Leftrightarrow \eta_{\mathbb{S}}A \in \mathfrak{T}$ as above. This yields $\mathcal{S}\mathcal{U}\mathbb{S} = \mathbb{S}$ by the last condition of v), which shows that $\mathbb{S}$ is an n-semisimple class.

ii)⇒i) We assume ii) for $\mathbb{S}$. Then $(S\varrho)$ yields $\eta_{\mathbb{S}}A \in \mathfrak{T} \Leftrightarrow A \in \mathbb{S}$ for all $A \in \mathfrak{H}$ by (7.1). We show (Sa) for $\mathbb{S}$ by way of contradiction, and assume (4.2) for some $A \in \mathfrak{H}$ but $A \notin \mathbb{S}$ and hence $\eta_{\mathbb{S}}A \notin \mathfrak{T}$. We write η for $\eta_{\mathbb{S}}$ and apply (4.2) to $B = \eta A \lhd A$. This yields $\eta A \twoheadrightarrow C = \eta A/K \in \mathbb{S}$ and $K \subset \eta A$ for some kernel K of ηA. Applying $(S\varrho)$ to $\eta A/K \in \mathbb{S}$, we obtain $\eta(\eta A) \subseteq K$. Now the condition $\eta\eta A = \eta A$ yields the contradiction $\eta A \subseteq K \subset \eta A$.

iv)⇒ii) We show that iv) for $\mathbb{S}$ implies $\eta_{\mathbb{S}}\eta_{\mathbb{S}}A = \eta_{\mathbb{S}}A$ for all $A \in \mathfrak{H}$, and we write again η for $\eta_{\mathbb{S}}$. Since $\eta\eta A \in \mathfrak{K}(A)$ is assumed, we have $(A/\eta\eta A)/(\eta A/\eta\eta A) \cong A/\eta A$ by Th. 2.12. Now $A/\eta A \in \mathbb{S}$ and $\eta A/\eta\eta A \in \mathbb{S}$, both by $(S\varrho)$ for $\mathbb{S}$, imply $A/\eta\eta A \in \mathbb{S}$ by (Se). The latter yields $\eta\eta A \supseteq \eta A$, again by $(S\varrho)$, and so $\eta\eta A = \eta A$. ◊

8. Some remarks and supplements to the new theory

As already noted in the previous sections, the assumptions (Rc) for an n-radical class, $(S\gamma)$ for an n-semisimple class, and $(\varrho\omega)$ for an n-radical operator demand nothing if the considered n-universal class $\mathfrak{H} \subseteq \mathfrak{S}^* \cup \mathfrak{G}$ satisfies $\mathfrak{H} \subseteq \mathfrak{S}^{\mathrm{idp}}$ or $\mathfrak{H} \subseteq \mathfrak{G}$. The first case allows to deal with idempotent semifields without involving groups. This was one reason for us to generalize here the theory given in [12], in which idempotent semifields can be considered only together with groups, regardless of the fact that such semifields have no groups as kernels. The second case means that the radical theory for semifields includes the radical theory for groups as a special case.

As a third special case, (Rc) demands nothing for those n-radical classes $\mathbb{R}$ of an arbitrary n-universal class $\mathfrak{H} \subseteq \mathfrak{S}^* \cup \mathfrak{G}$ which satisfy $\mathbb{R} \cap \mathfrak{H} \cap \mathfrak{G} \subseteq \mathfrak{T}$. Assuming the latter, also $(\varrho\omega)$ and $(S\gamma)$ are trivially satisfied for the corresponding n-radical operator $\varrho_{\mathbb{R}}$ and the n-semisimple class $\mathbb{S} = \mathcal{S}\mathbb{R}$. In all these cases, the radical theory of semifields developed so far is a further step closer to the radical theory of rings or groups. In particular, an n-radical class $\mathbb{R}$ of $\mathfrak{H}$ is then already defined by properties (Ra) and (Rb) and an n-semisimple class $\mathbb{S}$ of $\mathfrak{H}$ by (Sa) and (Sb).

Next we show that one needs really more than (Ra) and (Rb) to define a suitable concept of a radical class in an arbitrary universal class of proper semifields. Such a class $\mathbb{R}$, clearly, has to have the property that each $A \in \mathfrak{H}$ under consideration has a greatest $\mathbb{R}$-kernel $\varrho_{\mathbb{R}}A$ as

described in Th. 5.7, a property equivalent to $(R\varrho)$.

Theorem 8.1. a) *Let $\mathfrak{H}$ be an n-universal class of $\mathfrak{S}^* \cup \mathfrak{G}$ which contains a direct product $A = B_1 \times B_2$ of non-idempotent semifields. Then the properties (Ra) and (Rb) for a subclass $\mathbb{R}$ of $\mathfrak{H}$ do not imply $(R\varrho)$ for $\mathbb{R}$.*

b) *The same holds if $\mathfrak{H}$ contains a direct product $A = B \times C$ of a non-idempotent semifield B satisfying $\operatorname{hull}_B(\mathbb{H}e) = B$ with a non-trivial idempotent semifield C. In this case there is even a subclass $\mathbb{R}$ of $\mathfrak{H}$ which satisfy (Ra) and (Rb), but neither $(R\varrho)$ nor (Re) nor (Rc).*

Proof. a) At first we show that, for any n-universal class $\mathfrak{H}$, the subclass $\mathbb{R} = (\mathfrak{G} \cap \mathfrak{H}) \cup \mathfrak{T}$ satisfies (Ra) and (Rb). The latter is obvious, and (Ra) for $\mathbb{R}$ is clearly true for each group $A \in \mathfrak{H}$. If $A \in \mathfrak{H} \setminus \mathfrak{T}$ is a semifield, then by Prop. 2.14, A has a homomorphic image B which satisfies $C \notin \notin \mathfrak{G}$ for each $C \lhd B$, and so $C \notin \mathbb{R}$. Hence each $A \in (\mathfrak{H} \setminus \mathfrak{T}) \cap \mathfrak{S}^*$ does not satisfy the assumption (4.1) of (Ra), which completes the proof of (Ra) for $\mathbb{R}$. Now, we apply Ex. 2.4 to $A = B_1 \times B_2 \in \mathfrak{H}$. Then the natural projections $\varphi_i : A \to B_i$ have kernels

$$(8.1) \qquad K_1 = \{(e_1, b_2) \mid b_2 \in B_2\} \text{ and } K_2 = \{(b_1, e_2) \mid b_1 \in B\}.$$

By $B_i \in \mathfrak{S}^\diamond$, both K_1 and K_2 are groups, and hence in $\mathbb{R}$. Since $K_1 \vee K_2 = A \notin \mathbb{R}$, the considered class $\mathbb{R}$ does not satisfy $(R\varrho)$.

b) Here we consider the class $\mathbb{R}$ obtained from $\mathbb{M} = \mathfrak{S}^{\mathrm{idp}} \cap \mathfrak{H}$ by $\mathbb{R} = \mathcal{U}\mathbb{M}$. Since $\mathbb{M}$ is hereditary, $\mathbb{R}$ satisfies (Ra) and (Rb). Again, $\mathbb{R}$ contains both kernels $(K_1, \cdot) \cong (C, \cdot)$ and $(K_2, +, \cdot) \cong (B, +, \cdot)$ of $A = B \times C$, the former by $\mathfrak{G} \cap \mathfrak{H} \subseteq \mathbb{R}$, the latter by $\operatorname{hull}_B(\mathbb{H}e) = B$ which excludes non-trivial homomorphic images of $(B, +, \cdot)$ in $\mathbb{M}$. Moreover, $(C, +, \cdot) \in \mathbb{M}$ implies $(C, +, \cdot) \notin \mathbb{R}$, and so $(A, +, \cdot) \notin \mathbb{R}$. Now it follows, as above, that $\mathbb{R}$ does not satisfy $(R\varrho)$. Further, (Re) is disproved by $K_1 \in \mathbb{R}$ and $A/K_1 \cong (B, +, \cdot) \in \mathbb{R}$ and $A \notin \mathbb{R}$. The latter disproves also (Rc) for $\mathbb{R}$, since (Ra), (Rb) and (Rc) would imply (Re) by Prop. 5.5. $\Diamond$

The crucial point of this proof is, of course, that each of the considered classes $\mathbb{R}$ satisfy $(A, \cdot) \in \mathbb{R}$ and $(A, +, \cdot) \notin \mathbb{R}$ for a semifield $(A, +, \cdot) \in \mathfrak{H}$, which is just excluded by (Rc). An other indication to assume this property for n-radical classes is the following

Proposition 8.2. *Let $\mathfrak{H} \subseteq \mathfrak{S}^* \cup \mathfrak{G}$ be an n-universal class which is closed under taking finite direct products, and contains a subclass $\mathbb{R}$ which satisfies $(R\varrho)$ and $(R\eta)$ but not (Rc). Then each such subclass $\mathbb{R}$ does not contain non-idempotent semifields, that is, $\mathbb{R} \cap \mathfrak{S}^* \subseteq \mathfrak{S}^{\mathrm{idp}}$.*

Proof. By the last assumption on $\mathbb{R}$ there is a semifield B in $\mathfrak{H}$ which satisfies $(B,\cdot) \in \mathbb{R}$ and $(B,+,\cdot) \notin \mathbb{R}$. By way of contradiction we assume $(C,+,\cdot) \in \mathbb{R}$ for a semifield $C \in \mathfrak{S}^\diamond$, and consider the cases $B \in \mathfrak{S}^\diamond$ and $B \in \mathfrak{S}^{\mathrm{idp}}$. In the first case, we use the direct product $A = B_1 \times B_2$ for $B_1 = B_2 = B$. Then, as in the last proof, the kernels K_i of A given in (8.1) are again groups, obviously isomorphic to $(B,\cdot)$, and so in $\mathbb{R}$. Now $(R\varrho)$ for $\mathbb{R}$ yields $\varrho_{\mathbb{R}}A = A$ and thus $A \in \mathbb{R}$, which contradicts $(R\eta)$ for $\mathbb{R}$ by $(A,+,\cdot) \twoheadrightarrow (B,+,\cdot) \notin \mathbb{R}$. In the case $B \in \mathfrak{S}^{\mathrm{idp}}$ we consider the direct product

$$A = B_1 \times B_2 \quad \text{for } B_1 = B \in \mathfrak{S}^{\mathrm{idp}} \quad \text{and} \quad B_2 = C \in \mathfrak{S}^\diamond.$$

Now we obtain from Ex. 2.4 that K_1 is a semifield isomorphic to $(C,+ +,\cdot) \in \mathbb{R}$, whereas K_2 is a group isomorphic to $(B,\cdot) \in \mathbb{R}$. Again $(R\varrho)$ implies $\varrho_{\mathbb{R}}A = A$, and so $(A,+,\cdot) \twoheadrightarrow (B,+,\cdot) \in \mathbb{R}$, contradicting $(B,+,\cdot) \notin \mathbb{R}$. $\Diamond$

In this context we note that the assumption of (Rc) in Def. 5.1 corresponds to the assumption of $(\varrho\omega)$ and of $(S\gamma)$ in the Defs. 6.1 and 7.1. Indeed, a subclass $\mathbb{S}$ of $\mathfrak{H}$ satisfies (Sa) and (Sb) iff $\mathbb{R} = \mathcal{U}\mathbb{S}$ satisfies (Ra) and (Rb) by Prop. 4.4, and for these classes the equivalence of $(S\gamma)$ for $\mathbb{S}$ and (Rc) for $\mathbb{R}$ follows from Lemma 7.3. Moreover, $(\varrho\omega)$ is just the formulation of (Rc) in terms of the radical operator $\varrho_{\mathbb{R}}$ (cf. the third step in the proof of Th. 6.2).

We close this section with the following theorem and corresponding examples.

Theorem 8.3. *Let $\mathbb{R}$ and $\mathbb{S}$ be any subclasses of $\mathfrak{H} \subseteq \mathfrak{S}^* \cup \mathfrak{G}$. Then $\mathbb{R}$ is an n-radical class of $\mathfrak{H}$ and $\mathbb{S}$ the corresponding n-semisimple class $\mathcal{S}\mathbb{R}$ iff the following properties hold for all $A, B \in \mathfrak{H}$:*

a) $\mathbb{R} \cap \mathbb{S} \subseteq \mathfrak{T}$.

b) *$A \in \mathbb{R}$ and $A \twoheadrightarrow B$ imply $B \notin \mathbb{S}$, that is, $\mathbb{R} \subseteq \mathcal{U}\mathbb{S}$.*

c) *$A \in \mathbb{S}$ and $B \lhd A$ imply $B \notin \mathbb{R}$, that is, $\mathbb{S} \subseteq \mathcal{S}\mathbb{R}$.*

d) *There is a kernel $K \in \mathfrak{K}(A)$ satisfying $K \in \mathbb{R}$ and $A/K \in \mathbb{S}$.*

e) *$\mathbb{R}$ satisfies (Rc) or $\mathbb{S}$ satisfies $(S\gamma)$.*

Proof. We already know that all these properties hold for an n-radical class and its n-semisimple class $\mathbb{S} = \mathcal{S}\mathbb{R}$. For the converse we apply d) to each $A \in \mathcal{S}\mathbb{R}$. Since the latter yields $B \notin \mathbb{R}$ for all $B \lhd A$, we obtain $K \in \mathfrak{T}$, and hence $A \in \mathbb{S}$. This shows $\mathcal{S}\mathbb{R} \subseteq \mathbb{S}$, and so $\mathbb{S} = \mathcal{S}\mathbb{R}$ by c). Likewise one proves $\mathbb{R} = \mathcal{U}\mathbb{S}$ by d) and b). Now $\mathbb{R} = \mathcal{U}\mathcal{S}\mathbb{R}$ and $\mathbb{S} = \mathcal{S}\mathcal{U}\mathbb{S}$ imply (Ra) and (Rb) for $\mathbb{R}$ and (Sa) and (Sb) for $\mathbb{S}$ by (4.6) and (4.5). Hence, by Th. 7.2, $\mathbb{R}$ is an n-radical class iff $\mathbb{S} = \mathcal{S}\mathbb{R}$ is an n-semisimple

class, where the former holds whenever $\mathbb{R}$ satisfies (Rc), and the latter whenever $\mathbb{S}$ satisfies $(S\gamma)$. $\Diamond$

Example 8.4. a) *For each n-universal class* $\mathfrak{H} \subseteq \mathfrak{S}^* \cup \mathfrak{G}$,

$$\mathbb{R} = (\mathfrak{S}^{\mathrm{idp}} \cap \mathfrak{H}) \cup \mathfrak{T} \quad \textit{and} \quad \mathbb{S} = (\mathfrak{S}^{\circ} \cap \mathfrak{H}) \cup (\mathfrak{G} \cap \mathfrak{H}) \cup \mathfrak{T}$$

are corresponding n-radical and n-semisimple classes of $\mathfrak{H}$.

To show this, we check the properties listed in Th. 8.3. Whereas a) is clear, b) holds since $\mathbb{R}$ is homomorphically closed and c) since $\mathbb{S}$ is hereditary. For d) we use $\mathbb{R} \cup \mathbb{S} = \mathfrak{H}$ and choose $K = A$ for $A \in \mathbb{R}$ and $K \in \mathfrak{T}$ for $A \in \mathbb{S}$. Finally (Rc) holds for $\mathbb{R}$ by lack of non-trivial groups in $\mathbb{R}$.

b) We emphasize that $\mathbb{R} \cup \mathbb{S} = \mathfrak{H}$ *holds for the above classes and also* $\mathbb{R} \neq \mathfrak{T} \neq \mathbb{S}$ *by a suitable choice of* $\mathfrak{H}$, *in particular, for* $\mathfrak{H} = \mathfrak{S}^* \cup \mathfrak{G}$. *So the radical theory for semifields provides examples of "non-trivial complementary radical classes"* in the new meaning. This peculiar phenomenon cannot occur if all the trivial objects are isomorphic as in the case of rings or groups (cf. [9]).

Example 8.5. a) An idempotent semifield $(M, +, \cdot)$ is called *rectangular* if $(M, +)$ is a rectangular semigroup. The latter is equivalent to $a + b + a = a$ for all $a, b \in M$ and in turn to $a + b = b + a \Rightarrow a = b$. As shown in [11], each idempotent semifield A contains a greatest rectangular subsemifield M. The latter is a kernel of A and A/M is an additively commutative semifield.

b) Now, let $\mathfrak{H}$ be an n-universal class contained in $\mathfrak{S}^{\mathrm{idp}}$. Then

$$\mathbb{R} = \{A \in \mathfrak{H} \mid A \text{ is rectangular}\}$$

and

$$\mathbb{S} = \{A \in \mathfrak{H} \mid (A, +) \text{ is commutative}\}$$

are corresponding n-radical and n-semisimple classes of $\mathfrak{H}$. Using a), one obtains this easily from Th. 8.3. In particular, for each $A \in \mathfrak{H}$, the greatest rectangular subsemifield M is the $\mathbb{R}$-radical $\varrho_{\mathbb{R}} A = M$ of A.

c) More generally, one obtains that

$$\mathbb{R} = \{A \in \mathfrak{H} \cap \mathfrak{S}^{\mathrm{idp}} \mid A \text{ is rectangular}\} \cup \mathfrak{T}$$

is an n-radical class in each n-universal class $\mathfrak{H} \subseteq \mathfrak{S}^* \cup \mathfrak{G}$, *where*

$$\mathbb{S} = (\mathfrak{H} \setminus \mathfrak{S}^{\mathrm{idp}}) \cup \{A \in \mathfrak{H} \cap \mathfrak{S}^{\mathrm{idp}} \mid (A, +) \text{ is commutative}\}$$

is the corresponding n-semisimple class.

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