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ON SOME EXTENSIONS OF σ-FINITE MEASURES

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Abstract: We shall present that if (X, S, μ) is a measure space with a non-negative σ-finite measure then for every set $Y \subset X$ and every $c \in [\mu_*(Y), \mu^*(Y)]$ where μ_* and μ^* denote respectively the inner and outer measure corresponding μ, there exists and extension μ_c of μ such that $Y \in \operatorname{dom} \mu_c =$ $= \sigma(S \cup \{Y\})$ and $\mu_c(Y) = c$. The cardinality of the set of such extensions is also discussed.

Our paper is motivated by the results contained in paper [3], see also [2]. Let (X, S, μ) be an arbitrary measure space. Let $Y \subset X$ be an arbitrary subset of X. Then the functions

$$\mu_*(Y) = \sup \{\mu(W) : Y \supset W \in S\},$$
$$\mu^*(Y) = \inf \{\mu(Z) : Y \subset Z \in S\}$$

are called, respectively, inner and outer measure corresponding to μ. By the family $\sigma(K)$ we denote the smallest σ-field generated by family K.

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If measure μ is defined in a field we understand that μ is additive and in the case when μ is defined in a σ-field we assume that μ is σ-additive.

The following results were established in [3].

Theorem A (cf. Th.1, 4 in [3]). *Let (X, S, μ) be an arbitrary measure space where S is a σ-field. Then for every set $Y \subset X$ and $c = \mu_*(Y)$ or $c = \mu^*(Y)$ there exists an extension μ_c of μ such that $\mu_c(Y) = c$ and dom $\mu_c = \sigma(S \cup \{Y\})$.*

Theorem B (cf. Th.2, 4 in [3]). *Under the hypotheses of* Th. A *and $\mu^*(Y) < \infty$, for every $c \in [\mu_*(Y), \mu^*(Y)]$ there exists an extension μ_c of μ such that $\mu_c(Y) = c$ and* $\operatorname{dom} \mu_c = \sigma(S \cup \{Y\})$.

And also

Theorem C (cf. Th. 3 in [3]). *Let (X, S, μ) be a measurable space where S is a field, then for every $c \in [\mu_*(Y), \mu^*(Y)]$ there exists an extension μ_c of μ such that $\mu_c(Y) = c$ and dom μ_c is the field generated by $S \cup \{Y\}$.*

It was observed in [3], that Th. C does not work for the σ-fields and σ-additive measures.We shall show that this fact is true for σ-finite measures in the σ-fields. In fact Ths. A and B are proved in [3] via canonical extensions. In the sequel we shall deal only with measures on the σ-fields.

Let us recall Marczewski theorem.

Theorem M (cf. [4]). *Let (X, S, μ) be measurable space and J be σ-ideal of subsets of X such that $\mu_*(Y) = 0$ for every $Y \in J$. Then there exists extension μ' of measure μ with domain $S \triangle J = \{A \triangle Y; A \in \in S, Y \in J\}$, giving by formula $\mu'(A \triangle Y) = \mu(A)$.*

By this theorem it is easy to obtain Ths. A and B for the σ-finite measures.

Let W and Z be μ-measurable kernel and μ-measurable hull of Y, respectivelly. Putting $J = \mathbb{P}(Y - W)$ we have that for every set $A \in J$, $\mu_*(A) = 0$. Thus by Th. M, there exists extension μ' of measure μ such that $\mu'(Y) = \mu(W) = \mu_*(Y)$. Measure $\mu_c = \mu'/\sigma(S \cup \{Y\})$ has the desired property for the case $c = \mu_*(Y)$. In the similar way, considering σ-ideal $J = \mathbb{P}(Z - Y)$ we get the case for $c = \mu^*(Y)$.

Th. B is the consequence of Th. A by considering the convex combinations of two measures (cf. Th. 2 in [3])

Lemma 1. *Let (X, S, μ) be an arbitrary measurable space and $Y \subset X$ be such that $\mu_*(Y) = 0$ and $\mu^*(Y) = \infty$. Then for every set $Y_1 \subset Y$ such that $\mu^*(Y_1) < \infty$ and for every extension μ_1 of measure μ such* *that* $\operatorname{dom}\mu_1 = \sigma(S \cup \{Y_1\})$ *with property* $\mu_1(Y_1) < \infty$ *the following conditions are satisfied:*

1^0 $(\mu_1)_*(Y - Y_1) = 0$,
2^0 $(\mu_1)^*(Y - Y_1) = \infty$.

Proof. Since $\sigma(S \cup \{Y_1\}) = \{(A \cap Y_1) \cup (B - Y_1); A, B \in S\}$ then every set $W \in \sigma(S \cup \{Y_1\})$ and $W \subset Y - Y_1$ has the form $W = B - Y_1$, where $B \in S$.This implies that $B \subset B - Y_1 \cup Y_1 = W \cup Y_1 \subset Y$ and thus $\mu(B) = 0$. It follows that $(\mu_1)_*(Y - Y_1) = 0$.

Let $Z \in \sigma(S \cup \{Y_1\})$ be such that $Z \supset Y - Y_1$. It suffices to consider set Z of the form $Z = B - Y_1$, where $B \in S$. Since $\mu^*(Y_1) < < \infty$ then there exists a set $C \in S$ such that $Y_1 \subset C$ and $\mu(C) < \infty$. Then $B \supset B - Y_1 \supset Y - Y_1 \supset Y - C$ and thus $Y \subset B \cup C$. It follows that $(\mu_1)^*(Y - Y_1) = \infty$. ◊

Now, we fix that (X, S, μ) is measurable space with σ-finite measure μ.

Theorem 2. *For every* $Y \subset X$ *and every* $c \in [\mu_*(Y), \mu^*(Y)]$ *there exists extension* μ_c *of measure* μ *such that* $\mu_c(Y) = c$ *and dom* $\mu_c = = \sigma(S \cup \{Y\})$.

Proof. In view of Th. B the case $\mu^*(Y) = \infty$ is left. Let $c \in [\mu_*(Y), \mu^*(Y)]$, where $\mu^*(Y) = \infty$. The case $c = \mu_*(Y)$ or $c = \mu^*(Y)$ holds by Th. A. Let $c \in (\mu_*(Y), \mu^*(Y))$ and suppose that $\mu_*(Y) = 0$. By the σ-finite property of measure μ there exists a μ-measurable set C of finite measure such that $\mu^*(Y \cap C) > c$. By Th. B there exists extension μ_1 of measure μ such that $\mu_1(Y \cap C) = c$ and dom $\mu_1 = \sigma(S \cup \{Y \cap C\}) = S_1$. Let $Y_1 = Y \cap C$. By Lemma 1 $(\mu_1)_*(Y - Y_1) = 0$. Then again by Th. B there exists extension μ_2 of measure μ_1 such that dom $\mu_2 = \sigma(S_1 \cup \cup \{Y - Y_1\})$ and $\mu_2(Y - Y_1) = 0$. Since $\sigma(S_1 \cup \{Y\}) = \sigma(S_1 \cup \{Y - Y_1\})$ we can put $\mu_c = \mu_2$ and μ_c is the extension of the original measure μ and $\mu_c(Y) = \mu_2(Y) = \mu_2(Y_1) + \mu_2(Y - Y_1) = \mu_1(Y_1) + \mu_2(Y - Y_1) = c$. In the case, when $\mu_*(Y) > 0$ and $c \in (\mu_*(Y), \mu^*(Y))$ it is sufficient to take set $Y^* = Y - W$, where W is μ-kernel of Y and apply the previous result to finish the proof. ◊

Let μ_1 and μ_2 be two extensions of measure μ such that dom $\mu_1 = = $ dom $\mu_2 = \sigma(S \cup \{Y\})$ for some subset $Y \subset X$. It is obvious that any extension of a σ-finite measure is again σ-finite measure. Let us pay attention on the following

Lemma 3. *The following conditions are equivalent*

1^0 $\forall_{A \in S}$ $\mu_1(A \cap Y) = \mu_2(A \cap Y)$

$2^0 \quad \forall_{A\in S} \quad \mu_1(A-Y)=\mu_2(A-Y)$

Property 4. *The measures μ_1 and μ_2 are identical if and only if*

$$\forall_{\substack{A\in S\\ \mu(A)<\infty}} \quad \mu_1(A\cap Y)=\mu_2(A\cap Y)$$

Proof. Necessity is obvious. Sufficiency: Let $B\in\sigma(S\cup\{Y\})$. Then $B=(C\cap Y)\cup(D-Y)$, where $C,D\in S$. Hence in view of Lemma 3 and the fact of σ-finite property μ_1,μ_2

$\mu_1(B)=\mu_1(C\cap Y)+\mu_1(D-Y)=\mu_2(C\cap Y)+\mu_2(D-Y)=\mu_2(B)$.

It follows our assertion. $\Diamond$

Now we try to investigate how many different extensions of the original measure μ we may get to attach the value c equal to the measure of the set Y. Some unicity problems are also contained in [1].

For every $Y\subset X$ and $c\in[\mu_*(Y),\mu^*(Y)]$ we denote

$$\mathfrak{M}_c(Y)=\{\mu' : \mu' \text{ is extension of } \mu \text{ with domain}$$
$$\sigma(S\cup\{Y\})\ \&\ \mu'(Y)=c\},$$

$$\mathfrak{M}(Y)=\{\mu' : \mu' \text{ is extension of } \mu \text{ with domain} \sigma(S\cup\{Y\}).$$

We have that $\mathfrak{M}(Y)=\bigcup\{\mathfrak{M}_c(Y);\ c\in[\mu_*(Y),\mu^*(Y)]\}$.

Property 5. *If $c=\mu_*(Y)<\infty$ or $c=\mu^*(Y)<\infty$ then* card $\mathfrak{M}_c(Y)=$ $=1$.

Proof. Let us assume that $c=\mu_*(Y)<\infty$. By Th. 2 card $\mathfrak{M}_c(Y)\geq$ ≥ 1. Let $\mu_1,\mu_2\in\mathfrak{M}_c(Y)$. Let W be μ-kernel of Y. Then for every set $A\in S$ we have that $\mu_1(A\cap Y)=\mu(A\cap W)$. Similarly $\mu_2(A\cap Y)=$ $=\mu(A\cap W)$. By Prop. 4 we have that $\mu_1\equiv\mu_2$ and card $\mathfrak{M}_c(Y)=1$. The case $c=\mu^*(Y)<\infty$ can be proved similarly. $\Diamond$

Property 6. *For every $c_1,c_2\in(\mu_*(Y),\mu^*(Y))$* card $\mathfrak{M}_{c_1}(Y)=$ $=$ card $\mathfrak{M}_{c_2}(Y)$.

Proof. Let $c_1<c_2$. Let $c_3>c_2$ and $c_3\in(\mu_*(Y),\mu^*(Y)))$. Let us fix $\mu_{c_1}\in\mathfrak{M}_{c_1}(Y)$ and $\mu_{c_3}\in\mathfrak{M}_{c_3}(Y)$. There exists $\alpha\in(0,1)$ such that $c_2=\alpha c_1+(1-\alpha)c_3$. Putting $\mu_{c_2}=\alpha\mu_{c_1}+(1-\alpha)\mu_{c_3}$ we get that $\mu_{c_2}\in\mathfrak{M}_{c_2}(Y)$. Moreover, if $\mu'_{c_1},\mu''_{c_1}\in\mathfrak{M}_{c_1}(Y)$ and $\mu'_{c_1}\neq\mu''_{c_1}$ then $\mu'_{c_2}\neq\mu''_{c_2}$. Hence card $\mathfrak{M}_{c_1}(Y)\leq$ card $\mathfrak{M}_{c_2}(Y)$. Analogously card $\mathfrak{M}_{c_2}(Y)\leq$ card $\mathfrak{M}_{c_1}(Y)$. $\Diamond$

Remark. Example in [3] shows that Prop. 6 does not hold for the segment $[\mu_*(Y),\mu^*(Y))]$.

Property 7. *For every $c\in[\mu_*(Y),\mu^*(Y))]$* card $\mathfrak{M}_c(Y)=1$ *or* card $\mathfrak{M}_c(Y)\geq\mathfrak{c}$.

Proof. Let $c \in (\mu_*(Y), \mu^*(Y)))$. By Th. 2 card $\mathfrak{M}_c(Y) \geq 1$. Suppose that card $\mathfrak{M}_c(Y) > 1$. Let $\mu_1, \mu_2 \in \mathfrak{M}_c(Y)$ and $\mu_1 \neq \mu_2$. Then for every $\alpha \in [0,1]$ $\mu_\alpha = \alpha\mu_1 + (1-\alpha)\mu_2 \in \mathfrak{M}_c(Y)$ and by Prop. 4 we have that $\mu_\alpha \neq \mu_\beta$ for every $\alpha \neq \beta$ such that $\alpha, \beta \in [0,1]$. It means that card $\mathfrak{M}_c(Y) \geq \mathfrak{c}$. ◊

Corollary 8. *For every $Y \subset X$,* card $\mathfrak{M}(Y) = 1$ *or* card $\mathfrak{M}(Y) \geq \mathfrak{c}$.

Property 9. *If $\mu^*(Y) = \infty$, then for every $c \in [\mu_*(Y), \mu^*(Y)]$* card $\mathfrak{M}_c(Y) \geq \mathfrak{c}$.

Proof. Case 1. Let $c = \mu^*(Y)$. By Th. 2 there exists measure $\mu_c \in \in \mathfrak{M}_c(Y)$. Let $d \in (\mu_*(Y), \mu^*(Y))$ and $\mu_d \in \mathfrak{M}_d(Y)$. Putting for every $\alpha \in (0,1]$ $\mu_\alpha = \alpha\mu_c + (1-\alpha)\mu_d$ we get that $\mu_\alpha \in \mathfrak{M}_c(Y)$ and for every $\alpha, \beta \in (0,1]$ $\mu_\alpha \neq \mu_\beta$. Hence card $\mathfrak{M}_c(Y) \geq \mathfrak{c}$.

Case 2. Let $c \in (\mu_*(Y), \mu^*(Y))$. We suppose, by the same technique as in proof of Th. 2 that $\mu_*(Y) = 0$. Let c_1, c_2, c_3, c_4 be positive numbers such that $c_1 + c_2 = c_3 + c_4 = c$ and $c_1 \neq c_3$. Since $\mu^*(Y) = = \infty$ there exists S-measurable set C such that $c < \mu^*(Y_1) < \infty$, where $Y_1 = Y \cap C$. By Th. 1, there exist extensions μ_{c_1}, μ_{c_3} of measure μ such that dom μ_{c_1} =dom $\mu_{c_3} = \sigma(S \cup \{Y_1\})$ and $\mu_{c_1}(Y_1) = c_1$ and $\mu_{c_3}(Y_1) = c_3$. According to Lemma 3 $(\mu_{c_1})_*(Y - Y_1) = 0$, $\mu^*_{c_1}(Y - Y_1) = \infty, (\mu_{c_3})_*(Y - Y_1) = 0, \mu^*_{c_3}(Y - Y_1) = \infty$. There exists a set $D \in S$ such that for the set $Y_2 = D \cap (Y - Y_1)$ we have

$$c < \mu_{c_1}(Y_2) < \infty \text{ and } c < \mu_{c_3}(Y_2) < \infty.$$

Applying Th. 2 we get extension μ_{c_2} of measure μ_{c_1} and extension μ_{c_4} of measure μ_{c_3} such that dom μ_{c_2} =dom $\mu_{c_4} = \sigma(S_1 \cup \{Y_2\}) = = S_2$ and $\mu_{c_2}(Y_2) = c_2$ and $\mu_{c_4}(Y_2) = c_4$. By Lemma 1 we have $(\mu_{c_2})_*((Y - Y_1) - Y_2) = 0$ and $(\mu_{c_4})_*((Y - Y_1) - Y_2) = 0$. Applying Th. 2 we get extension μ'_c of measure μ_{c_2} and extension μ''_c of measure μ_{c_4} such that dom μ'_c = dom $\mu''_c = \sigma(S_2 \cup \{Y - (Y_1 \cup Y_2)\})$ and $\mu'_c(Y - (Y_1 \cup Y_2)) = \mu''_c(Y - (Y_1 \cup Y_2)) = 0$. Taking into account the form of the sets Y_1 and Y_2 we conclude that $\sigma(S_2 \cup \{Y - (Y_1 \cup Y_2)\}) = = \sigma(S \cup \{Y\})$ and $\mu'_c(Y) = \mu'_c(Y_1 \cup Y_2) + \mu'_c(Y - (Y_1 \cup Y_2)) = \mu'_c(Y_1) + + \mu'_c(Y_2) = \mu_{c_1}(Y_1) + \mu_{c_2}(Y_2) = c_1 + c_2 = c$. Also $\mu''_c(Y) = c$. Hence $\mu'_c, \mu''_c \in \mathfrak{M}(Y)$. Simultaneously $\mu'_c \neq \mu''_c$ because $\mu'_c(Y_1) = c_1$ and $\mu''_c(Y_1) = c_2$. The convex combination of measures μ'_c and μ''_c makes inequality card $\mathfrak{M}_c(Y) \geq \mathfrak{c}$. ◊

We finish with the following property.

Property 10. *If W and Z are μ-kernel and μ-hull of set Y, respectively, then the following conditions are satisfied:*

1^0 card $\mathfrak{M}(Y) = 1 \iff \mu(Z - W) = 0$
2^0 card $\mathfrak{M}(Y) \geq \mathfrak{c} \iff \mu(Z - W) > 0$.

Proof. It is sufficient to show that if $\mu(Z-W) = 0$ then card $\mathfrak{M}(Y) = 1$ and if $\mu(Z - W) > 0$ then card $\mathfrak{M}(Y) \geq \mathfrak{c}$. By Th. 2 card $\mathfrak{M}(Y) \geq 1$. If $\mu_1, \mu_2 \in \mathfrak{M}(Y)$ and $\mu_1(Z - W) = \mu_2(Z - W) = 0$ then by Prop. 4 $\mu_1 \equiv \mu_2$ and card $\mathfrak{M}(Y) = 1$. Let $\mu(Z - W) > 0$. Suppose firstly that $\mu(W) < \mu(Z)$. Since $\mu_*(Y) = \mu(W) < \mu(Z) = \mu^*(Y)$ we conclude by Th. 2 that card $\mathfrak{M}(Y) \geq \mathfrak{c}$. Suppose secondly that $\mu(W) = \mu(Z) = \infty$. By Th. A there exist measure $\mu_1, \mu_2 \in \mathfrak{M}(Y)$ such that $\mu_1(Y) = \mu(W)$, $\mu_2(Y) = \mu(Z)$ and $\mu_1(Y - W) = 0$, $\mu_2(Z - Y) = 0$. Hence we obtain that $0 < \mu(Z-W) = \mu_1(Z-W) = \mu_1((Z-Y)+\mu_2(Y-W) = \mu_1(Z-$ $-Y)$. Thus $\mu_1(Z - Y) > 0$. It means that $\mu_1 \neq \mu_2$. By Cor. 8 we get that card $\mathfrak{M}(Y) \geq \mathfrak{c}$. ◊

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