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ABOUT THE ATTRACTIVE POINTS OF THE ROOT FUNCTIONS ON THE RIEMANNIAN FOLIATION

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Dedicated to Professor W. Breckner on his 60th birthday

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Abstract: This paper is motivated by the work of H. Zeitler [3] who researched the Mandelbrot set and its main body for a function $f(z) = z^n + c$, where $n \in \mathbb{Z}$. An interesting connection between classical geometry and modern chaos theory, excursions into complex number iterations, amazing properties of the Mandelbrot set and impressive pictures inspired me to investigate the main body of the Mandelbrot set for the root function. How does the main body of that function look like in a complex plane or is it better to look for it somewhere else?

1. Introduction

Let $f(z) = z^{\frac{p}{q}} + c$, where $p, q \in \mathbb{N}$, $p < q$ and $\mathrm{GCD}(p, q) = 1$, be a function of a complex variable z. To ensure that this is a single-valued mapping we must examine it on a suitable Riemannian foliation

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R ($f : R \to R$). The Riemannian foliation (briefly foliation) R consists of q complex planes put one over the other so that the coordinate axes of all planes coincide. We cut each of these planes starting at the origin and then along the positive part of the real axe. We then connect the i-th plane with the $(i+1)$-th plane along the cut, $i = 1, 2, \dots, q-1$ and finally connect the q-th plane with the first one. How do the function f act on the described foliation? The first of the q planes in the foliation R is transformed into the area between angles 0 and $\frac{2\pi p}{q}$ in the first plane of the foliation which contains images of the function f. The second of the q planes is transformed into the area between angles $\frac{2\pi p}{q}$ and $\frac{4\pi p}{q}$, and so on. The last of the q planes in the foliation R is transformed into the area between angles $\frac{2\pi(q-1)p}{q}$ and $2\pi p$ and it is connected with the first one (area between angles 0 and $\frac{2\pi p}{q}$).

We consider a fixed z (for example $z = 0$) and iterate with respect to c, obtaining the following sequence of complex numbers:

$$f(0) = c,\ f^2(0) = c^{\frac{p}{q}} + c,\ f^3(0) = \left(c^{\frac{p}{q}} + c\right)^{\frac{p}{q}} + c, \dots .$$

We present this sequence as a set of points in the Riemannian foliation R. We define the Mandelbrot set M in respect of f and $z = 0$ as the set of $c \in R$ for which the sequence c, $c^{\frac{p}{q}} + c$, $\left(c^{\frac{p}{q}} + c\right)^{\frac{p}{q}} + c \dots$, does not tend to ∞ as n tends to ∞:

$$M = M(f, 0) = \{c \in R;\ f^s(0) \nrightarrow \infty, \text{ for } s \to \infty\}.$$

Definition 1.1. The set of all points $c \in M$ such that the above sequence for function f has exactly one attractive fixed point is called the main body H of M (for function f and initial point $z = 0$).

By the term attractive fixed point we mean a point w_0 which satisfies the following conditions:

(a) $f(w_0) = w_0$.

(b) There exists such a neighborhood U of the point w_0 that for every $w \in U$

$$\lim_{n \to \infty} f^n(w) = w_0.$$

The main results in the present paper are descriptions of the subsets of Riemannian foliation representing the main bodies of the irrational functions $f(z) = z^{\frac{p}{q}} + c$ with $p, q \in \mathbb{N}$, $p < q$, $q > 1$ and $\mathrm{GCD}(p, q) = 1$.

Remark 1.2. There are two variables used in our text: the variable z and the parameter c. We shall even jump between the z-plane (with c fixed) and the c-plane (with z fixed). The first one is denoted as *dynamical plane* and the second one as *parameter plane*. The c-plane is the native place of the Mandelbrot set.

2. Conditions for the existence of attracting fixed points

The following theorem indicates the conditions under which attractive fixed points of the function exist.

Theorem 2.1. *Suppose that $f: R \to R$, where R is a Riemannian foliation in a sence explained in Introduction, is defined by $f(z) = z^{\frac{p}{q}} + c$ where $p, q \in \mathbb{N}$ and $GCD(p,q) = 1$. If the function f is continuously differentiable in a neighborhood of the fixed point w_0 and if $|f'(w_0)| < 1$, then an open neighborhood U of the point w_0 exists, such that $\lim_{n\to\infty} f^n(w) = w_0$ for every element $w \in U$.*

Proof. By the assumption of the theorem the function f is continuously differentiable at the point w_0. So an open disc $K(w_0,\varepsilon)$, $\varepsilon > 0$ exists, such that $|f'(w)| \le A < 1$ for every $w \in K(w_0,\varepsilon)$. Because of the properties of analytic functions the following inequalities are valid:

(1)
$$\begin{aligned}
&|f(w) - w_0| = \\
&= \left| f(w_0) + f'(w_0)(w - w_0) + \frac{f''(w_0)}{2!}(w - w_0)^2 + \cdots - w_0 \right| = \\
&= \left| f'(w_0)(w - w_0) + \frac{f''(w_0)}{2!}(w - w_0)^2 + \cdots \right| \le \\
&\le |f'(w_0)(w - w_0)| + \left| \left(\sum_{k=2}^{\infty} \frac{f^{(k)}(w_0)}{k!}(w - w_0)^{k-2} \right)(w - w_0)^2 \right| = \\
&= |f'(w_0)|\,|w - w_0| + \left| \sum_{k=2}^{\infty} \frac{f^{(k)}(w_0)}{k!}(w - w_0)^{k-2} \right| |w - w_0|^2 < \\
&< A\,|w - w_0| + C\varepsilon\,|w - w_0| = (A + C\varepsilon)\,|w - w_0| < |w - w_0|.
\end{aligned}$$

To end the proof we have to show that the expression

$$\left|\sum_{k=2}^{\infty} \frac{f^{(k)}(w_0)}{k!} (w - w_0)^{k-2}\right|$$

is less than a constant C. The series inside the absolute value signs is the power series in the form $\sum\limits_{n=0}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$, which converges for $|z| < r$, $z \in \mathbb{C}$ (with r we denote the radius of convergence). The radius of convergence for the given series equals

$$r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|,$$

if the limit exists. In case of the function $f(z) = z^{\frac{p}{q}} + c$ where $p, q \in \mathbb{N}$, $p < q$, $q > 1$ and $\mathrm{GCD}(p, q) = 1$ the following result for the radius of convergence is obtained:

$$\begin{aligned} r &= \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \\ &= \lim_{n\to\infty} \left|\frac{\frac{p}{q}\left(\frac{p}{q}-1\right)\cdots\left(\frac{p}{q}-n+1\right) w_0^{\frac{p}{q}-n}\,(n+1)!}{n!\frac{p}{q}\left(\frac{p}{q}-1\right)\cdots\left(\frac{p}{q}-n+1\right)\left(\frac{p}{q}-n\right) w_0^{\frac{p}{q}-n-1}}\right| = \\ &= |w_0| \lim_{n\to\infty}\left|\frac{q\,(n+1)}{p-nq}\right| = |w_0|\,. \end{aligned}$$

We conclude that our series is convergent and the expression (1) is strictly less than

$$(A + C\varepsilon)\,|w - w_0| < |w - w_0|$$

for a suitable constant C.

With the description above we showed that the point $f(w)$ is closer to the point w_0 then the point w. We find out, in a similar way, that the point $f^2(w)$ is even closer to the point w_0 then the point $f(w)$ and so on. So we conclude that $\lim\limits_{n\to\infty} f^n(w) = w_0$ for every $w \in$ $\in K(w_0, \varepsilon)$. This ends the proof of the theorem. ◊

3. The border line of the main body for the function $\mathbf{f(z)=z^{\frac{p}{q}}+c}$, $\mathbf{p, q\in\mathbb{N}}$

Theorem 3.1. *The main body H of the Mandelbrot set M in respect of functions $f(z) = z^{\frac{p}{q}} + c$, $p, q \in \mathbb{N}$, $p < q$, $GCD(p, q) = 1$ is the set of* *points in the Riemannian plane R outside the curve represented by the equation*

$$c = \left(\frac{p}{q}\right)^{\frac{q}{q-p}} e^{it} - \left(\frac{p}{q}\right)^{\frac{p}{q-p}} e^{it\frac{p}{q}}, \quad t \in \mathbb{R}.$$

Remark 3.2. Let be $p = q'$, $q = p'$. With the following substitution of parameter

$$t = \frac{p'}{q'} s = \left(\frac{p}{q}\right)^{-1} s$$

we rearrange the equation of the curve from Th. 3.1:

$$c = \left(\frac{p}{q}\right)^{\frac{q}{q-p}} e^{it} - \left(\frac{p}{q}\right)^{\frac{p}{q-p}} e^{it\frac{p}{q}} = \left(\frac{q'}{p'}\right)^{\frac{p'}{p'-q'}} e^{is\frac{p'}{q'}} - \left(\frac{q'}{p'}\right)^{\frac{q'}{p'-q'}} e^{is} =$$

$$= -\left[\left(\frac{q'}{p'}\right)^{\frac{q'}{p'-q'}} e^{is} - \left(\frac{q'}{p'}\right)^{\frac{p'}{p'-q'}} e^{is\frac{p'}{q'}}\right]$$

This means that the second curve is the mirror image (over the origin) of the first curve.

From the following corollary it is evident that the mentioned equation represents the border curve of the main body H for the function $f(z) = z^{\frac{p}{q}} + c$, $p, q \in \mathbb{N}$, $p > q$, $\mathrm{GCD}(p, q) = 1$ (with the "$-$" sign).

Corollary 3.3. *The main body H of the set M in respect of functions* $f(z) = z^{\frac{p}{q}} + c$, $p, q \in \mathbb{N}$, $p > q$, $GCD(p, q) = 1$ *is the set of points in the foliation R within the curve represented by the equation* $c = \left(\frac{q}{p}\right)^{\frac{q}{p-q}} e^{it} - \left(\frac{q}{p}\right)^{\frac{p}{p-q}} e^{it\frac{p}{q}}$, $t \in \mathbb{R}$.

Let us explain why the transformation in the Remark 3.2 was done. In the first part of this paper we described the Riemannian foliation in case $p < q$. With our transformation we showed, that the curve from Th. 3.1 ($p < q$) and the curve in Cor. 3.3 ($p > q$) mean the same curve in different positions on the Riemannian foliation R: the second curve is obtained from the first one by reflection over the origin. So the Riemannian foliation in case $p < q$ can be used as a set of images for the function $f(z) = z^{\frac{p}{q}} + c$ in both cases: $p < q$ and $p > q$.

The proofs of Th. 3.1 and it's corollary for the case f_1: $\mathbb{C} \to \mathbb{C}$ where f_1 is one of the branches of the function f can be found in [2]. Th. 2.1 in the case of the Riemannian foliation as the domain and the set of images of the function $f(z) = z^{\frac{p}{q}} + c$, $p, q \in \mathbb{N}$, allow us to extend (in the same way) these results to the global case $f : R \to R$.

4. A connection between the dynamical and the parameter plane

The proofs of Th. 3.1 and it's corollary imply that the border line of the main body for the function $f(z) = z^{\frac{p}{q}} + c$, $p, q \in \mathbb{N}$, $\mathrm{GCD}(p, q) = 1$ is the unit circle $\beta(t) = e^{it}$ in the dynamical plane (for more detail see [2]): Because the main body H consists only of attractive fixed points we have to make sure that fixed point z (we will write $z = k\lambda$, $k \in \mathbb{R}^+$, due to further calculation) is of this type. By the stability criterion from Th. 2.1 involving the derivation of f we obtain

$$\left|f'(z)\right| = \left|\frac{p}{q} k^{\frac{p-q}{q}} \lambda^{\frac{p-q}{q}}\right| < 1.$$

Now we choose k such that $\frac{p}{q} k^{\frac{p-q}{q}} = 1$. This means $k = \left(\frac{p}{q}\right)^{\frac{q}{q-p}}$. Then we obtain $\left|\lambda^{\frac{p-q}{q}}\right| < 1$ and because $p < q$ further $|\lambda| > 1$.

We are going to investigate the transformation φ that maps the unit circle to the parameter plane which is the native plane of the main body of the Mandelbrot set. From the condition for the fixed points of

$$f(z) = z^{\frac{p}{q}} + c = z$$

we get the following prescription for the transformation φ:

$$c = \varphi(z) = z - z^{\frac{p}{q}}$$

for every $z \in \mathbb{C}$. We want the transformation φ from the dynamical plane to the parametric plane to be one-to-one. Therefore we should investigate our curve $c = \varphi(z)$ (or in a parametric form $c = \gamma(t)$) on a suitable Riemannian foliation. We have to be careful with the points that "contradict" the one-to-one property of the function:

$$z_1 \neq z_2 \text{ and } f(z_1) = f(z_2).$$

These points appear in the neighborhoods of the critical points. The point z_0 is a critical point for the transformation φ if $\varphi'(z_0) = 0$. If two different points in the neighborhood of the critical point are close enough to each other they might have the same value in respect of the transformation φ. The roots of the equation

$$\varphi'(z) = 1 - \frac{p}{q} z^{\frac{p-q}{q}} = 0, \quad \text{or} \quad z^{\frac{p-q}{q}} = \frac{q}{p}$$

are the points

$$z = \sqrt[q-p]{\left(\frac{p}{q}\right)^q}. \tag{2}$$

To make sure, that the transformation φ is one-to-one, we will eliminate the points $\varphi(z)$ from the parameter plane.

Beside these $|q-p|$ points, the point 0 should be removed because $0 = \varphi(0) = \varphi(1)$. Therefore $|q-p|+1$ points are to be cut out of the plane.

If we want the transformation φ^{-1} from the parametric to the dynamical plane to be one-to-one, another point should be removed: the point at infinity. The explanation of the statement is as follows: if we denote $\varphi(z) = w$, the next equalities are valid:

$$\left(\varphi \circ \varphi^{-1}\right)(w) = id(w) = w$$

and

$$\varphi'\left(\varphi^{-1}(w)\right)\left[\varphi^{-1}(w)\right]' = 1,$$

$$\left[\varphi^{-1}(w)\right]' = \frac{1}{\varphi'\left(\varphi^{-1}(w)\right)} = 0.$$

Hence w is a critical point for φ^{-1} when $\varphi'\left(\varphi^{-1}(w)\right) = \varphi'(z) = \infty$ and the only corresponding points are $z = 0$ in the case $p < q$ and $z = \infty$ in the case $p > q$.

5. The special Riemannian foliation - a home place for the main body of the function $\mathbf{f(z) = z^{\frac{p}{q}} + c}$, $\mathbf{p, q \in \mathbb{N}}$

We proceed with the description of a model of the Riemannian foliation on which our curves lay.

Our model of the Riemannian foliation looks like a tower build of complex planes laid one over the other (the coordinate axes of these planes coincide). The points $z = \sqrt[q-p]{\left(\frac{p}{q}\right)^q}$ are arranged symmetrically around the origin. These points are mapped by function φ into the points

$$\varphi(z) = z - z^{\frac{p}{q}} = \left(\frac{p}{q}\right)^{\frac{p}{q-p}} \left[\frac{p}{q} - 1\right]$$

which should be signed on each of the complex planes and connected with the point 0 by cuts. To remove the point at infinity we also make a cut from the origin along the positive part of the real axe.

We make these cuts on each of the complex planes that form the foliation. Then we connect every two neighboring planes as well as the last plane with the first one. The foliation consists of q planes if the number $|p-q|$ is odd. On the other hand, if the number $|p-q|$ is even, the foliation R is built of only $q-1$ planes (in this case the cut from the origin along the positive part of the real axe already eliminates one of the points $\varphi(z)$).

On the above described foliation our curve $c = \gamma(t)$ is continuous and it does not cross itself.

Remark 5.1. The curves created in Th. 3.1 and it's corollary lay on the described foliations. Each curve divides the foliation into two regions. Due to arbitrary convention the region which contains the origin is named interior of the curve and the remaining part of the foliation is called exterior of the curve. The function φ which maps the unit circle into the curve $c = \gamma(t)$ is continuous. The points inside the unit circle are mapped into the interior of the curve and the points laying outside the circle are mapped into the exterior of the curve on the foliation.

According to Remark 5.1 the $\varphi(z)$ values lay inside $(p > q)$ or outside $(p < q)$ of the area limited by the curve $c = \gamma(t)$. So we have to remove the point 0 *or* the point ∞. Hence in each case ($p > q$ or $p < q$) we eliminate $|q-p|+1$ points.

Because the described foliations are difficult to present in the space we will restrict ourselves to the projection of the curves $c = \gamma(t)$ on one of the complex planes which form the foliation.

6. Visualization for some special cases

At the end of the paper we illustrate the above results for functions $f(z) = z^{\frac{p}{q}} + c$, $p, q \in \mathbb{N}$ with some special values of the exponent $\frac{p}{q}$. The following pictures present the mentioned projections of the circle $\beta(t) = re^{it}$ (where r is a constant value, $r = 3$) mapped by the function φ (Figures 1 - 4):

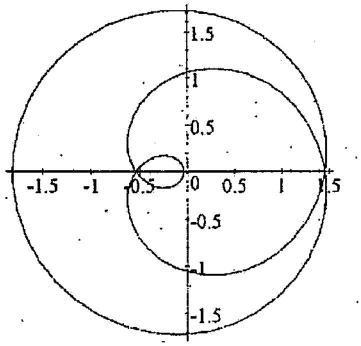

Fig. 1 $\left(\frac{p}{q}=\frac{2}{3}\right)$

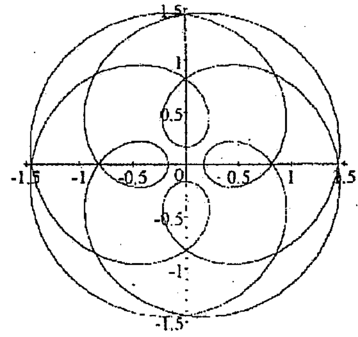

Fig. 2. $\left(\frac{p}{q}=\frac{3}{7}\right)$

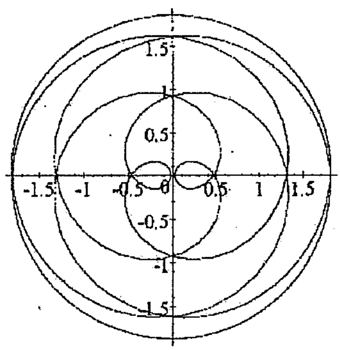

Fig. 3 $\left(\frac{p}{q}=\frac{5}{7}\right)$

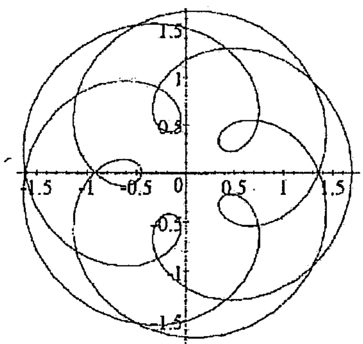

Fig. 4. $\left(\frac{p}{q}=\frac{3}{8}\right)$

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