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FAST FOURIER TRANSFORM FOR RATIONAL SYSTEMS

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Abstract: Conditionally orthogonal systems of rational functions are introduced and their product systems are examined. It is proved that under certain conditions these product systems form a discrete orthogonal system and the partial sums of their Fourier-series can be obtained in useful form. Also byorthogonal systems of this type are investigated.

Expansions by orthogonal and biorthogonal systems play an important role in mathematics and in applications. Several methods are available for constructing such systems. In a Hilbert space, for instance, the Gram-Schmidt method transforms a linearly independent system into an orthonormed one. Orthogonal polynomials, the Franklin–system and its generalizations, orthogonal systems consisting

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of rational functions (discrete Laguerre, Kautz, Malmquist–Takenaka systems) are examples that can be derived this way [2], [3].

In the middle of the 1970's the author introduced a new method for constructing orthogonal systems starting from some conditionally orthogonal functions [5], [6], [7]. Several classical systems, including the trigonometric, the Walsh system or the Vilenkin system, character systems of additive and multiplicative groups of local fields, UDMD and Walsh–similar systems, can all be constructed by using this method [10], [11],[13],[15], [16], [17]. One of the key concepts in our construction is the notion of product systems of conditionally orthogonal systems. These systems have important theoretical properties that are useful in numerical computations, too. For instance, Fourier–coefficients and partial sums can be computed by applying fast algorythms similar to FFT [8], [9], [12], [14].

In this paper we introduce conditionally orthogonal systems of rational functions, and examine their product systems. We prove that under certain conditions these product systems form a discrete orthogonal system and the partial sums of their Fourier–series can be obtained in a useful form. We also investigate biorthogonal systems of this type.

Expansion of this type can be used in control theory [2].

1. Introduction

In this section we recall some notions and results on UDMD systems introduced in [11], [13]. Fix a probability space $(X, \mathcal{A}, \mu)$. The conditional expectation (CE) of the function f with respect to the sub-σ-algebra $\mathcal{B} \subseteq \mathcal{A}$ is denoted by $E^{\mathcal{B}}f$. The L^q-space of $\mathcal{B}$-measurable functions will be denoted by $L^q(\mathcal{B}) := L^q(X, \mathcal{B}, \mu)$. Instead of $L^q(X, \mathcal{A}, \mu)$ we write L^q. It is well-known that for $1 \leq q \leq \infty$ the map $L^q \ni$ $\ni f \to E^{\mathcal{B}}f$ is a bounded linear projection onto $L^q(\mathcal{B})$ and $\|E^{\mathcal{B}}f\|_q \leq$ $\leq \|f\|_q$. The operator $E^{\mathcal{B}}$ is $\mathcal{B}$-homogeneous, i.e. if λ is $\mathcal{B}$-measurable and $f, \lambda f \in L^1$ then

$$E^{\mathcal{B}}(\lambda f) = \lambda E^{\mathcal{B}} f. \tag{1.1}$$

Furthermore, if $\mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$ are sub-σ-algebras of $\mathcal{A}$ then

$$E^{\mathcal{C}}(E^{\mathcal{B}}f) = E^{\mathcal{B}}(E^{\mathcal{C}}f) = E^{\mathcal{C}}f \tag{1.2}$$

for any $f \in L^1$ (see [4], [18]). We note that if $\mathcal{B} := \{X, \emptyset\}$ is the trivial σ-algebra then

$$E^{\mathcal{B}}f = \int_X f\,d\mu, \tag{1.3}$$

i.e. CE is a generalization of the integral.

The conditional expectation operator has a simple form if $\mathcal{B}$ is an atomic σ-algebra, i.e. if $\mathcal{B}$ is generated by the collection of pairwise disjoint sets $B_j \in \mathcal{B}$ $(j = 1, 2, \cdots, m)$:

$$\mathcal{B} := \sigma\{B_j : j = 1, 2, \cdots, m\},$$

$$B_i \cap B_j = \emptyset \quad (1 \le i < j \le m), \quad \cup_{j=1}^m B_j = X.$$

The sets B_j $(j = 1, \cdots, m)$ are called the atoms of $\mathcal{B}$ and the $\mathcal{B}$-measurable functions are exactly the step functions, constant on the B_j's. This m-dimensional space coincides with L^1. Denote the collection of atoms in $\mathcal{B}$ by $\hat{\mathcal{B}}$. Then the conditional expectation is of the form

$$(E^{\mathcal{B}}f)(x) = \frac{1}{\mu(B)} \int_B f\,d\mu \quad (x \in B \in \hat{\mathcal{B}}). \tag{1.4}$$

The fact that CE has properties similar to those of integral makes it possible to extend several concepts connected with the integral to CE. For example a finite or infinite system of functions $\phi_n \in L^2$ $(n \in \mathbb{N})$ is called a $\mathcal{B}$-orthonormal system ($E^{\mathcal{B}}$-ONS) if

$$E^{\mathcal{B}}(\phi_k \overline{\phi}_\ell) = \delta_{k\ell} \quad (k, \ell \in \mathbb{N}), \tag{1.5}$$

where $\delta_{k\ell}$ is the Kronecker-symbol. If $\mathcal{B} := \{X, \emptyset\}$ is the trivial σ-algebra then we get the usual definition of ONS. Moreover, (1.2) and (1.3) imply that each $E^{\mathcal{B}}$-ONS is an ONS in the usual sense.

Replacing the integral by CE in the definitions of Fourier-coefficients and Fourier partial sums we get the following: The function $E^{\mathcal{B}}(f\overline{\phi}_n)$ is called the n-th $\mathcal{B}$-Fourier coefficient of the function f with respect to the system $(\phi_n, n \in \mathbb{N})$. In the case $\mathcal{B} := \{X, \emptyset\}$ these notions coincide with the usual definitions of Fourier-coefficient. The concept of $\mathcal{B}$-biorthogonal systems and Fourier-coefficients with respect $\mathcal{B}$-biorthogonal systems can be defined in a similar way.

For the definition of product systems (see [1]) we fix a collection of function systems

$$\Phi_k := \{\phi_k^\ell : \ell = 0, 1, \cdots, p-1\} \subset L^2 \quad , \tag{1.6}$$

where $p \ge 2, p \in \mathbb{N}$ is a fixed number and $0 \le k < N \le \infty$. We shall use the expansion of natural numbers with respect to the base p. It is well-known that every number $m \in \mathbb{N}, m < p^N$ can uniquely be written in the form

$$m = \sum_{0 \le k < N} m_k p^k, \tag{1.7}$$

where $m_k \in \{0, 1, \cdots p-1\}$. Then for each $0 \le m < p^N$ we define the product

$$\psi_m := \prod_{0 \le k < N} \phi_k^{m_k}, \tag{1.8}$$

provided that in the case $N = \infty$ the infinite product in (1.8) converges. The system $\Psi = \{\psi_m : 0 \le m < p^N\}$ is called the product system of the systems Φ_k $(0 \le k < N)$. In the special case $p = 2$ (1.7) is the dyadic reprezentation of $m \in \mathbb{N}$.

In order to get orthonormed product systems we fix a stochastic basis, i.e. an increasing sequence of sub-σ-algebras of $\mathcal{A}$:

$$\mathcal{A}_0 := \{X, \emptyset\} \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_n \subset \cdots \subset \mathcal{A}, \tag{1.9}$$

and a sequence Φ_k $(0 \le k < N)$ of adapted conditionally orthonormal systems (AC-ONS). This means that the functions in Φ_k are $\mathcal{A}_{k+1}$-measurable and Φ_k is $\mathcal{A}_k$-orthonormed:

$$\text{(1.10)} \qquad \begin{aligned} &\text{i)} \quad \Phi_k \subset L^2(\mathcal{A}_{k+1}) \quad (k < N), \\ &\text{ii)} \quad E_k(\phi_k^i \overline{\phi_k^j}) = \delta_{ij} \quad (0 \le i, j < p, k < N), \end{aligned}$$

where E_k denotes the conditional expectation with respect to $\mathcal{A}_k$.

The stochastic basis (1.9) is called dyadic if for every $k < N$ the σ-algebra $\mathcal{A}_k$ is atomic and every atom $I \in \hat{\mathcal{A}}_k$ can be split into two atoms $I', I'' \in \hat{\mathcal{A}}_{k+1}$, such that $\mu(I') = \mu(I'')$. If $p = 2$ and $\phi_k^0 = 1$ $(0 \le k < N)$, then (1.10) ii) means that $\phi_k := \phi_k^1$ $(0 \le k < N)$ is a martingale difference sequence with respect to the stochastic base $(\mathcal{A}_{k+1}, k = 0, 1, \cdots)$. If $|\phi_k| = 1$ then the system $(\phi_k, k = 0, 1, \cdots)$ is called a system of unitary dyadic martingale differences, or UDMD-system and the system Ψ is called an UDMD product system.

It is known (see [8], [13], [14]) that conditions (1.10) imply that the product system Ψ is an ONS.

Theorem A. *Let Ψ be the product system of a finite* AC-ONS *satisfying* (1.10). *Then Ψ is an orthonormed system.*

In this paper we investigate only finite product systems, i.e. we suppose that $N < \infty$ and $\mathcal{A} = \mathcal{A}_N$ is the collection of subset of X. If

$p = 2$ then X has 2^N and the Fourier-coefficients with respect to the system Ψ can be written in the form

$$(1.11)\quad \hat{f}(k) = [f, \psi_k] = 2^{-N} \sum_{x \in X} f(x)\overline{\psi}_k(x) \quad (k = 0, 1, \cdots, 2^N - 1).$$

Furthermore, each function $f : X \to \mathbb{C}$ can be reconstructed from $\hat{f}$ by

$$(1.12)\qquad f(x) = \sum_{k=0}^{2^N-1} \hat{f}(k)\psi_k(x) \quad (x \in X).$$

In order to compute the Ψ-Fourier coefficients of a function f or to reconstruct f from $\hat{f}$ by formula (1.11) and (1.12) one needs $2^N \cdot 2^N$ multiplications and $2^N(2^N - 1)$ additions. In the trigonometric case, there is an algorithm which computes the discrete Fourier coefficients using $N2^N$ algebraic operations (additions or multiplications). This algorithm is called the Fast Fourier Transform or, more briefly, FFT. It was showed (see [9], [10], [12], [13], [14]) that such an algorithm exists for any Ψ-transform provided Ψ is a product system of systems satisfying (1.10)i).

For $0 \le m < 2^N$ denote

$$\hat{f}_n(m) := E_n(f\overline{\psi}_m) \quad (0 \le n < N)$$

the m-th $\mathcal{A}_n$ Fourier-coefficient with respect to the system Ψ and set

$$m^n := \sum_{k=n}^{N-1} m_k 2^k \quad (0 \le n < N).$$

By (1.1), (1.2), (1.8) and (1.10)ii)

$$(1.13)\qquad \hat{f}_n(m^n) = E_n(\overline{\phi}_n^{m_n} E_{n+1}(f\overline{\psi}_{m^{n+1}})) \quad (0 \le n < N),$$

where in the case $n = N$ we set $\psi_{m^N} = 1$, and consequently $E_N(f\overline{\psi}_{m^N}) = f$.

The $\mathcal{A}_n$ measurable function g is constant on each $I \in \hat{\mathcal{A}}_n$ atom. The value of g at the points of I will be denoted by $g(I)$. Especially, the value of $\hat{f}_n(m^n)$ at I is denoted by $\hat{f}_n(m^n, I)$. Let $I = I' \cup I''$ be the decomposition of $I \in \hat{\mathcal{A}}_n$ into two $\mathcal{A}_{n+1}$ atoms. Then (1.13) can be written in the form

(1.14)
$$\hat{f}_n(m^n, I) = \frac{\overline{\phi}_n^{m_n}(I')\hat{f}_{n+1}(m^{n+1}, I') + \overline{\phi}_n^{m_n}(I'')\hat{f}_{n+1}(m^{n+1}, I'')}{2}$$
$$(0 \le n < N, 0 \le m < 2^N, I = I' \cup I'', I \in \hat{\mathcal{A}}_n, I', I'' \in \hat{\mathcal{A}}_{n+1}).$$

The numbers of atoms in $\hat{\mathcal{A}}_n$ is 2^n and the numbers of m_n's is 2^{N-n}. Thus computing the E_n Fourier coefficients on the basis (1.14) we need 2^N operations. Starting with the function $\hat{f}_N = f$ and applying the recursions (1.14) we get the Fourier coefficients $\hat{f} = \hat{f}_0$ in N steps. The number of operations in this algorithm is $N2^N$. The algorithm suggested by (1.14) can be organized so that the calculations and the storage used is mimimized (see [14]).

2. Rational UDMD systems

In this section we investigate discrete conditionally orthonormal systems constructed by rational functions. To this end denote $\mathbb{C}$ the set of complex numbers and let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unite disc. In our construction the Blaschke functions

$$B_b(z) := \frac{z - b}{1 - \bar{b}z} \quad (z \in \mathbb{C}) \tag{2.1}$$

play a basic role. If the parameter b belongs to $\mathbb{D}$ then the restriction of B_b to $\mathbb{D}$ is a bijection of $\mathbb{D}$. Furthermore B_b is a $1-1$ map on the unite circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Functions of the form

$$G(z) = cB_{b_1}(z)B_{b_2}(z)\cdots B_{b_n}(z) \quad (z \in \mathbb{C}, c \in \mathbb{T}, n \in \mathbb{N}^* := \{1, 2, \cdots\})$$

are called Blaschke products of order n. It can be showed (see Lemma 1 in section 3), that for every $w \in \mathbb{T}$ the equation $G(z) = w$ has n pairwise distinct solutions belonging to $\mathbb{T}$.

For every couple $a = (a^{(1)}, a^{(2)}) \in \mathbb{D}^2 := \mathbb{D} \times \mathbb{D}$ we introduce the Blaschke products of order two:

$$A_a(z) := B_{a^{(1)}}(z)B_{a^{(2)}}(z) \quad (z \in \mathbb{C}, a = (a^{(1)}, a^{(2)}) \in \mathbb{D}^2). \tag{2.2}$$

For the definition of the discrete probability space we fix the sequence $a_n \in \mathbb{D}^2$ $(n \in \mathbb{N} := \mathbb{N}^* \cup \{0\})$ of couples and for all $n \in \mathbb{N}$ we introduce the functions

$$\varphi_0(z) := B_0(z) = z \quad (z \in \mathbb{C}), \tag{2.3}$$
$$\varphi_n := A_{a_{n-1}} \circ \cdots \circ A_{a_1} \circ A_{a_0} \quad (n \in \mathbb{N}^*),$$

which map $\mathbb{D}$ onto $\mathbb{D}$ and $\mathbb{T}$ onto $\mathbb{T}$. Here $\circ$ stands for composition of functions. It will be proved (see Lemma 2 in Section 3) that φ_n is a Blaschke product of order 2^n. Fix the numbers $N \in \mathbb{N}^*$, $w \in \mathbb{T}$ and introduce the set

$$X := X_N^w := \{w \in \mathbb{C} : \varphi_N(z) := w\}.$$

Then (see Lemma 1 and Lemma 2) $X \subset \mathbb{T}$ and X has 2^N elements. Denote $\mathcal{A} = \mathcal{A}_N$ the collection of subsets of X and denote μ the discrete probability measure defined by

$$\mu(H) := 2^{-N} \sum_{x \in H} 1 \quad (H \in \mathcal{A}). \tag{2.4}$$

For $n = 1, \cdots, N$ let $\mathcal{A}_n$ denote the σ-algebra generated by the function

$$\phi_{n-1} := \varphi_{N-n}.$$

Since $\phi_{n-1} = A_{a_{N-n-1}} \circ \phi_n$ $(n = 1, 2, \cdots, N)$ we have

$$\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_{N-1} \subset \mathcal{A}_N = \mathcal{A}. \tag{2.5}$$

It is easy to see that for $n = 1, 2, \cdots, N$ the range of ϕ_{n-1} is the set

$$Y_n := \{z \in \mathbb{C} : (A_{a_{N-1}} \circ A_{a_{N-2}} \circ \cdots \circ A_{a_{N-n}})(z) = w\}$$

with 2^n elements. Thus the σ-algebra $\mathcal{A}_n$ is atomic with the atoms

$$\phi_{n-1}^{-1}(y) := \{x \in X : \phi_{n-1}(x) = y\} \quad (y \in Y_n), \tag{2.6}$$
$$\hat{\mathcal{A}}_n := \{\phi_{n-1}^{-1}(y) : y \in Y_n\} (\subset \mathcal{A}_n).$$

Obviously $\mu(I) = 2^{-n}$ for every $I \in \hat{\mathcal{A}}_n$. Furthermore, every atom $I \in \hat{\mathcal{A}}_n$ is of the form

$$I = I' \cup I'', \quad \text{where} \quad I', I'' \in \hat{\mathcal{A}}_{n+1}, \quad I' \cap I'' = \emptyset, \quad \text{and} \tag{2.7}$$
$$A_{a_{N-n-1}}(\phi_n(I'')) = A_{a_{N-n-1}}(\phi_n(I')) = \phi_{n-1}(I),$$

and (1.10)ii) is equivalent to

$$E_n(\phi_n^k \overline{\phi}_n^\ell)(I) = \frac{1}{2}\Big((\phi_n^k \overline{\phi}_n^\ell)(I') + (\phi_n^k \overline{\phi}_n^\ell)(I'')\Big). \tag{2.8}$$

Let $p = 2$ and denote $\Psi := \Psi_N := (\psi_n, n = 0, 1, \cdots, 2^N - 1)$ the product system of

$$\Phi_n := \{1, \phi_n\} \quad (n = 0, 1, \cdots, N-1).$$

If $a_n^{(1)} = -a_n^{(2)}$ then $\phi_n(I') = -\phi_n(I'')$ by Lemma 1. Therefore, we have by (2.8) that the system Φ_n is $\mathcal{A}_n$-orthonormed, moreover ϕ_n $(n = = 0, 1, \cdots, N-1)$ is an UDMD system. This implies the following

Theorem 1. *Suppose that the generating sequence $a_n \in \mathbb{D}^2$ satisfies $a_n^{(1)} = -a_n^{(2)}$. Then Φ_n is an* AC-ONS *of rational functions and consequently the product system Ψ is a discrete rational orthonormal system. The Fourier coefficients with respect to the system Ψ can be computed by the the fast algorithm* (1.14) *using $O(N2^N)$ operations.*

In the general case a biortogonal system $\widetilde{\Psi}_N$ to Ψ_N can be constructed. Let $I = I' \cup I''$ the decomposition of the atom $I \in \hat{\mathcal{A}}_n$. Observe that by (2.7) and by Lemma 1 $\phi_n(I') \neq \phi_n(I'')$. To define the system $\widetilde{\Phi}_n := \{\widetilde{\phi}_n^0, \widetilde{\phi}_n^1\}$ biorthogonal to the system $\Phi_n := \{\phi_n^0, \phi_n^1\} := := \{1, \phi_n\}$ set $\gamma_I := 2/(\phi_n(I'') - \phi_n(I'))$ and let

$$\widetilde{\phi}_n^0(I') = \overline{\gamma}_I \overline{\phi}_n(I''), \quad \widetilde{\phi}_n^0(I'') = -\overline{\gamma}_I \overline{\phi}_n(I')$$
$$\widetilde{\phi}_n^1(I') = -\overline{\gamma}_I, \qquad \widetilde{\phi}_n^1(I'') = \overline{\gamma}_I$$

and consequently

$$E_n(\widetilde{\phi}_n^k \overline{\phi}_n^\ell) = \delta_{k\ell} \quad (k, \ell = 0, 1).$$

Then we get (see also [12])

Theorem 2. *The product system $\widetilde{\Psi}$ of the systems $\widetilde{\Phi}_n$ $(n = 0, 1, \cdots \cdots, N-1)$ is biorthogonal to Ψ, i.e.*

$$[\psi_k, \widetilde{\psi}_\ell] = \delta_{k\ell} \quad (0 \le k, \ell < 2^N).$$

The Fourier coefficients with respect to the system $\widetilde{\Psi}$ can be computed by the the fast algorithm (1.14) *using $O(N2^N)$ operations.*

3. Properties of the Blaschke products

In our constructions we used the following properties of the Blaschke products.

Lemma 1. *Let $b_1, b_2, \cdots, b_n \in \mathbb{D}$ $(n \in \mathbb{N}^*)$ and denote*

$$G(z) := B_{b_1}(z) B_{b_2}(z) \cdots B_{b_n}(z) \quad (z \in \mathbb{C})$$

a Blaschke product of order n. Then for every $w \in \mathbb{T}$ the equation

$$G(z) = w \tag{3.1}$$

has exactly n solutions, and the solution z_j $(j = 1, 2, \cdots, n)$ satisfy

$$z_j \in \mathbb{T}, \quad z_j \neq z_k \quad (1 \leq j, k \leq n, j \neq k).$$

If $n = 2$ *and* $b_1 = -b_2$ *then* $z_1 = -z_2$.

Proof. Equation (3.1) is eqivalent to

$$(1 - w\bar{b}_1 \cdots \bar{b}_n)z^n + \cdots + (-1)^n b_1 \cdot \cdots \cdot b_n - w = 0. \tag{3.2}$$

Consequently, it has at most n distinc solution. In order to prowe that the number of the solutions is n we use the fact that B_b is an $1-1$ map of $\mathbb{T}$ whenever $b \in \mathbb{D}$. Moreover, on $\mathbb{T}$ the map B_b is of the form

$$B_b(e^{it}) = e^{i\gamma_b(t)} \quad (t \in \mathbb{R}),$$

where $\gamma_b : \mathbb{R} \to \mathbb{R}$ is a strictly increasing continuous function such that $\gamma_b(t + 2\pi) = \gamma_b(t) + 2\pi \quad (t \in \mathbb{R})$ (see [2]). Set $w = e^{i\delta}$ and $z = e^{it}$ $(t, \delta \in [0, 2\pi))$. Then (3.1) is equivalent to

$$e^{i\gamma(t)} = e^{i\delta}, \quad \text{where} \quad \gamma = \gamma_{b_1} + \cdots + \gamma_{b_n}.$$

Obviously, $\gamma(2\pi) - \gamma(0) = n2\pi$. Let

$$k_0 := \min\{k \in \mathbb{Z} : \delta + k2\pi \geq \gamma(0)\},$$

and introduce the sequence $\tau_j := \delta + (k_0 + j)2\pi \ (j \in \mathbb{N})$. Then by definition $\gamma(0) \leq \tau_j < \gamma(2\pi) \ (0 \leq j < n)$. Consequently, the numbers $t_j := \gamma^{-1}(\tau_j) \ (0 \leq j < n)$ satisfy

$$0 \leq t_0 < t_1 < \cdots < t_{n-1} < 2\pi.$$

Set $z_j := e^{it_{j-1}} \ (j = 1, 2, \cdots, n)$. Then

$$G(z_j) = G(e^{it_{j-1}}) = e^{i\gamma(t_{j-1})} = e^{i\tau_{j-1}} = e^{i(\delta+(k_0+j-1)2\pi)} = e^{i\delta} = w,$$

and the numbers z_j are the solutions of (3.1).

If $n = 2$ then (3.2) is of the form

$$(1 - w\bar{b}_1\bar{b}_2)z^2 - (b_1 + b_2 - w(\bar{b}_1 + \bar{b}_2)z + b_1b_2 - w = 0.$$

Hence it is clear that if $b_1 = -b_2$ then the solutions satisfy $z_1 = -z_2$. ◊

The set of finite Blaschke products is closed under multiplication and it is also closed under compositions of functions $\circ$. Namely, the following lemma holds.

Lemma 2. *Let* $a_1, a_2, \cdots, a_n \in \mathbb{D}^2$ $(n \in \mathbb{N}^*)$. *Then the function*

$$F(z) := A_{a_1} \circ A_{a_2} \circ \cdots \circ A_{a_n} \tag{3.3}$$

is a Blaschke product of order 2^n, *i.e. there exist* 2^n *numbers* $\alpha_j \in \mathbb{D}$ $(j = 0, 1, \cdots, 2^n - 1)$ *and a number* $c \in \mathbb{T}$ *such that*

$$F = c \prod_{j=0}^{2^n-1} B_{\alpha_j}. \tag{3.4}$$

Proof. First we show that for every $b \in \mathbb{D}$ and $a = (a_1, a_2) \in \mathbb{D}^2$ there exist two numbers $\alpha_1, \alpha_2 \in \mathbb{D}$ and a number $\epsilon \in \mathbb{T}$ such that

$$B_b \circ A_a = \epsilon B_{\alpha_1} B_{\alpha_2}. \tag{3.5}$$

It is easy to see that

$$\begin{aligned} B_b(A_a(z)) &= \frac{(1 - b\overline{a}_1\overline{a}_2)z^2 - (a_1 + a_2 - b(\overline{a}_1 - \overline{a}_2))z + (a_1a_2 - b)}{(\overline{a}_1\overline{a}_2 - \overline{b})z^2 - (\overline{a}_1 + \overline{a}_2 - \overline{b}(a_1 + a_2)) + (1 - \overline{b}a_1a_2)} = \\ &= \epsilon \frac{z^2 - pz + q}{\overline{q}z^2 - \overline{p}z + 1}, \end{aligned}$$

where

$$p := \frac{a_1 + a_2 - b(\overline{a}_1 + \overline{a}_2)}{1 - b\overline{a}_1\overline{a}_2}, \quad q := \frac{a_1a_2 - b}{1 - b\overline{a}_1\overline{a}_2}, \quad \epsilon := \frac{1 - b\overline{a}_1\overline{a}_2}{1 - \overline{b}a_1a_2}.$$

Let α_1 and α_2 denote the roots of the equation $z^2 - pz + q = 0$. Then

$$z^2 - pz + q = (z - \alpha_1)(z - \alpha_2) = z^2 - (\alpha_1 + \alpha_2)z + \alpha_1\alpha_2 \quad (z \in \mathbb{C}).$$

Consequently

$$(1 - \overline{\alpha}_1 z)(1 - \overline{\alpha}_2 z) = 1 - \overline{p}z + \overline{q}z^2 \quad (z \in \mathbb{C}).$$

Thus we get

$$B_b(A_a(z)) = \epsilon \frac{(z - \alpha_1)(z - \alpha_2)}{(1 - \overline{\alpha}_1 z)(1 - \overline{\alpha}_2 z)} \quad (z \in \mathbb{C}).$$

Since the left hand side is bounded on the closed disc $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq \leq 1\}$ the poles of the right hand side are outside of $\overline{\mathbb{D}}$, i.e. $|\overline{\alpha}_j|^{-1} > 1$ $(j = 1, 2)$. Thus $\alpha_j \in \mathbb{T}$ $(j = 1, 2)$ and (3.5) is proved.

Let $n = 2$. Then

$$A_{a_1}(A_{a_2}(z)) = B_{a_1^{(1)}}(A_{a_2}(z)) B_{a_1^{(2)}}(A_{a_2}(z)) \quad (z \in \mathbb{C}).$$

Applying (3.5) we get that there exist $\epsilon_j \in \mathbb{T}$ $(j = 1, 2)$ and α_k $(k = = 0, 1, 2, 3)$ such that

$$A_{a_1}(A_{a_2}(z)) = \epsilon_1\epsilon_2 \prod_{k=0}^{3} B_{\alpha_k}(z) \quad (z \in \mathbb{C})$$

and (3.4) is proved for $n = 2$.

The general case can be proved in a similar way by induction. ◊

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