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COERCIVITY OF SET-VALUED MAPPINGS ON METRIC SPACES

A. **Kristály**

Department of Mathematics, Babeş-Bolyai University, RO-3400 Cluj-Napoca, Romania

Cs. **Varga**

Department of Mathematics, Babeş-Bolyai University, RO-3400 Cluj-Napoca, Romania

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Abstract: The paper establishes characterization of coercivity of set-valued mappings on metric spaces versus the Palais-Smale condition, introducing the notion of the slope. Comparisons with other Palais-Smale conditions are proved also.

1. Introduction

The relation between the coercivity and the suitable Palais-Smale condition was treated in many papers, see [8], [2], [4], [6], [9], [3], [5] and the references therein. The basic result is the following:

Let $(X, \|\cdot\|)$ be a Banach space and $f : X \to \mathbb{R}$ be bounded bellow, differentiable function which satisfies the Palais-Smale condition. Then

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E-mail addresses: akristal@math.ubbcluj.ro, csvarga@cs.ubbcluj.ro

f is coercive, that is, $f(x)$ goes to infinity as $\|x\|$ goes to infinity.

The above cited works are extensions of this result. The main object of this paper is to obtain a set-valued version of the above result in metric spaces. Here, we introduce the notion of slope of a set-valued mapping. The main tool in the proof is the well-known Ekeland's variational principle.

The paper is organized as follows. In Section 2 we introduce the slope of a set-valued mapping on a metric space and we compare it with the contingent derivative, see [1]. According to this new notion, we can define the corresponding Palais-Smale condition. Here, we treat also the relations between different Palais-Smale conditions. In Section 3 we establish the main result of this note, which states the equivalence between our Palais-Smale condition and coercivity. Of course, this result contains the above basic result and a special form of coercivity results from [8], [6] and [3].

2. Palais-Smale conditions

First, we recall some definitions.

Definition 2.1. Let X be a Banach space and $f : X \to \mathbb{R}$ be a continuous differentiable function. We say that f *satisfies condition* (PSB) (resp., condition (PS)), if whenever $\{u_n\} \subset X$ is a sequence such that $\{f(u_n)\}$ is bounded and $\|f'(u_n)\|_{X^*} \to 0$, then $\{u_n\}$ is bounded (resp., $\{u_n\}$ contains a convergent subsequence.)

The following class of functionals is introduced in [10] by A. Szulkin.

Let X be a normed space and $I : X \to (-\infty, +\infty]$ be a functional satisfying the following structural condition:

(H) $I = f + \psi$, with $f : X \to \mathbb{R}$ of class C^1 and $\psi : X \to$ $\to (-\infty, +\infty]$ proper, convex and lower semicontinuous.

Definition 2.2. The functional $I : X \to (-\infty, +\infty]$ in (H) *satisfies condition* $(Sz - PSB)$ (resp., $(Sz - PS)$), if whenever $\{u_n\} \subset X$ is a sequence such that $\{I(u_n)\}$ is bounded and

$$f'(u_n)(v - u_n) + \psi(v) - \psi(u_n) \geq -\varepsilon_n \|v - u_n\|, \ (\forall)\ v \in X,$$

for a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+$ with $\lim\limits_{n\to\infty} \varepsilon_n = 0$, then $\{u_n\}$ is bounded (resp., $\{u_n\}$ contains a convergent subsequence).

Our aim is to give a set-valued version of the above Palais-Smale conditions on metric spaces and to treat the connection between these notions. First of all, we need some notions and definitions from the set-valued analysis.

Let (M, d) be a metric space and $F : M \rightsquigarrow \mathbb{R}$ be a set-valued map with nonempty values. The graph of the map F is defined by

$$\operatorname{Graph}(F) = \{\ (u, c) \in M \times \mathbb{R} \mid c \in F(u)\ \}.$$

Definition 2.3 (see [1, Def. 1.4.6]). We say that

$$\operatorname*{Limsup}_{x' \to x} F(x') := \left\{ y \in \mathbb{R}\ \mid \liminf_{x' \to x} \operatorname{dist}(y, F(x')) = 0 \right\}$$

is the *upper limit* of $F(x')$ when $x' \to x$.

Definition 2.4. Let X be a normed vector space, K a subset of X and $x \in \overline{K}$ ($\overline{K}$ being the closure of K). The *contingent cone* $T_K(x)$ is defined by

$$T_K(x) = \{v \in X \mid \liminf_{h \to 0^+} \operatorname{dist}(x + hv, K)/h = 0\}.$$

Definition 2.5 [1, pp. 181]. Let X be a normed space, $F : X \rightsquigarrow \mathbb{R}$ be a set-valued map and $y \in F(x)$. The *contingent derivative* $DF(x, y)$ is defined by

$$\operatorname{Graph}(DF(x, y)) := T_{\operatorname{Graph}(F)}(x, y).$$

Definition 2.6. Let (M, d) be a metric space.

(i) $F : M \rightsquigarrow \mathbb{R}$ is *Lipschitz around* $x \in M$ if there exist a positive constant L and a neighborhood U of x such that

$$\forall\ x_1, x_2 \in U, \quad F(x_1) \subset F(x_2) + Ld(x_1, x_2)[-1, 1].$$

(ii) $F : M \rightsquigarrow \mathbb{R}$ is *upper semicontinuous at* x if for any neighborhood U of $F(x)$, $\exists\ \eta > 0$ such that for every $x' \in B_M(x, \eta) = \{y \in M : : d(x, y) < \eta\}$ we have $F(x') \subset U$.

(iii) F is *locally Lipschitz* (resp., *upper semicontinuous*) if it is Lipschitz around all $x \in M$ (resp., upper semicontinuous in all $x \in M$).

Clearly, if F is Lipschitz around x with compact values, then it is also upper semicontinuous at x, see [7].

Remark 2.1. Let X be a normed space. Using the above definitions and providing that F is Lipschitz around $x \in X$, it is possible to characterize the contingent derivative by

$$DF(x, y)(u) = \operatorname*{Limsup}_{h \to 0^+} \frac{F(x + hu) - y}{h},$$

see [1, Prop. 5.1.4].

Definition 2.7. Let (M, d) be a metric space and $F : M \rightsquigarrow \mathbb{R}$ be a set-valued map with non-empty values. Let $(x, y) \in \operatorname{Graph}(F)$. The *slope* $|\nabla F|(x, y)$ is defined by

$$|\nabla F|(x,y) := \operatorname*{Limsup}_{w\to x} \frac{F(w)-y}{d(x,w)}.$$

Now, we compare the slope and the contingent derivative.

Proposition 2.1. *Let X be a normed space and $F : X \rightsquigarrow \mathbb{R}$ be Lipschitz around $x \in X$. Then, for all $u \in X \setminus \{0\}$ and $y \in F(x)$ we have*

$$DF(x,y)(u) \subseteq |\nabla F|(x,y) \cdot \|u\|.$$

Proof. Let $u \neq 0$ be fixed and $v \in DF(x,y)(u)$. From the Remark 2.1., we have $\liminf\limits_{h\to 0^+} \operatorname{dist}\left(v, \frac{F(x+hu)-y}{h}\right) = 0$. This is equivalent by $\liminf\limits_{h\to 0^+} \operatorname{dist}\left(\frac{v}{\|u\|}, \frac{F(x+hu)-y}{h\|u\|}\right) = 0$. Let $w_h := x + hu$, $h > 0$. Clearly, $h \to 0^+$ iff $w_h \to x$. Therefore, $\liminf\limits_{w_h\to x} \operatorname{dist}\left(\frac{v}{\|u\|}, \frac{F(w_h)-y}{d(w_h,x)}\right) = 0$. From this, we obtain that $\liminf\limits_{w\to x} \operatorname{dist}\left(\frac{v}{\|u\|}, \frac{F(w)-y}{d(w,x)}\right) = 0$. Therefore, we get $\frac{v}{\|u\|} \in |\nabla F|(x,y)$. $\diamond$

Definition 2.8. Let (M,d) be a metric space and $g : M \to \mathbb{R}$ be a function. A subset M_0 of M is *g-bounded* if there exists $K > 0$ such that $|g(x)| \leq K, \forall x \in M_0$.

Now, we introduce the suitable Palais-Smale conditions to the contingent derivative resp., to the slope.

Definition 2.9. Let X be a normed space, $F : X \rightsquigarrow \mathbb{R}$ be a set-valued function with non-empty values and $g : X \to \mathbb{R}$ be a function. F *satisfies the condition* $(D - PSB - g)$ (resp. $(D - PS)$), if whenever $\{u_n, v_n\} \subset \operatorname{Graph}(F)$ is a sequence such that

$$DF(u_n,v_n)(u-u_n) + \varepsilon_n\|u-u_n\| \subseteq \mathbb{R}_+, \ \forall\, u \in X$$

for a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+$ with $\lim\limits_{n\to\infty} \varepsilon_n = 0$, and $\{v_n\}$ is bounded, then $\{u_n\}$ is g-bounded (resp., $\{u_n\}$ contains a convergent subsequence).

Definition 2.10. Let (M,d) be a metric space, $F : M \rightsquigarrow \mathbb{R}$ be a set-valued function with non-empty values and $g : M \to \mathbb{R}$ be a function. F *satisfies the condition* $(\nabla - PSB - g)$ (resp. $(\nabla - PS)$), if whenever $\{u_n, v_n\} \subset \operatorname{Graph}(F)$ is a sequence such that

$$|\nabla F|(u_n,v_n) + \varepsilon_n \subseteq \mathbb{R}_+$$

for a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+$ with $\lim\limits_{n\to\infty} \varepsilon_n = 0$, and $\{v_n\}$ is bounded, then $\{u_n\}$ is g-bounded (resp., $\{u_n\}$ contains a convergent subsequence).

Remark 2.2. Let $(X, \|\cdot\|$ be a normed space, and $F(x) = \{f(x)\}$ is single-valued, f being of class C^1.

(I) The contingent derivative reduces to the classical differential, i.e.

$$DF(x, f(x))(u) = f'(x)(u), \forall u \in X$$

see [1, Prop. 5.1.2]. Therefore the condition $(D - PSB - \|\cdot\|)$ (resp., $(D - PS)$) is exactly the $(Sz - PSB)$ (resp., $(Sz - PS)$) with $\psi \equiv 0$.

(II) Moreover, (PSB) (resp., (PS)) implies $(\nabla - PSB - \|\cdot\|)$ (resp., $(\nabla - PS)$). Indeed, since $F = f$ is of class C^1, then it is locally Lipschitz, therefore from Prop. 2.1. we have

$$f'(x)(u) \in |\nabla F|(x, f(x)) \cdot \|u\|, \ \forall u, x \in X \tag{2.1}$$

(u can be 0 also). Now, let a sequence $\{u_n\}$ such that $|\nabla F|(u_n, f(u_n)) + \varepsilon_n \subseteq \mathbb{R}_+$ for a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+$ with $\lim_{n\to\infty} \varepsilon_n = 0$ and $\{f(u_n)\}$ is bounded. Multiplying the above inclusion by $\|u - u_n\|$ and using the (2.1) we obtain that $f'(u_n)(u - u_n) + \varepsilon_n\|u - u_n\| \subseteq \mathbb{R}_+, \forall u \in X$, i.e. $f'(u_n)(u) + \varepsilon_n\|u\| \geq 0, \forall u \in X$. From this, we get $\|f'(u_n)\|_{X^*} \leq \varepsilon_n$. Since $\varepsilon_n \to 0$, we obtain the desired relations.

3. Coercivity result

In the sequel, we use the Ekeland variational principle to establish the main result of this paper. In its strong form, Ekelands's principle can be stated as follows:

Let (M, d) be a complete metric space and $\Phi : M \to \mathbb{R}$ be a lower semicontinuous function which is bounded below, say $a = \inf_M \Phi$. Let $\varepsilon > 0$ be given and $u \in M$ be such that $\Phi(u) \leq a + \varepsilon$.

Then, for any $\lambda > 0$, there exists $v \in M$ such that

(i) $\Phi(v) \leq \Phi(u)$,

(ii) $d(v, u) \leq \lambda$,

(iii) $\Phi(v) < \Phi(w) + (\varepsilon/\lambda)d(v, w), \ \forall w \neq v$.

Lemma 3.1. *Let (M, d) be a complete metric space, $F : M \rightsquigarrow \mathbb{R}$ be an upper semicontinuous set-valued mapping with compact and non-empty values, such that $\inf F(M) > -\infty$ and a Lipschitz continuous function $g : M \to \mathbb{R}$. Define $c = \liminf_{|g(u)|\to\infty} \min F(u)$. Then, if $c \in \mathbb{R}$, there exists a sequence $\{v_n\} \subset M$ such that:*

(i) $|g(v_n)| \to +\infty$,

(ii) $\min F(v_n) \to c$,

(iii) $|\nabla F|(v_n, \min F(v_n)) + \varepsilon_n \subseteq \mathbb{R}_+$, *where* $\varepsilon_n \to 0^+$.

Proof. From the definition of c, there exists a sequence $\{u_n\} \subset M$ such that

$$\min F(u_n) \leq c + \frac{1}{n} \tag{3.1}$$

and

$$|g(u_n)| \geq (L+1)n, \tag{3.2}$$

where $L > 0$ is the Lipschitz constant of g. The function $\Phi : M \to \mathbb{R}$, defined by $\Phi(u) = \min F(u)$, $u \in M$ is lower semicontinuous, (see [1, Th. 1.4.16] for $f : \text{Graph}(F) \to \mathbb{R}$, $f(x, y) = -y$). Now, we apply the Ekeland's principle for Φ, $\varepsilon'_n = c + \frac{1}{n} - \inf F(M)$, $u := u_n$ and $\lambda := n$. Therefore, there exists $v_n \in M$ such that

$$\min F(v_n) \leq \min F(u_n) \tag{3.3}$$

$$d(v_n, u_n) \leq n \tag{3.4}$$

$$\min F(v_n) < \min F(w) + (\varepsilon'_n/\lambda) d(v_n, w), \ \forall w \neq v_n. \tag{3.5}$$

From (3.4) and (3.2), we have $|g(v_n)| \geq |g(u_n)| - Ld(v_n, u_n) \geq$ $\geq (L+1)n - Ln = n$, i.e. $|g(v_n)| \to \infty$, which represents exactly (i).

From (3.3) and (3.1) we have $\min F(v_n) \leq c + \frac{1}{n}$. From the definition of c, we have $\min F(v_n) \to c$, exactly the (ii).

From (3.5), we have that $F(w) - \min F(v_n) + \varepsilon_n d(w, v_n) \subseteq \mathbb{R}_+$, $\forall\ w \in M \setminus \{v_n\}$, where $\varepsilon_n = \frac{\varepsilon'_n}{n}$. Clearly $\varepsilon_n \to 0^+$. Dividing by $d(w, v_n) > 0$ the above inclusion, we get

$$\frac{F(w) - \min F(v_n)}{d(w, v_n)} + \varepsilon_n \subseteq \mathbb{R}_+, \ \forall\ w \in M \setminus \{v_n\}.$$

Taking the upper limit of the above inclusion when $w \to v_n$, we get $|\nabla F|(v_n, \min F(v_n)) + \varepsilon_n \subseteq \mathbb{R}_+$, which is exactly the (iii). Thus the proof of lemma is complete. $\Diamond$

Definition 3.1. The set-valued function $F : M \rightsquigarrow \mathbb{R}$ is *g-coercive*, if $\min F(u) \to \infty$ as $|g(u)| \to \infty$.

The main result of this paper

Theorem 3.1. *Let (M, d) be a complete metric space, $F : M \rightsquigarrow \mathbb{R}$ be an upper semicontinuous set-valued mapping with compact and non-empty values, such that $\inf F(M) > -\infty$ and a Lipschitz continuous* *function $g : M \to \mathbb{R}$. F satisfies condition $(\nabla - PSB - g)$ if and only if F is g-coercive.*

Proof. Suppose that F is not g-coercive, i.e. let $c = \liminf_{|g(u)|\to\infty} \min F(u)$ finite. Then by Lemma 3.1., there exists a sequence $\{v_n\}$ such that

(i) $|g(v_n)| \to \infty$,

(ii) $\min F(v_n) \to c$,

(iii) $\nabla F(v_n, \min F(v_n)) + \varepsilon_n \subseteq \mathbb{R}_+$, with $\varepsilon_n \to 0^+$.

From (ii) and (iii), using the condition $(\nabla - PSB - g)$, we obtain that the sequence $\{v_n\}$ is g-bounded which contradicts (i).

Conversely, let us suppose that condition $(\nabla - PSB - g)$ not holds. Therefore, there exists a sequence $\{u_n\} \subset M$ such that $\nabla F(u_n, v_n) + + \varepsilon_n \subseteq \mathbb{R}_+$, with $\varepsilon_n \to 0$, $v_n \in F(u_n)$, $\{v_n\}$ bounded and $\{u_n\}$ is not g-bounded, i.e. $|g(u_n)| \to \infty$. Using the g-coercivity of F, we obtain that $\min F(u_n) \to \infty$, therefore $\{v_n\}$ is unbounded which is a contradiction. $\diamondsuit$

In a similar way it is possible to state the following

Theorem 3.2. *Let $(X, \|\cdot\|)$ be a Banach space, $F : X \rightsquigarrow \mathbb{R}$ be a locally Lipschitz set-valued mapping with compact and non-empty values, such that $\inf F(X) > -\infty$ and a Lipschitz continuous function $g : X \to \mathbb{R}$. F satisfies condition $(D - PSB - g)$ if and only if F is g-coercive.*

Corollary 3.1. *Under the assumptions from Th. 3.2, we can state that conditions $(\nabla - PSB - g)$ and $(D - PSB - g)$ are equivalent.*

Remark 3.1. Similar result as Th. 3.2. was obtained by authors in [7].

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