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FLOWS FOR THE STOCHASTIC NAVIER-STOKES EQUATION

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Dedicated to my father Prof. Dr. Wolfgang Breckner, who taught me to see the beauty of studying and teaching mathematics.

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Abstract: The purpose of this paper is to transform a stochastic equation of Navier-Stokes type with multiplicative noise into a random partial differential equation, which can be solved pathwise. We also derive the existence of a stochastic flow and of the perfect cocycle for the considered equation.

1. Introduction

In the present paper we consider a stochastic Navier-Stokes equation of the type

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$$
(1) \quad \begin{aligned} X_{s,t} = y + \int_s^t \Big(-\mathcal{A}X_{s,r} + \mathcal{B}(X_{s,r}, X_{s,r}) + F(X_{s,r})\Big) dr + \\ + \sum_{j=1}^{m} \int_s^t \mathcal{C}_r^j X_{s,r} \circ dW_r^j, \quad 0 \leq s \leq t, \end{aligned}
$$

where the stochastic integral is in the sense of Stratonovich. We transform (1) into a random partial differential equation of the form

$$
(2) \qquad \Psi_{s,t} = y + \int_s^t \mathcal{G}_{s,r}\left(\Psi_{s,r}\right) dr, \quad s \leq t
$$

by using a bijective process $\Lambda_{s,t}$. We will prove that equation (1) generates a perfect stochastic flow $\left(\Phi_{s,t}\right)_{s \leq t}$, which is given by

(3) $\Phi_{s,t}(\leq, \cdot) = \Lambda_{s,t}(\leq, \cdot) \circ \Psi_{0,t-s}(\theta_s \leq, \cdot)$ for all $0 \leq s \leq t,\ \omega \in \Omega,$

where $\Psi_{s,t}$ is the solution of the random equation (2). Moreover, we derive that $\left(\Phi_{0,t}\right)_{0 \leq t}$ is a perfect cocycle for (1).

The transformation of (1) into (2) can also be done by using a stationary coordinate change Λ_t, i.e. the cocycle of (1) is given by the following conjugation relation

$$
\Phi_{0,t}(\omega, \cdot) = \Lambda_0(\theta_t \omega, \cdot) \circ \Psi_{0,t}(\omega, \cdot) \circ \Lambda_0^{-1}(\omega, \cdot) \quad \text{for all } t \geq 0,\ \omega \in \Omega,
$$

as presented in [9]. The method of stationary coordinate changes is useful when one wants to obtain for stochastic differential equations results which involve aspects of ergodic theory, because it is much easier to study them in the framework of random differential equations, which describe the motion along a stationary vector field (see [12], [11], [15] etc).

The problem of existence of stochastic flows and cocycles for stochastic partial differential equations was solved just in some special cases, for example in [8] for linear equations, in [9] for nonlinear parabolic equations, in [3] for the stochastic Navier-Stokes equation (on the torus). The method presented in this paper is much easier than the nonstandard analysis method for the stochastic Navier-Stokes equation from [3]. It is applicable to the two dimensional Navier-Stokes equation on $\mathbb{R}^2$ and it can be easily adapted for the same equation considered on the torus.

Nonstationary transformations of linear and semilinear stochastic partial differential equations are the subjects of the papers of Da Prato, Iannelli and Tubaro [6] and Da Prato and Tubaro [7], but without investigating flow or cocycle properties.

Only in special cases the transformation of stochastic partial differential equations into random ones has been performed, in order to prove existence of random attractors, as in [5] (for reaction diffusion equations with additive noise, and for stochastic Navier-Stokes equation with additive and with multiplicative noise), Crauel, Debussche and Flandoli [4] (for the stochastic Navier-Stokes equation with additive noise, the white noise driven Burgers equation and the stochastic nonlinear wave equation), Keller and Schmalfuß [13] (for stochastic hyperbolic equations).

To prove the existence of the stochastic flows, perfect cocycles and of random global attractors one does not need the stationary transformation, a nonstationary transformation is also helpful. In the present paper we give the proof for the first two aspects, the existence of the random attractor will be the subject for a future paper. The present paper has the following structure: in Section 2 there are given the assumptions for the equation, some definitions and preliminaries. Section 3 contains the main results of the paper, i.e. in Th. 3.2 the transformation of (1) into a random equation (2), which can be solved pathwise, while the perfect flow and cocycle properties for $\left(\Phi_{s,t}\right)_{s\leq t}$ are proved in Th. 3.4.

2. Assumptions and preliminaries

Notations: We will use the same spaces as mentioned in the book [temam] of R. Temam. Let $\mathcal{D}(\mathbb{R}^d)$ be the space of all $\mathcal{C}^\infty$ vector functions from $\mathbb{R}^d$ to $\mathbb{R}^d$ with compact support contained in $\mathbb{R}^d$. We consider also

$$\mathcal{V} := \{u \in \mathcal{D}(\mathbb{R}^d) \,:\, \text{div}\ u = 0\}, \quad V := \text{the closure of } \mathcal{V} \text{ in } \mathcal{H}^1(\mathbb{R}^d),$$
$$H := \text{the closure of } \mathcal{V} \text{ in } l^2(\mathbb{R}^d).$$

The norm in V is

$$\|v\|_V^2 := \int_{\mathbb{R}^d} \Big[\sum_{i=1}^{d} \Big| \frac{\partial v(x)}{\partial x_i} \Big|_d^2 + |v(x)|_d^2 \Big] dx$$

and in H

$$\|h\|_H^2 := \int_{\mathbb{R}^d} |h(x)|_d^2 dx,$$

where $|\cdot|_d$ is the Euclidian norm in $\mathbb{R}^d$. Let V^* be the dual of V. We denote the dual pairing between $.,.\Big\rangle$. We identify H with its dual space H^*, so we have the continuous dense embeddings

$$V \hookrightarrow H \hookrightarrow V^*.$$

We also have $\Big\langle x, y \Big\rangle = \Big(x, y \Big)_H$ and $\|y\|_H \leq \|y\|_V$ for all $x \in H$ and $y \in V$.

Now we state the **assumptions** about the equation that will be investigated:

(H$_1$) $\Big(W_t\Big)_{0\leq t} = \Big(W_t^1, \dots, W_t^m\Big)_{0\leq t}$ is a m-dimensional Brownian motion on the completed Wiener space $(\Omega, \mathcal{F}, P)$, $(\theta_t)_{0\leq t}$ is the Wiener shift

$$\theta_t \omega = \omega(\cdot + t) - \omega(t)$$

and the filtration $\mathcal{F}_t := \sigma\{W_s : \ 0 \leq s \leq t\}, 0 \leq t$ is assumed to be completed by the P-completion of $\mathcal{F}$.

(H$_2$) $\mathcal{A} : V \to V^*$ is a linear, continuous and (weakly) coercive operator such that for every $v \in V$ and for some constants $\mu, \nu > > 0, \lambda \in \mathbb{R}$ we have

$$\|\mathcal{A}v\|_{V^*} \leq \alpha \|v\|_V \text{ and } \Big\langle \mathcal{A}v, v \Big\rangle \geq \nu \|v\|_V^2 - \lambda \|v\|_H^2.$$

(H$_3$) $\mathcal{B} : V \times V \to V^*$ is a bilinear operator such that $\Big\langle \mathcal{B}(u,v), v \Big\rangle = 0$ and there exists a positive constant β such that

$$\Big| \Big\langle \mathcal{B}(u,v), z \Big\rangle \Big|^2 \leq \beta \|z\|_V^2 \|u\|_V \|u\|_H \|v\|_V \|v\|_H \text{ for all } u, v, z \in V.$$

(H$_4$) $F : H \to H$ is a continuous mapping such that for all $u, v \in H$

$$\|F(u) - F(v)\|_H \leq \gamma \|u - v\|_H,$$

with γ positive constant.

(H$_5$) for each $j \in \{1, \dots, m\}, t \geq 0$ the mappings $\mathcal{C}_t^j : V \to H$ are given by

$$C^j_t u(x) := b^j_t u(x) + \sum_{i=1}^{d} c^{i,j}_t \frac{\partial u(x)}{\partial x_i} \quad \text{for all} \quad u \in V,$$

where $\left(b^j_t\right)_{0\leq t}$ is a real-valued $\mathcal{F}_t$-adapted stochastic process and $\left(c^j_t\right)_{0\leq t}$ is a $\mathbb{R}^d$-valued $\mathcal{F}_t$-adapted stochastic process with the stationarity property

$$b^j_{t+s}(\leq) = b^j_t(\theta_s \leq), \quad c^j_{t+s}(\leq) = c^j_t(\theta_s \leq), \quad s,t \in \mathbb{R}_+, \omega \in \Omega$$

and

$$E\int_0^\infty |b^j_t|^2 dt < \infty, \quad E\int_0^\infty |c^j_t|^2_d dt < \infty.$$

We investigate the stochastic evolution equation

(4)

$$\left(X_{s,t}, v\right)_H = \left(h, v\right)_H - \int_s^t \left\langle \mathcal{A}X_{s,r}, v\right\rangle dr + \int_s^t \left\langle \mathcal{B}\left(X_{s,r}, X_{s,r}\right), v\right\rangle dr + \int_s^t \left(F(X_{s,r}), v\right)_H dr + \sum_{j=1}^{m} \int_s^t \left(C^j_r X_{s,r}, v\right)_H \circ dW^j_r$$

for all $t \geq s \geq 0, v \in V, h \in H$, a.e. $\omega \in \Omega$. The stochastic integral is in the sense of Stratonovich.

Typical example is the two dimensional Navier-Stokes equation

$$du(t,x) = \left(\nu \sum_{i=1}^{2} \frac{\partial^2 u(t,x)}{\partial x_i^2} - \sum_{i=1}^{2} u_i(t,x)\frac{\partial u(t,x)}{\partial x_i} + f(t,x) - \nabla p(t,x)\right)dt + \sum_{j=1}^{m}\left(b^j u(t,x) + \sum_{i=1}^{2} c^{i,j}\frac{\partial u(t,x)}{\partial x_i}\right) \circ dW^j_t,$$

$$\text{div } u = 0, \ u(0,x) = h(x), \ x \in \mathbb{R}^2,$$

where u is the velocity field, $\nu > 0$ is the viscosity, f is an external force and p is the pressure. After projection on divergence free fields (i.e. on H) the pressure term p disappears and the other terms may be written in the form described in the abstract assumptions and the equation can be written as a stochastic evolution equation like (4).

Remark 2.1. (1) The operators $\mathcal{C}_t^j$ are linear and continuous from V to H, but they can uniquely be extended to mappings from H to V^* by considering

$$(5) \qquad \left\langle \mathcal{C}_t^j h, v \right\rangle = -(h, c_t^j \nabla v)_H + b_t^j (h, v)_H \quad \text{for all} \quad h \in H, v \in V.$$

(2) The operator $\mathcal{C}_t^j : V \to H$ admits an adjoint $(\mathcal{C}_t^j)^* : H \to V^*$ given by

$$\left\langle (\mathcal{C}_t^j)^* h, v \right\rangle = \left(h, \mathcal{C}_t^j v \right)_H, \quad h \in H, v \in V.$$

(3) From the assumption on F it follows that for all $u \in H$

$$\|F(u)\|_H \leq \|F(0)\|_H + \gamma \|u\|_H.$$

Without loss of generality we can assume that F satisfies (H_4) and

$$\|F(u)\|_H \leq \gamma (1 + \|u\|_H) \text{ for all } u \in H.$$

Definition 2.2. We say that equation (4) generates a *perfect stochastic flow*, if there exists a process $\left(\Phi_{s,t} \right)_{s \leq t}$ such that for all $0 \leq s \leq u \leq$ $\leq t, \omega \in \Omega$ the operator $\Phi_{s,t}(\omega, \cdot) : H \to H$ is continuous and satisfies the properties

$$(6) \qquad \Phi_{s,t} = \Phi_{u,t} \circ \Phi_{s,u},$$

$\Phi_{s,s}$ is the identity map on H and

$$\Phi_{s,t}(\omega, h) = X_{s,t}(\omega) \quad \text{for a.e.} \quad \omega \in \Omega,$$

where $X_{s,t}$ denotes the solution of (4) at time t with initial value h at time s.

The semiflow $\left(\Phi_{0,t} \right)_{0 \leq t}$ is a *perfect cocycle* over θ if for all $\omega \in$ $\in \Omega, t \geq 0$ the map $\Phi_{0,t}(\leq, \cdot) : H \to H$ is continuous, $\Phi_{0,0}(\leq, \cdot)$ is the identity on H and

$$\Phi_{0,s+t}(\leq, \cdot) = \Phi_{0,t}(\theta_s \leq, \cdot) \circ \Phi_{0,s}(\leq, \cdot) \quad \text{for all } 0 \leq s, t.$$

The aim of our paper is to prove that equation (4) admits a perfect stochastic flow and a perfect cocycle. This will be done by transforming (4) into a random equation of type (2), solving this equation pathwise and then obtain the flow for (4).

3. The transformation

First we investigate auxilliary processes closely related to the diffusion part of equation (4).

We consider the following $\mathbb{R}^d$-valued process

$$(7) \qquad \xi_{s,t}(x) = x + \sum_{j=1}^{m} \int_s^t c_r^j \circ dW_r^j, \quad x \in \mathbb{R}^d, 0 \le s \le t,$$

and the linear equation in $\mathbb{R}$

$$(8) \qquad \eta_{s,t} = 1 + \sum_{j=1}^{m} \int_s^t b_r^j \eta_{s,r} \circ dW_r^j \quad 0 \le s \le t.$$

We can find modifications of these processes such that for all $0 \le s \le$ $\le t, \omega \in \Omega$

$$(9) \qquad \begin{aligned} \xi_{s,t}(\le, \cdot) &= \xi_{0,t-s}(\theta_s \le, \cdot), \quad \xi_{0,t} = \xi_{s,t} \circ \xi_{0,s}, \\ \eta_{s,t}(\le, \cdot) &= \eta_{0,t-s}(\theta_s \le, \cdot), \quad \eta_{0,t} = \eta_{s,t} \cdot \eta_{0,s}. \end{aligned}$$

For each $0 \le s \le t, \omega \in \Omega$ we define

$$(10) \qquad \Lambda_{s,t} : H \to H \quad \text{by} \quad \Lambda_{s,t} h := h(\xi_{s,t})\eta_{s,t}.$$

This operator is correctly defined, because

$$\operatorname{div} \Lambda_{s,t} h = \eta_{s,t} \sum_{i,k=1}^{d} \frac{\partial h_i}{\partial x_k}(\xi_{s,t}) \frac{\partial \xi_{s,t}^k}{\partial x_i} = 0 \quad \text{for } h \in \mathcal{V}$$

and $\mathcal{V}$ is dense in H. Obviously $\Lambda_{s,t}$ is a linear and continuous mapping. Its adjoint $\Lambda_{s,t}^* : H \to H$ is given by

$$(11) \qquad {\Lambda^*}_{s,t} h = h(\xi_{s,t}^{-1})\eta_{s,t} \ ,$$

since

$$\left(h, {\Lambda^*}_{s,t} l\right)_H = \left(\Lambda_{s,t} h, l\right)_H = \int_{\mathbb{R}^d} h(\xi_{s,t}(x)) l(x) \eta_{s,t} dx =$$
$$= \int_{\mathbb{R}^d} h(y) l(\xi_{s,t}(y)) \eta_{s,t} dy$$

with $h, l \in H$ (we used that the determinant of the differential of $\xi_{s,t}$ is equal to 1).

Theorem 3.1. (i) *For fixed $0 \le s \le t, \omega \in \Omega$ the operator $\Lambda_{s,t}$ is bijective and the inverse of the adjoint is given by*

$$(12) \qquad ({\Lambda^*}_{s,t})^{-1} h = \frac{h(\xi_{s,t})}{\eta_{s,t}} \ \textit{for all } h \in H.$$

(ii) *For fixed $t \in \mathbb{R}, \omega \in \Omega$ the restrictions $\Lambda_{s,t}\big|_V, {\Lambda^*}_{s,t}\big|_V$ are continuous and bijective operators from V to V. If $s, t \in [0, T]$ ($T \ge 0$ is fixed), then*

$$(13)\qquad \|\Lambda_{s,t}h\|_H \le K_T\|h\|_H, \quad \|\Lambda_{s,t}v\|_V \le K_T\|v\|_V$$

and

$$(14)\qquad \|(\Lambda^*{}_{s,t})^{-1}h\|_H \le \hat{K}_T\|h\|_H, \quad \|(\Lambda^*{}_{s,t})^{-1}v\|_V \le \hat{K}_T\|v\|_V$$

for all $h \in H, v \in V$, *where* $K_T = \sup\limits_{s,t\in[0,T]} \eta^2_{s,t}$ *and* $\hat{K}_T = \sup\limits_{s,t\in[0,T]} \frac{1}{\eta^2_{s,t}}$.

(iii) *For all* $0 \le s \le u \le t, r \in \mathbb{R}_+, \omega \in \Omega$ *it holds*

$$(15)\qquad \Lambda_{s,t} = \Lambda_{u,t} \circ \Lambda_{s,u} \quad \text{and} \quad \Lambda_{s+r,t+r}(\le,\cdot) = \Lambda_{s,t}(\theta_r \le,\cdot).$$

(iv) *Let* $h \in H$. *The process* $\left(\Lambda_{s,t}\right)_{s\le t}$ *satisfies*

$$(16)\qquad \Lambda_{s,t}h = h + \int_s^t \mathcal{C}^j_r \Lambda_{s,r} h \circ dW^j_r, \quad 0 \le s \le t,$$

and $\left(\Lambda^*{}_{s,t}\right)_{s\le t}$ *satisfies*

$$(17)\qquad \Lambda^*{}_{s,t}h = h + \int_s^t \Lambda^*{}_{s,r}(\mathcal{C}^j_r)^* h \circ dW^j_r, \quad 0 \le s \le t.$$

Proof. (i) By calculation it is easily verifyed that $(\Lambda^*{}_{s,t})^{-1}$ given in (12) the inverse operator of $\Lambda^*{}_{s,t}$ from (11) is.

(ii) For all $h \in H$ we have by (10)

$$(18)\qquad \|\Lambda_{s,t}h\|^2_H = \int_{\mathbb{R}^d} |h(y)|^2_d \eta^2_{s,t} dy.$$

For $v \in \mathcal{V}$ we write

$$(19)\ \|\Lambda_{s,t}v\|^2_V = \int_{\mathbb{R}^d} \left(\sum_{i=1}^d \left| \frac{\partial}{\partial x_i} \left[\Lambda_{s,t} v(x) \right] \right|^2_d + \left| \Lambda_{s,t} v(x) \right|^2_d \right) dx \le$$

$$\le K_T \int_{\mathbb{R}^d} \left(\sum_{i=1}^d \left| \frac{\partial v}{\partial x_k}(\xi_{s,t}(x)) \right|^2_d + |v(x)|^2_d \right) dx = K_T \|v\|^2_V,$$

where $K_T = \sup\limits_{s,t\in[0,T]} \eta^2_{s,t}$. By using (7) we have

$$\text{div } \Lambda_{s,t} v = \eta_{s,t} \sum_{i,k=1}^d \frac{\partial v_i}{\partial x_k}(\xi_{s,t}) \frac{\partial \xi^k_{s,t}}{\partial x_i} = 0.$$

But $\mathcal{V}$ is dense in V, which imply together with (19) that $\Lambda_{s,t}$ is also an operator from V to V. From the similarity of the expressions of $\Lambda_{s,t}$

and $(\Lambda^*{}_{s,t})^{-1}$ it follows that the properties (14) hold, where $\hat{K}_T := \sup_{s,t\in[0,T]} \frac{1}{\eta^2_{s,t}}$.

(iii) The properties follow by (10) and (9).

(iv) For (16) with $h \in \mathcal{V}$ one uses the so-called characteristics method (see [14] p. 297 or [10] Prop. 3.2) and the density of $\mathcal{V}$ in H together with the continuity property proved in (ii). For (17) one uses (16), the definition of an adjoint operator and also Remark 2.1. ◊

We define for all $\omega \in \Omega$, $t \geq 0$ the random operator $\mathcal{G}_t : V \to V^*$ by

(20) $$\mathcal{G}_{s,t} := -\Lambda^{-1}_{s,t} \circ \mathcal{A} \circ \Lambda_{s,t} + \Lambda^{-1}_{s,t} \circ \mathcal{B} \circ \Lambda_{s,t}u, \Lambda_{s,t} + \Lambda^{-1}_{s,t} \circ F \circ \Lambda_{s,t}.$$

We want to point out that $\Lambda^{-1}_{s,t}$ appearing in the terms containing the operator $\mathcal{A}$ and respective $\mathcal{B}$ is $\Lambda^{-1}_{s,t} : V^* \to V^*$ given by $\left\langle \Lambda^{-1}_{s,t} v^*, v \right\rangle := \left\langle v^*, \Lambda^*{}_{s,t})^{-1} v \right\rangle$, where we use that $(\Lambda^*{}_{s,t})^{-1}$ is the inverse of the adjoint of $\Lambda_{s,t}$ restricted to the space V.

Let $\omega \in \Omega$ and $T \geq 0$ arbitrary. We consider the random (pathwise) evolution equation

(21) $$\Big(\Psi_{s,t}(y), v\Big)_H = \Big(y, v\Big)_H + \int_s^t \Big\langle \mathcal{G}_{s,r}(\Psi_{s,r}(y)), v \Big\rangle dr,$$

$$v \in V, y \in H, 0 \leq s \leq t \leq T.$$

Theorem 3.2. (i) *Equation* (21) *has a unique solution* $\Psi_{s,\cdot} \in \mathcal{L}^2([s,T]; V)$. *Moreover* $\Psi_{s,\cdot} \in \mathcal{C}([s,T]; H)$.

(ii) *For all* $0 \leq s \leq t \leq T, \omega \in \Omega$ *the operator* $\Psi_{s,t} : H \to H$ *is continuous.*

Proof. (i) For each fixed $\omega \in \Omega$ the evolution equation (21) can be solved by using the classic deterministic theory for partial differential equations of Navier-Stokes type (as in [16], Chapter III). The classic method for such equations is the Galerkin method, which we will use in the following.

For a more simple writing we will take $s := 0$ and use the notation $\Psi_t := \Psi_{0,t}$.

Step 1: We mention some properties for the operators $\mathcal{A}$ and $\mathcal{B}$: for $a \in \mathbb{R}$, $u, v \in V$ we have the properties

(22) $$\left\langle \mathcal{A}av, \frac{v}{a} \right\rangle \geq \nu \|v\|^2_V - \lambda \|v\|^2_H,$$

and

$$\left\langle \mathcal{B}(au, av), \frac{v}{a} \right\rangle = 0 \tag{23}$$

which follow by the assumptions on $\mathcal{A}$ and $\mathcal{B}$. By (22), (23), (10), (12), assumptions on F (also Remark 2.1) and Th. 3.1 we get

$$\begin{aligned} &\left\langle \Lambda_{s,t}^{-1} \mathcal{A} \Lambda_{s,t} v, v \right\rangle = \left\langle \mathcal{A} \Lambda_{s,t} v, (\Lambda_{s,t}^*)^{-1} v \right\rangle = \\ &= \left\langle \mathcal{A}(\eta_{s,t} v(\xi_{s,t})), \frac{v(\xi_{s,t})}{\eta_{s,t}} \right\rangle \geq \nu \|v\|_V^2 - \lambda \|v\|_H^2, \end{aligned} \tag{24}$$

$$\left| \left\langle \Lambda_{s,t}^{-1} \mathcal{B}(\Lambda_{s,t} u, \Lambda_{s,t} v), z \right\rangle \right|^2 \leq \beta K_T^2 \|z\|_V^2 \|u\|_V \|u\|_H \|v\|_V \|v\|_H, \tag{25}$$

$$\left\langle \Lambda_{s,t}^{-1} \mathcal{B}(\Lambda_{s,t} u, \Lambda_{s,t} v), v \right\rangle = 0, \tag{26}$$

$$\left| \left(\Lambda_{s,t}^{-1} F(\Lambda_{s,t} h), l \right)_H \right| \leq \gamma \|l\|_H (\hat{K}_T + \|h\|_H), \tag{27}$$

$$\left| \left(\Lambda_{s,t}^{-1} F(\Lambda_{s,t} h) - \Lambda_{s,t}^{-1} F(\Lambda_{s,t} l), h - l \right)_H \right| \leq \gamma \|h - l\|_H^2 \tag{28}$$

for all $s, t \in [0, T]$, $u, v, z \in V, h, l \in H$.

Step 2: Let $y \in H$ be fixed. Since V is separable and $\mathcal{V}$ is dense in V, there exists a basis in V of elements $e_1, \ldots, e_m, \ldots \in \mathcal{V}$. Let

$$\psi_t^n := \sum_{k=1}^n C_t^{k,n} e_k$$

and let $y_n \in \operatorname{span}\{e_1, \ldots, e_n\}$ such that $\lim_{n\to\infty} \|y_n - y\|_H = 0$. There exists $M > 0$ such that for all $n \in \mathbb{N}$

$$\|y_n\|_H^2 \leq M. \tag{29}$$

We define the Galerkin equations corresponding to (21)

$$\begin{aligned} \left(\psi_t^n, e_j \right)_H = y_n - &\int_0^t \left\langle \Lambda_{0,r}^{-1} \mathcal{A}(\Lambda_{0,r} \psi_r^n), e_j \right\rangle dr + \\ &+ \int_0^t \left\langle \Lambda_{0,r}^{-1} \mathcal{B}(\Lambda_{0,r} \psi_r^n, \Lambda_{0,r} \psi_r^n), e_j \right\rangle dr + \\ &+ \int_0^t \left(\Lambda_{0,r}^{-1} F(\Lambda_{0,r} \psi_r^n), e_j \right)_H dr, \quad j \in \{1, \ldots, n\}. \end{aligned} \tag{30}$$

Hence for each $j \in \{1, \ldots, n\}$ we have

(31)
$$\sum_{k=1}^{n}\Big(e_k,e_j\Big)_H\frac{\mathrm{d}C_t^{k,n}}{\mathrm{d}t}=-\sum_{k=1}^{n}\Big\langle\Lambda_{0,t}^{-1}\mathcal{A}(\Lambda_{0,t}e_k),e_j\Big\rangle C_t^{k,n}+$$
$$+\sum_{k,l=1}^{n}\Big\langle\Lambda_{0,t}^{-1}\mathcal{B}(\Lambda_{0,t}e_k,\Lambda_{0,t}e_l),e_j\Big\rangle C_t^{k,n}C_t^{l,n}+$$
$$+\Big(\Lambda_{0,t}^{-1}F(\sum_{k=1}^{n}C_t^{k,n}\Lambda_{0,t}e_k),e_j\Big)_H,$$

with the initial condition
$$\sum_{k=1}^{n}\Big(e_k,e_j\Big)_H C_0^{k,n}=\Big(y_n,e_j\Big)_H.$$

Since $e_1,\dots,e_n$ are linearly independent, we get
$$\det\Big(\Big(e_k,e_j\Big)_H\Big)_{k,j}\neq 0$$

and the system (31) can be solved with the unknown vector $(C_t^{1,n},\dots$ $\dots,C_t^{n,n})$ as a system of ordinary differential equations (see [18], Lemma 30.4, p. 776; Problem 30.2 p. 799).

Step 3: We multiply (30) by $C_t^{j,n}$ and add these equations for $j=1$ to m. Taking into account (24), (26), and (27) we get
$$\sup_{r\in[0,t]}\|\psi_r^n\|_H^2+$$
$$+2\nu\int_0^t\|\psi_r^n\|_V^2dr\le\|y_n\|_H^2+\gamma\hat{K}_Tt+(2\gamma+\gamma\hat{K}_T+2\lambda)\int_0^t\|\psi_r^n\|_H^2dr.$$

By the Gronwall Lemma and by (29) it follows that

(32)
$$\sup_{t\in[0,T]}\|\psi_t^n\|_H^2\le(\gamma\hat{K}_TT+M)e^{(2\gamma+\gamma\hat{K}_T+2\lambda)T}$$

and then

(33)
$$\int_0^T\|\psi_t^n\|_V^2dt\le\frac{\gamma\hat{K}_TT+M}{2\nu}e^{(2\gamma+\gamma\hat{K}_T+2\lambda)T}.$$

We also have that $\left\{\int_0^T\left\|\frac{d\psi_t^n}{dt}\right\|_{V^*}^2dt\right\}_n$ is a bounded sequence in $\mathcal{L}^2([0,T];V^*)$, since the properties of $\mathcal{B}$, $\Lambda_{0,t}$ imply

$$\int_0^T \|\Lambda_{0,t}^{-1}\mathcal{B}(\Lambda_{0,t}\psi_t^n, \Lambda_{0,t}\psi_t^n)\|_{V^*}^2 dt \le \beta K_T^2 \sup_{t\in[0,T]} \|\psi_t^n\|_H^2 \int_0^T \|\psi_t^n\|_V^2 dt,$$

Step 4: There exist $\psi \in l^2([0,T];V)$ and a subsequence (n_k) of (n) such that

$$\text{(34)} \qquad \mathrm{w} - \lim_{k\to\infty} \psi^{n_k} = \psi \quad \text{in } l^2([0,T];V) \quad \text{and in} \quad l^2([0,T];H),$$

where w $-$ lim denotes the weak convergence.

Let $j \in \{1,\ldots,n\}$ fixed and let $G_j := \operatorname{supp} e_j$, which is a compact subset of $\mathbb{R}^d$. We consider the evolution triple $(H^1(G_j), l^2(G_j), H^{-1}(G_j))$, where the embedding $H^1(G_j) \hookrightarrow l^2(G_j)$ is compact. We use the estimates from Step 2 and apply a compactness criterion (see [16], Th. 2.1, p. 271) by using that $\left\{\psi^{n_k}\Big|_{G_j}\right\}_k \subset l^2([0,T];l^2(G_j))$ and $\left\{\frac{d\psi_t^{n_k}\big|_{G_j}}{dt}\right\}_k \subset l^2([0,T];H^{-1}(G_j))$ are bounded sequences. We obtain that $\left\{\psi^{n_k}\Big|_{G_j}\right\}_k$ is relatively compact in $l^2([0,T];l^2(G_j))$. Hence there exist a subsequence of (n_k), which we will also denote by (n_k), and $\hat{\psi} \in l^2([0,T];l^2(G_j))$ such that

$$\text{(35)} \qquad \lim_{k\to\infty} \psi^{n_k}\Big|_{G_j} = \hat{\psi} \quad \text{in} \quad l^2([0,T];l^2(G_j)).$$

Then by (34) it follows

$$\text{(36)} \qquad \psi_r = \hat{\psi}_r \quad \text{for a.e. } r \in [0,T].$$

Now we prove that

$$\lim_{k\to\infty}\int_0^T \left|\left\langle \Lambda_{0,r}^{-1}\mathcal{B}(\Lambda_{0,r}\psi_r^{n_k}, \Lambda_{0,r}\psi_r^{n_k}) - \Lambda_{0,r}^{-1}\mathcal{B}(\Lambda_{0,r}\psi_r, \Lambda_{0,r}\psi_r), e_j\right\rangle f(r)\right| dr = $$
$$\text{(37)} \qquad = 0$$

for all $f \in l^\infty([0,T];\mathbb{R})$, this is a dense subspace of $l^2([0,T];\mathbb{R})$.

By using the properties of $\mathcal{B}$

$$\int_0^T \Big|\Big\langle \Lambda_{0,r}^{-1}\mathcal{B}(\Lambda_{0,r}\psi_r^{n_k},\Lambda_{0,r}\psi_r^{n_k}) - \Lambda_{0,r}^{-1}\mathcal{B}(\Lambda_{0,r}\psi_r,\Lambda_{0,r}\psi_r), e_j\Big\rangle f(r)\Big| dr \le$$

$$\le \sqrt{2\beta}K_T\|e_j\|_V \sup_{r\in[0,T]}|f(r)|\Big(\int_0^T \Big\|\psi_r^{n_k}\Big|_{G_j} - \hat{\psi}_r\Big\|^2_{l^2(G_j)} dr\times$$

$$\times\int_0^T \|\psi_r^{n_k}-\psi_r\|_V^2 dr\Big)^{\frac{1}{4}}\Big(\int_0^T \|\psi_r^{n_k}\|_V^2 + \|\psi_r\|_V^2) dr\Big)^{\frac{1}{2}}.$$

Then by (35), (33) follow (37).

Now we pass to the limit in (30) in the space $l^2([0,T];\mathbb{R})$ use (34), (37) to get

$$(38) \qquad \Big(\psi_t, e_j\Big)_H = \Big(y, e_j\Big)_H - \int_0^t \Big\langle \Lambda_{0,r}^{-1}\mathcal{A}(\Lambda_{0,r}\psi_r), e_j\Big\rangle dr +$$

$$+\int_0^t \Big\langle \Lambda_{0,r}^{-1}\mathcal{B}(\Lambda_{0,r}\psi_r,\Lambda_{0,r}\psi_r), e_j\Big\rangle dr + \int_0^t \Big(\Lambda_{0,r}^{-1}F(\Lambda_{0,r}\psi_r), e_j\Big)_H dr$$

for a.e. $t\in[0,T]$ and all $j\in\{1,\dots,n\}$.

But the right-hand side of (38) is continuous in t, so we can identify ψ_t with a process Ψ_t continuous in t such that (38) holds for all $t \in$ $\in [0,T]$. This process is the solution of (21).

Step 5: To prove the uniqueness of the solution we consider Ψ_t and $\hat{\Psi}_t$ to be two solutions for (21) starting in $y\in H$ at time 0. Then by (21), (24), (25), (26), (28) we have

$$\|\Psi_t(y)-\hat{\Psi}_t(y)\|_H^2 + 2\nu\int_0^t \|\Psi_r(y)-\hat{\Psi}_r(y)\|_V^2 dr \le$$

$$\le \frac{3\nu}{2}\int_0^t \|\Psi_r(y)-\hat{\Psi}_r(y)\|_V^2 dr +$$

$$+\frac{\beta^2 K_T^4}{2\nu^3}\int_0^t \|\Psi_r(y)\|_V^2\|\Psi_r(y)\|_H^2\|\Psi_r(y)-\hat{\Psi}_r(y)\|_H^2 dr+$$

$$+2(\gamma+\lambda)\int_0^t \|\Psi_r(y)-\hat{\Psi}_r(y)\|_H^2 dr.$$

Using the Gronwall Lemma (see for example [1]) it follows

$$\|\Psi_t(y)-\hat{\Psi}_t(y)\|_H^2 \le 0,$$

hence $\Psi_t(y)=\hat{\Psi}_t(y)$ for all $0\le t\le T$.

(ii) To prove the continuous dependence on the initial data we consider $u,y\in H$. Then by (21), (24), (25), (28)

$$\|\Psi_t(u)-\Psi_t(y)\|_H^2+2\nu\int_0^t \|\Psi_r(u)-\Psi_r(y)\|_V^2 dr \le \|u-y\|_H^2+$$

$$+\frac{3\nu}{2}\int_0^t \|\Psi_r(u)-\Psi_r(y)\|_V^2 dr+$$

$$+\frac{\beta^2 K_T^4}{2\nu^3}\int_0^t \|\Psi_r(y)\|_V^2\|\Psi_r(y)\|_H^2\|\Psi_r(u)-\Psi_r(y)\|_H^2 dr+$$

$$+2(\gamma+\lambda)\int_0^t \|\Psi_r(u)-\Psi_r(y)\|_H^2 dr.$$

Using the Gronwall Lemma (e.g. [1]), (32) and (33) it follows

$$\|\Psi_t(u)-\Psi_t(y)\|_H^2 \le \|u-y\|_H^2\times$$

$$\times\exp\left\{(\gamma+\lambda)t+\frac{\beta^2K_T^4(\gamma\hat{K}_T t+M)^2}{2\nu^3}\exp\{2(2\gamma+\gamma\hat{K}_T+2\lambda)t\}\right\}.$$

The above inequality shows the continuous dependence on the initial data. $\Diamond$

Remark 3.3. If we try to consider more general $\mathcal{C}_t^j$, e.g. $b_t^j, c_t^{i,j}$ depending on x, then the property $\Lambda_{s,t}$ maps V into V from Th. 3.1 could fail. Some nonlinear operators $\mathcal{C}_t^j$ lead to problems in defining $\mathcal{G}_{s,t}$ (we used the linearity of $\Lambda_{s,t}$ and the possibility to define the inverse of its adjoint) and also in proving that (21) admits a solution, because we have to verify coercitivity and monotonicity conditions for $\mathcal{G}_{s,t}$.

For for all $0 \leq s \leq t$, $\omega \in \Omega$ we define the stochastic process $\left(\Phi_{s,t}\right)_{s \leq t}$ by

$$\Phi_{s,t}(\leq, \cdot) := \Lambda_{s,t}(\leq, \cdot) \circ \Psi_{0,t-s}(\theta_s \leq, \cdot). \tag{39}$$

Theorem 3.4. (i) *The process* $\left(\Phi_{s,t}\right)_{s \leq t}$ *is a perfect stochastic flow associated to* (4).

(ii) *The process* $\left(\Phi_{0,t}\right)_{0 \leq t}$ *is a perfect cocycle.*

Proof. (i) Let now $0 \leq s \leq t$ arbitrary. We take T sufficiently large such that $s, t \in [0, T]$. Let $y \in H$.

Let $\Psi_{s,t}$ be the unique solution of

$$\Psi_{s,t}(y) = y + \int_s^t \mathcal{G}_{s,r}(\leq, \Psi_{s,r}(y))dr. \tag{40}$$

The existence and uniqueness of such a process can be proved analogously to the proof of (i) in Th. 3.2. By (21) we get

$$\begin{aligned} \Psi_{0,t-s}(\theta_s \leq, y) &= y + \int_0^{t-s} \mathcal{G}_{0,r}(\theta_s \leq, \Psi_{0,r}(\theta_s \leq, y))dr = \\ &= y + \int_s^t \mathcal{G}_{0,r-s}(\theta_s \leq, \Psi_{0,r-s}(\theta_s \leq, y))dr. \end{aligned}$$

Then by (15) we have

$$\Psi_{0,t-s}(\theta_s \leq, y) = y + \int_s^t \mathcal{G}_{s,r}(\leq, \Psi_{0,r-s}(\theta_s \leq, y))dr \tag{41}$$

and by the uniqueness of the solution of (40) it follows that for all $0 \leq s \leq t, \omega \in \Omega$

$$\Psi_{s,t}(\leq, \cdot) = \Psi_{0,t-s}(\theta_s \leq, \cdot). \tag{42}$$

We use (40), (10), (20) and apply the Itô formula to obtain

$$\Big(\Psi_{s,t}(y), {\Lambda^*}_{s,t}v\Big)_H = \Big(y, v\Big)_H - \int\limits_s^t \Big\langle \mathcal{A}\Lambda_{s,r}\Psi_{s,r}(y), v\Big\rangle dr +$$

$$+ \int\limits_s^t \Big\langle \mathcal{B}(\Lambda_{s,r}\Psi_{s,r}(y), \Lambda_{s,r}\Psi_{s,r}(y)), v\Big\rangle dr + \int\limits_s^t \Big(F(\Lambda_{s,r}\Psi_{s,r}(y)), v\Big)_H dr +$$

$$+ \sum_{j=1}^m \int\limits_s^t \Big\langle {\Lambda^*}_{s,r}(\mathcal{C}_r^j)^* v, \Psi_{s,r}(y)\Big\rangle \circ dW_r^j$$

for all $v \in V, t \in [0,T]$ and a.e. $\omega \in \Omega$. Then

$$\Big(\Lambda_{s,t}(\Psi_{s,t}(y)), v\Big)_H = \Big(y, v\Big)_H - \int\limits_s^t \Big\langle \mathcal{A}\Lambda_{s,r}\Psi_{s,r}(y), v\Big\rangle dr +$$

$$+ \int\limits_s^t \Big\langle \mathcal{B}(\Lambda_{s,r}\Psi_{s,r}(y), \Lambda_{s,r}\Psi_{s,r}(y)), v\Big\rangle dr + \int\limits_s^t \Big(F(\Lambda_{s,r}\Psi_{s,r}(y)), v\Big)_H dr +$$

$$+ \sum_{j=1}^m \int\limits_s^t \Big\langle \mathcal{C}_r^j \Lambda_{s,r}\Psi_{s,r}(y), v\Big\rangle \circ dW_r^j$$

for all $v \in V, t \in [0,T]$ and a.e $\omega \in \Omega$. But the equation above has almost surely unique solution (proved for example by Gronwall's Lemma). Hence by (39) and (42)

$$\Phi_{s,t}(y) = X_{s,t}(y) \quad \text{a.e.} \quad \omega \in \Omega.$$

To prove that relation

$$\Phi_{s,t} = \Phi_{u,t} \circ \Phi_{s,u} \tag{43}$$

holds for all $s \le u \le t$, $\omega \in \Omega$ is equivalent to show that

$$\Lambda_{s,u} \circ \Psi_{s,t} = \Psi_{u,t} \circ \Lambda_{s,u} \circ \Psi_{s,u}, \tag{44}$$

because by (39), (42) and (15) we can write

$$\Phi_{s,t} = \Lambda_{s,t} \circ \Psi_{s,t} = \Lambda_{u,t} \circ \Lambda_{s,u} \circ \Psi_{s,t}$$

and

$$\Phi_{u,t} \circ \Phi_{s,u} = \Lambda_{u,t} \circ \Psi_{u,t} \circ \Lambda_{s,u} \circ \Psi_{s,u}.$$

From (40) we have

$$\Big(\Psi_{u,t}(\Lambda_{s,u}\circ\Psi_{s,u}(y)),v\Big)_H=$$
$$=\Big(\Psi_{s,u}(y),{\Lambda^*}_{s,u}v\Big)_H+\int_u^t\Big\langle\mathcal{G}_{u,r}(\Psi_{u,r}(\Lambda_{s,u}\circ\Psi_{s,u}(y))),v\Big\rangle dr \tag{45}$$

and

$$\Big(\Lambda_{s,u}\circ\Psi_{s,t}(y),v\Big)_H=\Big(\Psi_{s,t}(y),{\Lambda^*}_{s,u}v\Big)_H=$$
$$=\Big(\Psi_{s,u}(y),{\Lambda^*}_{s,u}v\Big)_H+\int_u^t\Big\langle\mathcal{G}_{s,r}(\Psi_{s,r}(y)),{\Lambda^*}_{s,u}v\Big\rangle dr. \tag{46}$$

By (15) from Th. 3.1-(iii) we have

$$\Lambda_{s,r}=\Lambda_{u,r}\circ\Lambda_{s,u}\quad\text{and}\quad({\Lambda^*}_{s,r})^{-1}\circ{\Lambda^*}_{s,u}=({\Lambda^*}_{u,r})^{-1}.$$

Then by (15) we have

$$\Big(\Lambda_{s,u}\circ\Psi_{s,t}(y),v\Big)_H=\Big(\Psi_{s,u}(y),{\Lambda^*}_{s,u}v\Big)_H+$$
$$+\int_u^t\Big\langle\mathcal{G}_{u,r}(\Lambda_{s,u}\circ\Psi_{s,r}(y)),v\Big\rangle dr. \tag{47}$$

We use now (45) and (47), as soon as the uniqueness of the solution of equation (40) to obtain (44).

(ii) Relations (15), (42), (43) imply imediatly the perfect cocycle property. ◊

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