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# A NOTE ON LIE SEMIALGEBRAS IN $\mathfrak{sl}$(2,R)

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**Dedicated to may father Prof. Wolfgang W. Breckner whose rigorousness and elegance in proving and presenting mathematical results I have always admired: The heart of the mathematical experience is the mathematics itself.**

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**Abstract**: We give an elementary, pure algebraical proof of the following assertion stated in [2]: The Lie wedge of an exponential subsemigroup of $\mathrm{Sl}(2,\mathbb{R})$ with inner points is the intersection of at most four half-space semialgebras.

**1. Introduction.** Lie semialgebras play an important role in the structure theory of subsemigroups of Lie groups. In [3], Hofmann and Ruppert classify the reduced exponential Lie semigroups and show in the process that the Lie wedge of any exponential Lie semigroup is a Lie semialgebra. The semisimple part of such a Lie semialgebra is the direct sum of Lie semialgebras in $\mathfrak{sl}(2,\mathbb{R})$ which do not meet the interior of the standard double cone.

Nowadays the assertion stated first in [2] that the generating Lie semialgebras of $\mathfrak{sl}(2,\mathbb{R})$ not meeting the interior of the standard double cone are necessarily polyhedral and, in fact, the intersection of at most

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four half-spaces, is a classical result. The proof of this result was left to the reader as exercise E.II.1 of [2]. Though this property of the Lie semialgebras of $\mathfrak{sl}(2,\mathbb{R})$ is fundamental and intuitively clear for everyone working in the theory of subsemigroups of Lie groups, there is no direct proof of it in the literature. In [1] this property is obtained as a consequence of some results concerning so-called rectangular domains in $\mathfrak{sl}(2,\mathbb{R})$. In the present paper we offer an elementary and direct proof of this property.

**2. Notations and basic facts.** (cf. [2], p.105ff) We denote the Killing form of $\mathfrak{sl}(2,\mathbb{R})$ by $\kappa$. Following [2], let

$$H=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad P=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\quad Q=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}.$$

The matrix of $\kappa$ with respect to the basis $\{H,P,Q\}$ is

$$\begin{pmatrix}8 & 0 & 0\\ 0 & 0 & 4\\ 0 & 4 & 0\end{pmatrix}.$$

For an element $X\in\mathfrak{sl}(2,\mathbb{R})$, the equality $\kappa(X,X)=-8\det(X)$ holds.

The Killing form determines the so-called standard double cone (or light cone):

$$\begin{aligned}\mathcal{W}:&=\{X\in\mathfrak{sl}(2,\mathbb{R})\mid \kappa(X,X)\le 0\}=\\ &=\{hH+pP+qQ\mid h,p,q\in\mathbb{R},\ h^2+pq\le 0\}.\end{aligned}$$

The boundary of $\mathcal{W}$ is then

$$\partial\mathcal{W}=\{X\in\mathfrak{sl}(2,\mathbb{R})\mid \kappa(X,X)=0\}.$$

The double cone is the union of the two cones $\mathcal{W}^+$ and $\mathcal{W}^-=-\mathcal{W}^+$, where

$$\begin{aligned}\mathcal{W}^+ :&=\{hH+pP+qQ\in\mathcal{W}\mid p\ge 0,\ q\le 0\},\\ \mathcal{W}^- :&=\{hH+pP+qQ\mid p\le 0,\ q\ge 0\}.\end{aligned}$$

(Note that $\mathcal{W}^+$ and $\mathcal{W}^-$ are obtained from the two connected components of $\mathcal{W}\setminus\{0\}$ by reinserting 0.)

**3. Lie wedges.** Recall that if $S$ is a closed subsemigroup with $\mathbf{1}\in$ $\in S$ of a Lie group $G$ with Lie algebra $\mathfrak{g}$ then the Lie wedge of $S$ is the set of all $X\in\mathfrak{g}$ with $\exp(\mathbb{R}_0^+X)\subseteq S$, where $\mathbb{R}_0^+$ denotes the set of nonnegative reals.

**4. Lie semialgebras.** (cf. [2], p.86) Let $W$ be a wedge in a Lie algebra $\mathfrak{g}$. Then $W$ is called a Lie semialgebra if it is a local semigroup with respect to the Campbell-Hausdorff multiplication $*$, i.e., there exists a Campbell-Hausdorff neighborhood $B$ such that $(B\cap W)*(B\cap W) \subseteq W$.

**4. Half-space semialgebras in $\mathfrak{sl}(2,\mathbb{R})$.** A subset of a Lie algebra is called a half-space semialgebra if it is simultaneously a half-space and a Lie semialgebra.

For an element $X \in \partial\mathcal{W} \setminus \{0\}$ let

$$X^* := \{Y \in \mathfrak{sl}(2,\mathbb{R}) \mid \kappa(X,Y) \geq 0\}.$$

Cf.[2], p.109, the set $X^*$ is a half-space semialgebra for every $X \in \partial\mathcal{W} \setminus \{0\}$, and conversely, all half-space semialgebras are of the form $X^*$ with a suitable $X \in \partial\mathcal{W}$.

The following assertions can be checked by straightforward computation (see also II.3.6 of [2]):

(i) Every half-space semialgebra $X^*$ determined by an $X \in \mathcal{W}^+ \cap \partial\mathcal{W} \setminus \{0\}$ is conjugate to $P^* = \{hH + pP + qQ \mid q \geq 0\}$. Similarly, every half-space $X^*$ with $X \in \mathcal{W}^- \cap \partial\mathcal{W} \setminus \{0\}$ is conjugate to $Q^* = \{hH + pP + qQ \mid p \geq 0\}$.

(ii) By (i), every half-space semialgebra $X^*$ with $X \in \mathcal{W}^+ \cap \partial\mathcal{W} \setminus \{0\}$ (resp., $X \in \mathcal{W}^- \cap \partial\mathcal{W} \setminus \{0\}$) contains $\mathcal{W}^+$ (resp., $\mathcal{W}^-$).

**6. The fundamental theorem on Lie semialgebras in $\mathfrak{sl}(2,\mathbb{R})$.** (cf. II.3.7 of [2]) *Every generating Lie semialgebra in* $\mathfrak{sl}(2,\mathbb{R})$ *is the intersection of a family of half-space semialgebras of the form* $X^*, X \in \partial\mathcal{W} \setminus \{0\}$.

**7. Consequences of the fundamental theorem.** Let $W$ be a generating Lie semialgebra in $\mathfrak{sl}(2,\mathbb{R})$. By the above theorem, there is a nonvoid subset $\mathcal{F}$ of $\partial\mathcal{W} \setminus \{0\}$ such that $W = \bigcap_{X\in\mathcal{F}} X^*$. Denote by $\mathcal{X} := \mathcal{F} \cap \mathcal{W}^+$ and by $\mathcal{Y} := \mathcal{F} \cap \mathcal{W}^-$. Then exactly one of the following situations holds:

(1) $\mathcal{X} = \emptyset$. In this case $\mathcal{W}^- \subseteq W$ by 5(ii).

(2) $\mathcal{Y} = \emptyset$. In this case $\mathcal{W}^+ \subseteq W$ by 5(i).

(3) $\mathcal{X} \neq \emptyset$ and $\mathcal{Y} \neq \emptyset$. In this case $W$ is the intersection of conjugates of the Lie semialgebra $\mathfrak{sl}(2,\mathbb{R})^+ = P^* \cap Q^* = \{hH + pP + qQ \mid p \geq 0, q \geq 0\}$. (Note that for $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ the generating Lie-semialgebra $X^* \cap Y^*$ is the image of $\mathfrak{sl}(2,\mathbb{R})^+$ under the inner automorphism $e^{s\,\mathrm{ad}\,P} \circ e^{t\,\mathrm{ad}(P-Q)}$ for suitable $s, t \in \mathbb{R}$).

**8. Remark.** The Lie semialgebras described in assertion (3) of 7 are mapped under the exponential function homeomorphically onto a subsemigroup of $\mathrm{Sl}(2,\mathbb{R})$, thus these Lie semialgebras occur as the Lie wedges of three dimensional exponential Lie subsemigroups of $\mathrm{Sl}(2,\mathbb{R})$ (see also 3.8 of [1]). Recall that a closed subsemigroup $S$ of a Lie group $G$ is called exponential if it is the exponential image $\exp(W)$ of its Lie wedge $W$.

**9. Problem.** Let $W$ be a generating Lie semialgebra in $\mathfrak{sl}(2,\mathbb{R})$ which is the intersection of conjugates of the Lie semialgebra $\mathfrak{sl}(2,\mathbb{R})^+$. (Equivalently, $W$ is the Lie wedge of an exponential Lie subsemigroup of $\mathrm{Sl}(2,\mathbb{R})$) with inner points). Then $W$ is the intersection of at most four half-space semialgebras. This assertion was formulated first in [2] (p.110; its proof was left to the reader as exercise EII.1). In [1] it is proved with the aid of the notion of rectangular domains in $\mathfrak{sl}(2,\mathbb{R})$ (see 5.5 and 5.6). In the following, we shall give a direct proof of this assertion. For this we need the following proposition.

**Proposition 10.** *Let $W$ be a generating Lie semialgebra in $\mathfrak{sl}(2,\mathbb{R})$.*

(i) *Suppose that*

$$W = \Big(\bigcap_{X \in \mathcal{X}} X^*\Big) \cap Y^*,$$

*where $\emptyset \neq \mathcal{X} \subseteq \mathcal{W}^+ \cap \partial\mathcal{W} \setminus \{0\}$ and $Y \in \mathcal{W}^- \cap \partial\mathcal{W} \setminus \{0\}$. Then there exist $X_1, X_2 \in \mathcal{W}^+ \cap \partial\mathcal{W} \setminus \{0\}$ such that $W = X_1^* \cap X_2^* \cap Y^*$.*

(ii) *Suppose that*

$$W = X^* \cap \Big(\bigcap_{Y \in \mathcal{Y}} Y^*\Big),$$

*where $X \in \mathcal{W}^+ \cap \partial\mathcal{W} \setminus \{0\}$ and $\emptyset \neq \mathcal{Y} \subseteq \mathcal{W}^- \cap \partial\mathcal{W} \setminus \{0\}$. Then there exist $Y_1, Y_2 \in \mathcal{W}^- \cap \partial\mathcal{W} \setminus \{0\}$ such that $W = X^* \cap Y_1^* \cap Y_2^*$.*

**Proof.** (i) By 5(i), we can assume without any loss of generality that $Y = Q$, thus $Y^* = \{hH + pP + qQ \mid p \geq 0\}$. For every $X \in \mathcal{X}$, let $X = h_x H + p_x P + q_x Q$. Thus $h_x^2 + p_x q_x = 0$, $p_x \geq 0$ and $q_x \leq 0$. We have $p_x \neq 0$. Indeed, the equality $p_x = 0$ implies $h_x = 0$ and $q_x < 0$, hence $X^* = (q_x Q)^* = -Q^*$. Thus $W \subseteq (-Q^*) \cap Q^*$, a contradiction, since $W$ is generating. So $p_x > 0$ for every $X \in \mathcal{X}$.

We show that the set

$$\left\{\frac{h_x}{p_x} \mid X \in \mathcal{X}\right\}$$

is bounded. For this, we first observe that if $hH + pP + qQ \in W$, then $h^2 + pq \geq 0$ (since $W$ is contained in a conjugate of $\mathfrak{sl}(2,\mathbb{R})^+$). Consider an element $hH + pP + qQ$ of $\mathfrak{sl}(2,\mathbb{R})$. This element lies in $W = (\bigcap_{X\in\mathcal{X}} X^*) \cap Q^*$ if and only if $p \geq 0$ and $2h_x h + p_x q + q_x p \geq 0$ for every $X \in \mathcal{X}$. Since $p_x > 0$ and $p_x q_x = -h_x^2$, the matrix $hH + pP + qQ$ belongs to $W$ if and only if $p > 0$ and

$$-\left(\frac{h_x}{p_x}\right)^2 p + 2\frac{h_x}{p_x}h + q \geq 0 \text{ for every } X \in \mathcal{X}.$$

Denote by $f : \mathbb{R} \to \mathbb{R}$ the function defined by $f(u) = -pu^2 + 2hu + q$ and by $\Delta := h^2 + pq$. Taking into account that $p \geq 0$, the inequality $f\left(\frac{h_x}{p_x}\right) \geq 0$ holds if and only if one of the following two conditions is satisfied:

(1) $p = 0$ and $2h\frac{h_x}{p_x} + q \geq 0$,

(2) $p > 0$ and $\frac{h-\sqrt{\Delta}}{p} \leq \frac{h_x}{p_x} \leq \frac{h+\sqrt{\Delta}}{p}$.

Thus $hH+pP+qQ \in W$ if and only if exactly one of the conditions (1) and (2) holds for every $X \in \mathcal{X}$.

The wedge $W$ contains elements $hH + pP + qQ$ with $p > 0$ (otherwise $W$ would be a subset of the subalgebra spanned by $H$ and $Q$, a contradiction to the fact that $W$ is generating). So the above condition (2) implies that the set $\left\{\frac{h_x}{p_x} \mid X \in \mathcal{X}\right\}$ is bounded. Let $\alpha$ be the infimum and $\beta$ be the supremum of this set. Set $X_1 := \alpha H + P - \alpha^2 Q$ and $X_2 := \beta H + P - \beta^2 Q$. In virtue of (1) and (2) the equality $W = X_1^* \cap X_2^* \cap Q^*$ follows.

Assertion (ii) follows from (i) by multiplication with $-1$. ◊

**Theorem 11.** *A generating Lie semialgebra* $W$ *in* $\mathfrak{sl}(2,\mathbb{R})$ *which is the intersection of some conjugates of* $\mathfrak{sl}(2,\mathbb{R})^+$ *is the intersection of at most four half-space semialgebras.*

**Proof.** By assertion (3) of 7, we have that

$$W = (\bigcap_{X\in\mathcal{X}} X^*) \cap (\bigcap_{Y\in\mathcal{Y}} Y^*),$$

where $\emptyset \neq \mathcal{X} \subseteq \mathcal{W}^+ \cap \partial\mathcal{W} \setminus \{0\}$ and $\emptyset \neq \mathcal{Y} \subseteq \mathcal{W}^- \cap \partial\mathcal{W} \setminus \{0\}$. Since $\mathcal{Y} \neq \emptyset$, assertion (i) of 10 implies the existence of $X_1, X_2 \in \mathcal{W}^+ \cap \partial\mathcal{W} \setminus \{0\}$ such that

$$W = X_1^* \cap X_2^* \cap \big(\bigcap_{Y \in \mathcal{Y}} Y^*\big).$$

Applying now assertion (ii) of 10, there exist $Y_1, Y_2 \in \mathcal{W}^- \cap \partial\mathcal{W} \setminus \{0\}$ such that

$$W = X_1^* \cap X_2^* \cap Y_1^* \cap Y_2^*.$$

Thus $W$ is the intersection of at most four half-space semialgebras. ◊

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