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EXPONENTIALITY IN CATEGORIES OF PARTIAL ALGEBRAS

Josef **Šlapal**

Departement of Mathematics, Technical University of Brno, Technická 2, 616 69 Brno, Czech Republic

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Abstract: In categories of partial algebras we investigate the property of exponentiality, i.e., existence of function spaces for some pairs of objects. By this way we discover two new cartesian closed subcategories in every category of partial algebras of the same type.

It is well known that cartesian closed categories have useful applications to many branches of mathematics, especially to theoretic computer science where they form models of an important abstract programming language (the so-called typed λ-calculus). It is therefore worthwhile to look for some new cartesian closed categories, and this is precisely what we do in this note. We continue with the study of exponentiality, a more general property than cartesian closedness, in categories of partial algebras started in [11]. As consequences of our results we obtain two new cartesian closed categories of partial algebras of

E-mail address: slapal@um.fme.rutbr.cz
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the same type together with descriptions of their function spaces. These categories lie, with respect to the full categorical inclusion, between the two cartesian closed categories of partial algebras discovered in [11]. For the convenience of the reader we repeat the relevant material from [11] without proofs, thus making our exposition self-contained.

For the categorical terminology used see e.g. [3]. Throughout the paper, all categories are considered to be constructs, i.e., categories of structured sets and structure-compatible maps.

Definition 1. [9] Let $\mathbf{K}$ be a category with finite products and $\mathbf{S}, \mathbf{T}$ be full isomorphism closed subcategories of $\mathbf{K}$. Let $\mathbf{T}$ be finitely productive in $\mathbf{K}$. We say that $\mathbf{T}$ is *exponential* for $\mathbf{S}$ in $\mathbf{K}$ provided that for any two objects $A \in \mathbf{S}$ and $B \in \mathbf{T}$ there exists an object $A^B \in \mathbf{K}$ with $|A^B| = \mathrm{Mor}_{\mathbf{K}}(B, A)$ such that

(i) $A^B \in \mathbf{S} \cap \mathbf{T}$

(ii) the pair (A^B, e), where $e : B \times A^B \to A$ is the evaluation map (given by $e(y, f) = f(y)$), is a co-universal map for A with respect to the functor $B \times - : \mathbf{T} \to \mathbf{K}$.

If a category $\mathbf{T}$ is exponential for $\mathbf{K}$ in $\mathbf{K}$, then $\mathbf{T}$ will be called an *exponential subcategory* of $\mathbf{K}$ (cf. [7]). If $\mathbf{K}$ is an exponential subcategory of itself, then $\mathbf{K}$ is cartesian closed [2], i.e., the functor $B \times - : \mathbf{K} \to \mathbf{K}$ has a right adjoint for each object $B \in \mathbf{K}$ (and vice versa whenever in $\mathbf{K}$ all constant maps are morphisms). Especially, if $\mathbf{T}$ is exponential for $\mathbf{S}$ in $\mathbf{K}$ and if also $\mathbf{S}$ is finitely productive in $\mathbf{K}$, then $\mathbf{S} \cap \mathbf{T}$ is cartesian closed.

The objects A^B from Def. 1 are called *function spaces.* In [9] it is shown that function spaces fulfil the first exponential law, i.e. the law $(A^B)^C \simeq A^{B\times C}$ (where $\simeq$ denotes the isomorphism in $\mathbf{K}$), and that they are unique up to the isomorphisms that are (carried by) identity maps - hence unique whenever $\mathbf{K}$ is transportable.

As for partial algebras, the fundamental concepts used are taken from [1]. Throughout the paper, Ω will designate an arbitrary, but fixed set, and τ will designate an arbitrary, but fixed family of sets $\tau = (K_\lambda; \lambda \in \Omega)$. The family τ will be called a type. By a partial algebra of type τ we understand a pair $\langle X, (p_\lambda; \lambda \in \Omega)\rangle$ where X is a set and p_λ is a partial K_λ-ary operation on X (i.e. a partial map $p_\lambda : X^{K_\lambda} \to X$) for each $\lambda \in \Omega$. For any $\lambda \in \Omega$ we denote by D_{p_λ} the domain of the operation p_λ, i.e. the subset of X^{K_λ} having the property that $p_\lambda(x_k; k \in K_\lambda)$ is defined iff $(x_k; k \in K_\lambda) \in D_{p_\lambda}$. If $G = \langle X, (p_\lambda; \lambda \in$

$\in \Omega)\rangle$ and $H = \langle Y, (q_\lambda; \lambda \in \Omega)\rangle$ are partial algebras of type τ, then by a homomorphism of G into H we mean any map $f : X \to Y$ such that $p_\lambda(x_k; k \in K_\lambda) = x \Rightarrow q_\lambda(f(x_k); k \in K_\lambda) = f(x)$ for each $\lambda \in \in \Omega$. The set of all homomorphisms from G into H will be denoted by $\mathrm{Hom}(G, H)$. We denote by Pal_τ the category of all partial algebras of type τ with homomorphisms as morphisms. Obviously, Pal_τ is a transportable category with products (given by the direct products — see [1]).

Definition 2 [11]. Let $G = \langle X, (p_\lambda; \lambda \in \Omega)\rangle$, $H = \langle Y, (q_\lambda; \lambda \in \Omega)\rangle$ be partial algebras of type τ. The *power* of G and H is a partial algebra $[H, G] = \langle \mathrm{Hom}(H, G), (r_\lambda; \lambda \in \Omega)\rangle$ of type τ where, for each $\lambda \in \Omega$, r_λ is the K_λ-ary partial operation on $\mathrm{Hom}(H, G)$ given by $r_\lambda(f_k; k \in K_\lambda) = = f$ iff $f \in \mathrm{Hom}(H, G)$ is a unique homomorphism with the property that $q_\lambda(y_k; k \in K_\lambda) = y \Rightarrow p_\lambda(f_k(y_k); k \in K_\lambda) = f(y)$.

Definition 3. Let $G = \langle X, (p_\lambda; \lambda \in \Omega)\rangle$, $H = \langle Y, (q_\lambda; \lambda \in \Omega)\rangle$ be partial algebras of type τ and $S = \langle Z, (r_\lambda; \lambda \in \Omega)\rangle$ any partial algebra of type τ with $Z = \mathrm{Hom}(H, G)$. We say that S has

(1) *point-wise structure* if for any $\lambda \in \Omega$, any $(f_k; k \in K_\lambda) \in Z^{K_\lambda}$ and any $f \in Z$ there holds $r_\lambda(f_k; k \in K_\lambda) = f \Leftrightarrow p_\lambda(f_k(y); k \in \in K_\lambda) = f(y)$ for each $y \in Y$,

(2) *direct structure* if S is a subalgebra of the direct power $G^{|H|}$.

Remark 1. Clearly, S has direct structure if and only if it has point-wise structure and for any $\lambda \in \Omega$ and any $(f_k; k \in K_\lambda) \in Z^{K_\lambda}$, whenever $p_\lambda(f_k(y); k \in K_\lambda) \in D_{p_\lambda}$ for each $y \in Y$, the map $f : Y \to X$ given by $f(y) = p_\lambda(f_k(y); k \in K_\lambda)$ fulfils $f \in \mathrm{Hom}(H, G)$.

Definition 4. A partial algebra $\langle X, (p_\lambda; \lambda \in \Omega)\rangle$ of type τ is called *idempotent* if for any $x \in X$ and any $\lambda \in \Omega$ the family $(x_k; k \in K_\lambda)$ given by $x_k = x$ for each $k \in K_\lambda$ fulfils $(x_k; k \in K_\lambda) \in D_{p_\lambda}$ and $p_\lambda(x_k; k \in K_\lambda) = x$.

Let X, K, L be sets. By a $K \times L$-matrix over X we understand any map $M : K \times L \to X$, i.e. $M = (x_{kl}; k \in K, l \in L)$ where $x_{kl} \in X$ whenever $k \in K$ and $l \in L$.

Definition 5. Let $G = \langle X, (p_\lambda; \lambda \in \Omega)\rangle$ be a partial algebra of type τ.

(1) Let $\lambda, \mu \in \Omega$ and let $M = (x_{kl}; k \in K_\lambda, l \in K_\mu)$ be a $K_\lambda \times \times K_\mu$-matrix over X. We say that p_λ and p_μ fulfil the *interchange law* on M if from $(x_{kl}; l \in K_\mu) \in D_{p_\mu}$ for each $k \in K_\lambda$, $(p_\mu(x_{kl}; l \in K_\mu); k \in \in K_\lambda) \in D_{p_\lambda}$ and $(x_{kl}; k \in K_\lambda) \in D_{p_\lambda}$ for each $l \in K_\mu$ it follows that $(p_\lambda(x_{kl}; k \in K_\lambda); l \in K_\mu) \in D_{p_\mu}$ and $p_\lambda(p_\mu(x_{kl}; l \in K_\mu); k \in K_\lambda) =$

$= p_\mu(p_\lambda(x_{kl}; k \in K_\lambda); l \in K_\mu)$. If p_λ and p_μ fulfil the interchange law on M for each pair $\lambda, \mu \in \Omega$ and each $K_\lambda \times K_\mu$-matrix M over X, then G is said to fulfil the *interchange law.*

(2) Let $\lambda \in \Omega$ and let $M = (x_{kl}; k \in K_\lambda, l \in K_\lambda)$ be a $K_\lambda \times \times K_\lambda$-matrix over X. We say that p_λ has the *diagonal property* on M if from $(x_{kl}; l \in K_\lambda) \in D_{p_\lambda}$ for each $k \in K_\lambda$ and $(p_\lambda(x_{kl}; l \in K_\lambda); k \in \in K_\lambda) \in D_{p_\lambda}$ it follows that $(x_{kk}; k \in K_\lambda) \in D_{p_\lambda}$ and $p_\lambda(p_\lambda(x_{kl}; l \in \in K_\lambda); k \in K_\lambda) = p_\lambda(x_{kk}; k \in K_\lambda)$. If p_λ has the diagonal property on M for each $\lambda \in \Omega$ and each $K_\lambda \times K_\lambda$-matrix M over X, then G is called *diagonal.*

(3) G is called *weakly diagonal* if for any $\lambda \in \Omega$ and any $K_\lambda \times K_\lambda$-matrix $M = (x_{kl}; k \in K_\lambda, l \in K_\lambda)$ over X with $(x_{kl}; k \in K_\lambda) \in D_{p_\lambda}$ for each $l \in K_\lambda$, whenever p_λ and p_λ fulfil the interchange law on M, p_λ has the diagonal property on M.

Remark 2. a) For total algebras the interchange law coincides with the commutativity studied in [5]. Diagonal idempotent total algebras are investigated in [8]. It is evident that the diagonality implies the weak diagonality.

b) Let $G = \langle X, (p_\lambda; \lambda \in \Omega)\rangle$ be a partial algebra of type τ. Let G be called medial (cf. [4]) if p_λ and p_λ fulfil the interchange law on M for each $\lambda \in \Omega$ and each $K_\lambda \times K_\lambda$-matrix M over X. Clearly, if G fulfils the interchange law, then it is medial, and vice versa whenever card $\Omega = 1$.

We denote by $IPal_\tau$, $CPal_\tau$, $DPal_\tau$, or $WPal_\tau$ respectively the full subcategory of Pal_τ whose objects are precisely the partial algebras of type τ that are idempotent, fulfil the interchange law, are diagonal, or are weakly diagonal respectively. Further, we put $DIPal_\tau = DPal_\tau \cap \cap IPal_\tau$, $CDIPal_\tau = CPal_\tau \cap DIPal_\tau$, etc.

We will need the following result from [11]:

Theorem 1. *$IPal_\tau$ is an exponential subcategory of Pal_τ and the corresponding function spaces G^H coincide with the powers $[H, G]$.*

Theorem 2. *$IPal_\tau$ is exponential for $WPal_\tau$ in Pal_τ and the corresponding function spaces have point-wise structure.*

Proof. Let $G = \langle X, (p_\lambda; \lambda \in \Omega)\rangle$ be a $WPal_\tau$-object and $H = = \langle Y, (q_\lambda; \lambda \in \Omega)\rangle$ an $IPal_\tau$-object. We will show that the power $[H, G]$ has point-wise structure. To this end, let $[H, G] = \langle Z, (r_\lambda; \lambda \in \in \Omega)\rangle$. Next, let $\lambda \in \Omega$, $(f_k; k \in K_\lambda) \in Z^{K_\lambda}$ and $f \in Z$. If we suppose that $r_\lambda(f_k; k \in K_\lambda) = f$, then the idempotency of H implies $p_\lambda(f_k(y); k \in K_\lambda) = f(y)$ for each $y \in Y$. Conversely, suppose that

$p_\lambda(f_k(y); k \in K_\lambda) = f(y)$ for each $y \in Y$. Let $(y_k; k \in K_\lambda) \in D_{q_\lambda}$, $q_\lambda(y_k; k \in K_\lambda) = y$, and let M be the $K_\lambda \times K_\lambda$-matrix over X given by $K = (m_{kl}; k \in K_\lambda, l \in K_\lambda)$ where $m_{kl} = f_k(y_l)$ whenever $k, l \in K_\lambda$. As $f_k \in \mathrm{Hom}(H, G)$ for each $k \in K_\lambda$, we have $p_\lambda(f_k(y_l); l \in K_\lambda) = f_k(y)$ for each $k \in K_\lambda$. Further, there holds $p_\lambda(f_k(y_l); k \in K_\lambda) = f(y_l)$ for each $l \in K_\lambda$. Consequently, we get $p_\lambda(p_\lambda(f_k(y_l); l \in K_\lambda); k \in K_\lambda) = = p_\lambda(f_k(y); k \in K_\lambda) = f(y) = f(q_\lambda(y_l; l \in K_\lambda)) = p_\lambda(f(y_l); l \in K_\lambda) = = p_\lambda(p_\lambda(f_k(y_l); k \in K_\lambda); l \in K_\lambda)$ which means that p_λ fulfils the interchange law on M. As G is weakly diagonal, p_λ has the diagonal property on M, i.e., $p_\lambda(f_k(y_k); k \in K_\lambda) = p_\lambda(p_\lambda(f_k(y_l); l \in K_\lambda); k \in \in K_\lambda) = f(y)$. Because the idempotency of H implies the uniqueness of f, we have $r_\lambda(f_k; k \in K_\lambda) = f$. Hence $[H, G]$ has point-wise structure. As it can easily be seen that $[H, G]$ is weakly diagonal, the statement follows from Th. 1. ◊

Corollary 1. *$IPal_\tau$ is exponential for $DPal_\tau$ in Pal_τ and the corresponding function spaces have point-wise structure.*

Proof. It can easily be seen that the power $[H, G]$ is diagonal whenever G is diagonal. Now the statement follows from Th. 2. ◊

Corollary 2. *$IPal_\tau$ is exponential for $CDPal_\tau$ in Pal_τ and the corresponding function spaces have point-wise structure.*

Proof. Clearly, $[H, G]$ fulfils the interchange law whenever G fulfils the interchange law . Thus, the statement follows from Cor. 1. ◊

In [11] the following strengthening of Cor. 2 is proved:

Theorem 3. *$IPal_\tau$ is exponential for $CDPal_\tau$ in Pal_τ and the corresponding function spaces have direct structure.*

Remark 3. a) Let $\subseteq$ denote the full categorical inclusion. Then we have $CDIPal_\tau \subseteq DIPal_\tau \subseteq WIPal_\tau \subseteq IPal_\tau$. According to the previous considerations, all the four categories are cartesian closed with function spaces given by powers. In all of them, excluding $IPal_\tau$, the function spaces have point-wise structure. Moreover, the function spaces in $CDIPal_\tau$ have direct structure. (For $CDIPal_\tau$ and $IPal_\tau$ these results are proved in [11]).

b) The power of two total algebras (of the same type) is not a total algebra in general, and the same is true even if the power has point-wise structure. On the other hand, if this power has direct structure, then it is a total algebra. Thus, Th. 3 remains valid also when restricting our considerations to total algebras (cf. [10]).

Example. Some examples of idempotent algebras, algebras fulfilling the interchange law, and diagonal algebras can be found in [11]. As an example of a weakly diagonal algebra we can take a partial rectangular band, i.e., a partial mono-binary algebra $G = \langle X \times Y, *\rangle$ where X, Y are sets and $(x_1, y_1) * (x_2, y_2)$ is defined iff $y_1 = x_2$ and then $(x_1, y_1) * * (x_2, y_2) = (x_1, y_2)$. If $\mathrm{card}(X \cap Y) \geq 2$, then G is not diagonal.

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