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THE RADICALNESS OF POLYNOMIAL RINGS OVER NIL RINGS

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Dedicated to my teacher Professor R. Wiegandt on his 70-th birthday

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Abstract: The main purpose of this note is to give the exact upper bound of approximating Köthe's Problem by radicals. We construct and characterize the smallest radical ℓ such that $A[x] \in \ell$ for every nil ring A and show that this improves the approximation given in [1].

1. In this note associative rings and Kurosh–Amitsur radicals will be considered. As usual, $I \triangleleft A$ and $L \triangleleft_\ell A$ denote that I is an ideal and L is a left ideal in A, respectively.

A class $\mathcal{M}$ of rings is said to be *regular*, if every nonzero ideal of a ring in $\mathcal{M}$ has a nonzero homomorphic image in $\mathcal{M}$. Starting from a regular (in particular, hereditary) class $\mathcal{M}$ of rings the *upper radical operator* $\mathcal{U}$ yields a radical class:

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$$\mathcal{UM} = \{A \mid A \text{ has no nonzero homomorphic image in } \mathcal{M}\}.$$

For a radical class γ the *semisimple operator* $\mathcal{S}$ gives its semisimple class:

$$\mathcal{S}\gamma = \{A \mid A \text{ has no nonzero ideal in } \gamma\}.$$

Köthe's Problem: Is the sum of two nil left ideals nil?

It has been posed in 1930 at the genesis of radical theory [6]. This problem has many equivalent formulations. One of the most interesting one, which stimulated many further studies, is the following due to Krempa [7].

Does $A \in \mathcal{N}$ imply that the polynomial ring $A[x]$ in indeterminate x over A is in $\mathcal{J}$, where $\mathcal{N}$ and $\mathcal{J}$ denote the classes of nil rings and Jacobson radical rings, respectively?

In [9] it has been proved that $A \in \mathcal{N}$ implies $A[x] \in \mathcal{G}$, where $\mathcal{G}$ stands for the Brown–McCoy radical.

We consider two natural radicals:

• *The antiregular radical* $\mathcal{U}\nu$. This is the upper radical determined by the class ν of all von Neumann regular rings.

• *The uniformly strongly prime radical* $\mathcal{u}$. A ring A is said to be *uniformly strongly prime*, if there exists a finite subset F of A, called a *uniform insulator*, such that $xFy \neq 0$ whenever $0 \neq x, y \in A$. The uniformly stongly prime radical is the upper radical determined by the class of uniformly strongly prime rings [8].

In [2] it has been proved that $A \in \mathcal{N}$ implies $A[x] \in \mathcal{U}\nu \cap \mathcal{G} \cap \mathcal{u}$ (see [2, Cor. 3.5]).

We recall also some statements we shall need in the sequel.

The *upper radical* $\mathcal{N}_s$ determined by the class of rings which contain no nonzero nil left ideals or, equivalently, no nonzero nil right ideals is called the *lower strong radical determined by* $\mathcal{N}$ (see [1] and [2]).

The *Behrens radical* $\mathcal{B}$ is the upper radical determined by the class of all subdirectly irreducible rings having a nonzero idempotent in their heart.

Recently, in [1] the following has been proved.

Proposition 1.1. *$A \in \mathcal{N}_s$ implies $A[x] \in \mathcal{B}$.*

Proposition 1.2 [2, Th. 3.4]. *$\mathcal{N}_s \subseteq \mathcal{u}$.*

We say that a ring A has *bounded index of nilpotency* if there is a positive integer m such that $a^m = 0$ for each nilpotent element a of A [4].

Proposition 1.3 [5, Th. 10.8.2]. *Let A be PI algebra of degree d.Let $A(1)$ be the sum of the nilpotent ideals of A, and B any nil subalgebra of A. Then $B^m \subseteq A(1)$ where $m = [d/2]$.*

Proposition 1.4 [3, Th. 6.53]. *If in a ring A there exists a fixed positive integer n such that $x^n = 0$ for every $x \in A$, then A is locally nilpotent.*

The *Baer radical* β is the upper radical determined by the class of prime rings. A prime ring A is said to be $*$-*ring* if its every proper homomorphic image A' is in β. We denote by $M(A)$ the infinite matrix ring which has only finitely many nonzero entries from A.

Proposition 1.5 [12, Lemma 7]. *If A is a $*$-ring, then $M(A)$ is a $*$-ring with trivial center.*

A class M of rings is said to be *principally left hereditary* if $a \in \in A \in \mathcal{M}$, then $Aa \in \mathcal{M}$.

Proposition 1.6 [13, Th. 5.1]. *The Behrens radical $\mathcal{B}$ is the largest principally left hereditary subclass of the Brown–McCoy radical class $\mathcal{G}$ in fact, $\mathcal{MG} = \mathcal{B}$ where*

$$\mathcal{MG} = \{\ A\ | Aa \in \mathcal{G} \textit{ for every } a \in A\}.$$

2. We set

$$\mathcal{M} = \left\{ A \;\middle|\; \begin{array}{l} A \text{ has no nonzero locally nilpotent ideals and} \\ \text{every nil subring } S \text{ of } A \text{ is locally nilpotent} \end{array} \right\},$$

$$\mathcal{M}_0 = \left\{ A \;\middle|\; \begin{array}{l} A \text{ has no nonzero nil ideals and} \\ \text{all nilpotent elements form a subring in } A \end{array} \right\}.$$

Lemma 2.1. *$\mathcal{M}$ and $\mathcal{M}_0$ are*

a) *hereditary classes of rings;*

b) *both consist of semiprime rings;*

c) *both contain no nonzero nilrings.*

Proof. Trivial. $\Diamond$

Recall that a radical σ is said to be *left strong* if $\sigma(L) = L \triangleleft_\ell A$ implies $L \subseteq \sigma(A)$. *Right strong* radical is defined correspondingly.

Proposition 2.2. *$\gamma = \mathcal{UM}$ and $\delta = \mathcal{UM}_0$ are left and right strong and so is $\gamma \cap \delta$.*

Proof. Let $\gamma(L) = L \triangleleft_\ell A$, and $L \not\subseteq \gamma(A)$. Then we have

$$0 \neq \gamma\left(\frac{L+\gamma(A)}{\gamma(A)}\right) = \frac{L+\gamma(A)}{\gamma(A)} \triangleleft_\ell \frac{A}{\gamma(A)} \in \mathcal{S}\gamma.$$

Hence, we can choose $\gamma(A)=0$ and so $B = L + LA \in \mathcal{S}\gamma$. Therefore B has a nonzero homomorphic image B/I in $\mathcal{M}$. Let $\langle I\rangle$ be the ideal of A, generated by I. By Andrunakievich Lemma $\langle I\rangle^3 \subseteq I \subseteq \langle I\rangle$ and so by Lemma 2.1 a) and b) $\langle I\rangle = I$. Thus it follows that $I \triangleleft A$. Hence $L \not\subseteq I$. Again we can choose $B \in \mathcal{M}$. By Lemma 2.1 c) $\mathcal{N} \subseteq \gamma$ and so also the locally nilpotent radical $\mathcal{L}$ is contained in γ. Since $\mathcal{L}$ is left strong, we have $\mathcal{L}(L) \neq L$ and so $0 \neq L/\mathcal{L}(L) \in \gamma$. Hence $L/\mathcal{L}(L)$ has a non-locally nilpotent and nil subring $\overline{S}$. Let $S/\mathcal{L}(L) = \overline{S}$,then S is a nil subring of B which is not locally nilpotent, contradicting $B \in \mathcal{M}$. For δ the proof is similar. $\Diamond$

Corollary 2.3. $\mathcal{N}_s \subseteq \gamma \cap \delta \cap \mathcal{B} \cap u$.

Proof. $\mathcal{N}_s \subseteq \mathcal{B} \cap u$ follows from Props. 1.1 and 1.2. Since $N \subseteq \gamma \cap \delta$, by Prop. 2.2 we get $\mathcal{N}_s \subseteq \gamma \cap \delta$. $\Diamond$

Lemma 2.4. *If for a ring A the factor ring $A[x]/I$ is a prime (semiprime) ring, then there exist a prime (semiprime) ring B and an ideal J of $B[x]$ such that $A[x]/I \cong B[x]/J$ and $B \cap J = 0$.*

Proof. Let $H = A \cap I \triangleleft A$. Since $H^2[x] = (A\cap I)^2[x] \subseteq I$ and $(H[x])^2 \subseteq \subseteq H^2[x] \subseteq I$. We claim that $H[x] \subseteq I$. Suppose that $H[x] \not\subseteq I$. Then $I \subset H[x] + I$ and $H^2[x] \subseteq (H[x]+I)^2 \subseteq I$ by $H^2[x] \subseteq I$. Since I is a semiprime ideal, we conclude $H[x] \subseteq I$. So

$$\frac{I}{H[x]} \triangleleft \frac{A[x]}{H[x]} \overset{f}{\cong} (A/H)[x],$$

where f is an isomorphism of $(A[x])/(H[x])$ onto $(A/H)[x]$ such that

$$f\left(\sum_{i=0}^{n} a_i x^i + H[x]\right) = \sum_{i=0}^{n}(a_i + H)x^i, \quad \text{for } a_i \in A.$$

Choose $B = A/H$ and $J = f(I/H[x])$. Then we have

$$\frac{B[x]}{J} \cong \frac{A[x]/H[x]}{I/H[x]} \cong \frac{A[x]}{I},$$

and we claim that $B \cap J = 0$. If $B \cap J \neq 0$ then $0 \neq B \cap J = = H_1/H$, and $H \subset H_1 \triangleleft A$. Let $0 \neq h \in H_1 \setminus H$. Since $H[x] \subseteq I$ and $h + H[x] = f^{-1}(h+H) \in f^{-1}(J) = I + H[x] = I$. We get $h \in I$ and so $H_1 + H[x] \subseteq I$. Thus $H_1 \subseteq I$, contradicting $A \cap I = H$.

Now, we shall show that B is semiprime. If B is not semiprime then there exists an ideal H_1 of B such that $H \subset H_1$ and $H_1^2 \subseteq H$.

Hence $H_1^2[x] \subseteq H[x]$. So $H_1^2[x] \subseteq I$, and as above we have $H_1[x] \subseteq I$. Hence it follows $I_1 \subseteq I$, and so $H_1 \subseteq I \cap A = H$ implying $H_1 = H$, a contradiction.

Let $A[x]/I$ be a prime ring. If $H \subset H_1 \triangleleft A$ and $H \subset H_2 \triangleleft A$ and $H_1 H_2 \subseteq H$, then $(H_1 \cap H_2)^2 \subseteq H_1 H_2 \subseteq H$. It follows again that $H_1 \cap H_2 \subseteq I$, and so $H_1 \cap H_2 \subseteq H$.

Put $\overline{H}_1 = H_1/H$ and $\overline{H}_2 = H_2/H$, then $\overline{H}_1 \cap \overline{H}_2 = 0$. We have

$$\frac{H_1[x]}{H[x]} \cong (H_1/H)[x] = \overline{H}_1[x] \triangleleft B[x]$$

and

$$\frac{H_2[x]}{H[x]} \cong \frac{H_2[x]}{H[x]} = \overline{H}_2[x] \triangleleft B[x].$$

and also $\overline{H}_1[x] \cap \overline{H}_2[x] = 0$.

Since I is a prime ideal of $A[x]$ and

$$H_1[x] H_2[x] \subseteq H_1[x] \cap H_2[x] \subseteq I,$$

we conclude that either $H_1[x] \subseteq I$ or $H_2[x] \subseteq I$, and so either $H_1[x] \subseteq$ $\subseteq H[x]$ or $H_2[x] \subseteq H[x]$. Hence either $H_1 \subseteq H$ or $H_2 \subseteq H$, a contradiction. ◊

Corollary 2.5. *Let A and B be rings as in* Lemma 2.4. *If A is nil ring, then B is nil ring.* ◊

A ring A is said to be an *n-ring* if A is not a homomorphic image of the polynomial ring $B[x]$ for any nil subring B of A.

Put $n(x) = \{A \mid A$ has no nonzero accessible subring B which is n-ring$\}$. Denote by ℓ the lower radical generated by the class $\{A[x] \mid A$ is a nil ring$\}$.

Theorem 2.6. $\mathcal{U}n(x) = \ell$.

Proof. $\mathcal{U}n(x) \subseteq \ell$: Let $A \in \mathcal{U}n(x)$. then every homomorphic image A' has a nil subring $B \subseteq A'$, such that $B[x]/I \cong I_n \triangleleft \cdots \triangleleft A'$. Therefore $I_n \in \ell$. Hence $\ell(A') \neq 0$. If $\mathcal{U}n(x) \not\subseteq \ell$, then there exists a nonzero ring $A \in \mathcal{U}n(x) \cap \mathcal{S}\ell$. As above $\ell(A) \neq 0$, a contradiction.

$\ell \subseteq \mathcal{U}n(x)$: Let $A \in \ell \backslash \mathcal{U}n(x)$. Then A has a nonzero homomorphic image A' in $n(x)$. Since $A' \in \ell$, there exists an accessible subring $I_n \triangleleft \cdots \triangleleft A'$, which is a homomorphic image of $B[x]$, where B is a nil ring. Suppose $I_n \cong B[x]/I$. By Lemma 2.1 $I_n \cong B[x]/I$ is semiprime ring.

By Cor. 2.5, there exists a nil ring B' such that

$$B[x]/I \cong B'[x]/J \text{ and } B' \cap J = 0.$$

Since $B' \cap J = 0$, we have

$$B' \cong \frac{B'}{B' \cap J} \cong \frac{B' + J}{J} \subseteq \frac{B'[x]}{J} \cong \frac{B[x]}{I} \cong I_n.$$

So I_n contains a nil subring S which is isomorphic to B' and so $S[x] \cong$ $\cong B'[x]$. Hence I_n is a homomorphic image of $S[x]$. Therefore $I_n \notin n(x)$ and so $A' \notin n(x)$, a contradiction. ◊

Corollary 2.7. *Let σ be a radical. If $A \in \mathcal{N}$ imply $A[x] \in \sigma$ then $\ell \subseteq \sigma$.*

Lemma 2.8. *Let A be a semiprime commutative ring. Then every nil subring S of $M(A)$ is locally nilpotent.*

Proof. Since A is commutative, for any natural number n the standard polynomial S_{2n} actually is an identity of matrix ring $M_n(A)$ (see [10,6.1.17]). By Prop. 1.3, $M_n(A)$ has bounded index. Let m be the smallest among these indices.

Put

$${}_nM(A) = \{(a_{ij}) \mid a_{ij} \in A \text{ and } a_{ij} = 0 \text{ for } j > n\}$$

and

$$V = \{B \in {}_nM(A) \mid a_{ij} = 0 \text{ for } i, j \leq n\}.$$

Clearly $V \triangleleft_n M(A)$ and ${}_nM(A)/V \cong M_n(A)$. Let $B \in {}_nM(A)$ be a nilpotent element, then $B^m \in V$. Since $V^2 = 0$, also $B^{2m} = 0$. Hence ${}_nM(A)$ is of bounded index. For any $s \in S$, there exists natural number n, such that $s \in {}_nM(A)$. Since ${}_nM(A)$ is a left ideal of $M(A)$, also ${}_nM(A) \cap S \triangleleft_\ell S$. Therefore ${}_nM(A) \cap S$ is of bounded index nil ring. By Prop. 1.4, ${}_nM(A) \cap S$ is locally nilpotent. Since the locally nilpotent radical is left strong, S has a locally nilpotent ideal I_s of Swhich is $s \in I_s$, and so S is locally nilpotent. ◊

Theorem 2.9. $\ell = \mathcal{U}n(x) \subseteq \mathcal{B} \cap u \cap \gamma \cap \delta \subset \mathcal{B} \cap u \cap \delta.$

Proof. By Prop. 1.1 and Cor. 2.7, we get $\mathcal{U}n(x) \subseteq \mathcal{B} \cap u$. Let $A \in$ $\in \mathcal{U}n(x) \setminus \gamma$. Then there exists a nonzero homomorphic image A' of A in $\mathcal{M}$. Since $A' \in \mathcal{U}n(x)$, A' has a nonzero accessible subring I such that $I \cong B[x]/J$ and for a nil subring B of I by Lemma 2.4. Since $\mathcal{M}$ is hereditary, $I \in \mathcal{M}$. Hence B is locally nilpotent and so $B[x]/J$. Therefore I is locally nilpotent, a contradiction. It follows $\mathcal{U}n(x) \subseteq \gamma$.

Let $A \in \mathcal{U}n(x) \setminus \delta$. As above, we get an accessible subring I of $A' \in \mathcal{M}_0$ and so $I \in \mathcal{M}_0$ and $I \cong B[x]/J$. Since B is nil, for the semigroup $\{ax^n \mid a \in B, 0 \leq n \in \mathbb{Z}\}$ every element is nilpotent. The subring B' of $B[x]/J$ generated by the set $\{ax^n + J\}$ is isomorpic to I, because $\{ax^n + J\}$ are generators of $B[x]/J$. Hence I is nil ring. Again a contradiction. Thus, it follows $\mathcal{U}n(x) \subseteq \gamma \cap \delta$. Let us consider the ring

$$A = \left\{ \frac{2x}{2y+1} \mid x, y \in \mathbb{Z}, (2x, 2y+1) = 1 \right\}.$$

We know that A is a commutative $*$-ring (see [12]). We consider the ring $M(A)$. Since $M_n(A)$ is a Jacobson radical ring, one can easily check that also $M(A)$ is a quasi-regular ring. Hence $M(A) \in \mathcal{B}$. Let $a_1, \ldots, a_s \in M(A)$. Then there exists $n \in N$, such that $a_1, \ldots, a_s \in$ $\in M_n(A)$. Let V be as in the proof of Lemma 2.8, then $M_n(A) \cdot V = 0$ and $V \neq 0$. Hence $M(A)$ has no finite subset F, such that $xFy \neq$ $\neq 0$ $\forall x, y \neq 0$, $x, y \in M(A)$. By Prop. 1.4 $M(A)$ is a $*$-ring. Hence $M(A) \in \mathcal{U}$. Since $M(A)$ is not nil, $M(A)$ has no nonzero nil ideal.

Put $(x)_{ij} = (x_{k\ell}) = \begin{cases} x & \text{if } i = k \text{ and } j = \ell \\ 0 & \text{otherwise} \end{cases}$.

Clearly $(x)_{21}$ and $(y)_{12}$ are nilpotent for any $x, y \in A$. If $x \neq 0 \neq$ $\neq y$, then $(x)_{21}(y_{12})$ is not nilpotent. Therefore, since $M(A)$ is a $*$-ring, $M(A) \in \delta$. It follows $M(A) \in \mathcal{B} \cap \mathcal{U} \cap \delta$. By Lemma 2.8 any nil subring S of $M(A)$ is locally nilpotent and so $M(A) \notin \gamma$. $\Diamond$

Corollary 2.10. *The radical ℓ gives the best approximation of Köthe's Problem from above:*

$$A \in \mathcal{N} \Rightarrow A[x] \in \ell$$

and this improves the approximation

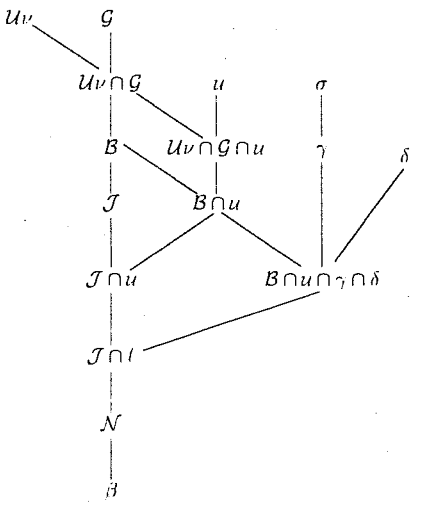
$$A \in \mathcal{N} \Rightarrow A[x] \in \mathcal{B} \cap \mathcal{U}.$$

Proof. The first statement follows from Th. 2.6, the second one follows from Th. 2.9. $\Diamond$

Remark. Obviously $\mathcal{N} \subseteq \mathcal{N}_s$ and $\mathcal{N} \subset \ell$. If Köthe's Problem has a positive solution, then $\mathcal{N} = \mathcal{N}_s$ and $\mathcal{N}_s \subset \ell$. However, $\mathcal{N}_s \not\subset \ell$ would mean that there exists a nil semisimple ring having a nonzero one-sided nil ideal, that is, Köthe's Problem has a negative solution.

We denote by σ, the upper radical generated by the class

$$\left\{ A \;\middle|\; \begin{array}{l} A \text{ has no nonzero locally nilpotent ideals and} \\ \text{all nilpotent elements have bounded nilpotency index.} \end{array} \right\}$$



Proposition 2.11. 1) $\mathcal{L} \subset \mathcal{N} \subset \mathcal{J} \cap \ell \subseteq \ell \subset \sigma$.

2) *If $R \in \sigma$ is a PI ring, then R is locally nilpotent.*

Proof. 1) Since $M(A)$ is not of bounded nilpotency index $M(A) \in \sigma$ and $M(A) \notin \ell$ by Th. 2.9, and $\mathcal{N} \subset J \cap \ell$ follows from [11, Th. 8].

2) If R is not locally nilpotent, then $R/\mathcal{L}(R) \neq 0$, where $\mathcal{L}(R)$ is locally nilpotent radical of R. Since R is a PI-ring, we get that $R/$ $/\mathcal{L}(R)$ is a PI-ring and semiprime. By Prop. 1.3 $R/\mathcal{L}(R)$ is of bounded nilpotency index. Hence $R/\mathcal{L}(R) \in \sigma \cap S\sigma = 0$, a contradiction. $\Diamond$

A *normal radical* r may be defined as left strong and principally left hereditary radical. In [13] it has been proved that here left strongness can be replaced by the weaker condition of *principally left strongness* (that is $r(L) = L \triangleleft_\ell A$ and for any $a \in L$, $La \in \gamma \Rightarrow L \subseteq r(A)$). An N-*radical* r may be defined as a normal radical containing the Baer radical β.

Set

$$\ell^\diamond = \{A \in \ell \mid Aa \in \ell, \text{ for any } a \in A\}.$$

Proposition 2.12. $\mathcal{N} \subseteq \ell^\diamond \subseteq \mathcal{B} \cap \mathcal{U} \cap \overline{\gamma \cap \delta}$, *where $\overline{\gamma \cap \delta}$ is largest N-radical in $\gamma \cap \delta$.*

Proof. Clearly $\mathcal{N} \subseteq \ell^\diamond$, since $\mathcal{N}$ is left hereditary. Let $A \in \ell^\diamond$, then $Aa \in \ell$, for any $a \in A$. By Prop. 1.6 $A \in \mathcal{B}$. Since $\gamma \cap \delta$ is left-strong, $L \triangleleft_\ell A$ implies $L \in \ell$ and so $L \in \gamma \cap \delta$. By [14, Th. 15], $A \in \overline{\gamma \cap \delta}$. $\Diamond$

Finally we give the position of the radicals considered in this note.

If Köthe's Problem has a positive solution, then

$$\mathcal{N} = \mathcal{N}_s \subset \ell \subset \mathcal{J}.$$

Moreover, $\mathcal{J} \not\subseteq \mathcal{B} \cap u \cap \gamma \cap \delta$, but if $\mathcal{B} \cap u \cap \gamma \cap \delta \subseteqq \mathcal{J}$ then $\mathcal{N} = \mathcal{N}_s$ and Köthe's Problem has a positive solution. Köthe's Problem has a positive solution if and only if $\ell(A[x]) = \mathcal{J}(A[x])$, for any ring A.

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References

[1] BEIDAR, K. I., FONG, Y. and PUCZYŁOWSKI, E. R.: Polynomial rings over nil rings cannot be homomorphically mapped onto rings with non-zero idempotents, *J. Algebra*, to appear.

[2] BEIDAR, K. I., PUCZYŁOWSKI, E. R. and WIEGANDT, R.: Radicals and Polynomial rings, *J. Austral. Math. Soc.*, to appear.

[3] DIVINSKY, N.: Rings and radicals, Allen & Unwin, London, 1965.

[4] HANNAH, J.: Quotient rings of semisimple rings with bounded index, *Glasgow Math. J.* **23** (1982), 53–64.

[5] JACOBSON, N.: Structure of rings, Amer. Math. Soc. Coll. Publ. 37 Providence, 1964.

[6] KÖTHE, G.: Die Struktur der Ringe, deren Restklassenring nach dem Radikal vollständig reduzibel ist, *Math. Zeitschr.* **32** (1930), 161–186.

[7] KREMPA, J.: Logical connections among some open problems in non-commutative rings, *Fund. Math.* **76** (1972), 121–130.

[8] OLSON, D. M.: A uniformly strongly prime radical, *J. Austral. Math. Soc. A* **43** (1987), 95–102.

[9] PUCZYŁOWSKI, E. R. and SMOKTUNOWICZ, A.: On maximal ideals and the Brown–McCoy radical of polynomial rings, *Comm. in Algebra* **26** (1998), 2473–2482.

[10] ROWEN, L. H.: Ring Theory II, Academic Press, 1988.

[11] SMOKTUNOWICZ, A.: Polynomial rings over nil rings need not be nil, *J. Algebra*, **233** (2000), 427-436.

[12] TUMURBAT, S. and WIEGANDT, R.: A note on special radicals and partitions of simple rings, *Comm. in Algebra*, to appear.

[13] TUMURBAT, S. and WIEGANDT, R.: Principally left hereditary and principally left strong radicals, *Algebra Colloquium* **8** (2001), 409–418.

[14] TUMURBAT, S. and ZAND, Z.: Hereditariness, strongness and a relationship between Brown–McCoy and Behrens radicals, *Beiträge Alg. und Geom.*, **42** (2001), 275–280.