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ON THE OSCILLATORY NATURE OF PERTURBED SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH DAMPING

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Abstract: In this paper, we present some criteria on the oscillation of solutions of the differential equation with damping $(p(x)y')' + q(x)f(x,y,y')y' + g(x,y) = R(x,x,x')$, $x \in [x_0, \infty)$, under suitable assumptions.

Consider the second order nonlinear differential equation with damping:

(1) $(p(x)y')' + q(x)f(x,y,y')y' + g(x,y) = R(x,y,y'), \quad x \in [x_0, \infty),$

where the functions involved are continuous and satisfying $p : [x_0, \infty) \to \to \mathbb{R}$, with $x_0 \geq 0$; $q : [x_0, \infty) \to [0, \infty)$; $f : [x_0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$; $g : : [x_0, \infty) \times \mathbb{R} \to \mathbb{R}$ and $R : [x_0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$. For all $y \neq 0$ and for $x \in [x_0, \infty)$ we assume that there exists continuous functions $k : \mathbb{R} \to \mathbb{R}$ and $r, s : [x_0, \infty) \to \mathbb{R}$ such that

(i) $yk(y) > 0$, $k'(y) \geq c > 0$; and

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$$\frac{g(x,y)}{k(y)} \geq r(x), \ \frac{R(x,y,y')}{k(y)} \leq s(x) \text{ for } y \neq 0.$$

A regular solution of (1) which is defined for all large x is called *oscillatory*, if it has no last zero, otherwise it is said to be *nonoscillatory.* Thus a nonoscillatory solution is eventually positive or negative. The equation (1) is *oscillatory* if all its solutions are oscillatory. We say that (1) is *sublinear* if $k(y)$ satisfies

$$\text{(ii)} \ \ 0 < \int_0^{\varepsilon} \frac{dz}{k(z)} < \infty, \quad 0 < \int_0^{-\varepsilon} \frac{dz}{k(z)} < \infty, \quad \varepsilon > 0.$$

(1) is *superlinear* if $k(y)$ satisfies

$$\text{(iii)} \ \ 0 < \int_{\varepsilon}^{\infty} \frac{dz}{k(z)} < \infty, \quad 0 < \int_{-\varepsilon}^{-\infty} \frac{dz}{k(z)} < \infty, \quad \varepsilon > 0.$$

(1) is a *mixed type* if $k(y)$ satisfies

$$\text{(iv)} \ \ 0 < \int_0^{\infty} \frac{dz}{k(z)} < \infty, \quad 0 < \int_0^{-\infty} \frac{dz}{k(z)} < \infty.$$

Yeh [24] considered equation (1) with $p \equiv 1$ and $f \equiv 1$, that is

$$y'' + q(x)y' + r(x)y = 0, \tag{2}$$

where $q, r \in C([x_0, \infty), (-\infty, \infty))$, $yg(y) > 0$ and $g'(y) \geq k > 0$ for $y \neq 0$ and proved that

$$\operatorname*{Lim\,sup}_{x\to\infty} \int_{x_0}^{x} (t-s)^{n-1} sq(s)ds = \infty,$$

$$\operatorname*{Lim}_{x\to\infty} \frac{1}{x^{n-1}} \int_{x_0}^{x} s \left[(t-s) \left(h(s) - \frac{1}{n} \right) + n - 1 \right]^2 (t-s)^{n-3} ds < \infty,$$

for some integer $n \geq 3$ are sufficient conditions in the oscillation of (2).

In [23] Yan has discussed the oscillatory behaviour of regular solutions of linear equation $(py')' + q(x)y' + r(x)y = 0$, under initial assumptions on p, q and r, and considering that

$$\int^{+\infty} \frac{dx}{p(x)} = +\infty.$$

Nagabuchi and Yamamoto (see [17]) extended and improves Yeh's result to the equation

(3) $$(p(x)y')' + q(x)y' + r(x)g(y) = 0.$$

Recently, by exploiting more fully a simple "completing square" and a differential inequality, Elabassy [5] has given sufficient conditions for the oscillation of a broad class of second order nonlinear equations of the type

$$(p(x)y')' + h(x)f(y)y' + \Psi(x,y) = H(x,y,y'),$$

under suitable conditions. It is clear that the above equation is a simple case of equation (1). For the other related results, we refer the reader to Butter [2], Elabassy [4], Elbert [6], Elbert and Kusano [7], Grace and Lalli [8], [9], Graef, Rankin and Spikes [10] and Nápoles [18]. In this paper, we present some criteria on the oscillation of solutions of the differential equation (1) under suitable assumptions. As a consequence we are able to extend and improve some well known oscillation results. The accuracy of these results have been illustrated by some examples.

Theorem 1. *Suppose that* (i)–(ii) *hold. In addition, assume that*

(v) *f is bounded from below, i.e. $f \geq -c$, $c > 0$ for $(x, y, y') \in \mathbb{R}^3$,*

(vi) *p is a bounded function for $x \geq x_0$, i.e., $0 < p(x) \leq a$, $a > 0$,*

(vii) *there exists a continuously differentiable function $u(x)$ on $[x_0, \infty)$ such that $u(x) > 0$, $u'(x) \geq 0$ and $u''(x) \leq 0$ on $[x_0, \infty)$, and $\gamma(x) = u'(x)p(x) + cu(x)q(x) \geq 0$, $\gamma'(x) \leq 0$ for $x \geq x_0$,*

(viii) $\lim\limits_{x\to\infty} \inf \int\limits_{x_0}^{x} u(z)(r(z) - s(z))dz > -\infty,$

(ix) $\lim\limits_{x\to\infty} \sup \left(\int\limits_{x_0}^{x} \dfrac{dz}{u(z)} \right)^{-1} \int\limits_{x_0}^{x} \frac{1}{u(z)} \int\limits_{x_0}^{z} u(v)(r(v) - s(v))dvdz = \infty.$

Then equation (1) *is oscillatory.*

Proof. Let y be a nonoscillatory solution of the differential equation (1). Without loss of generality, this solution can be supposed to be such that $y(x) \neq 0$ on $[X, \infty)$ for some $X \geq x_0$. Furthermore, we observe that the substitution $z = -y$ transforms (1) into the equation

$$(pz')' + qf^*z' + g^* = R^*,$$

where $f^* = f(x, -z, -z')$, $g^* = -g(x, -z)$, $R^* = R(x, -z, -z')$ are subject to same conditions as f, g and R respectively. That remark is valid for the function k. So, there is no loss of generality to assume that $y(x) > 0$ for all $x \geq X$.

Let $w(t)$ be defined by

$$w(x)=\frac{u(x)p(x)y'(x)}{k(y(x))} \text{ for all } x\geq X.$$

From (1) and the above expression we obtain that

$$w'(x)\leq\frac{\gamma(x)y'(x)}{k(y(x))}+u(x)(s(x)-r(x))-\frac{w^2(x)k'(y(x))}{u(x)p(x)}.$$

Consequently, integrating from X to x, we obtain

$$\int_X^x u(z)(r(z)-s(z))dz\leq$$

(4)

$$\leq -w(x)+w(X)+\int_X^x\frac{\gamma(z)y'(z)}{k(y(z))}dz-\int_X^x\frac{w^2(z)k'(y(z))}{u(z)p(z)}dz.$$

The integral $\int_X^x\frac{\gamma(z)y'(z)}{k(y(z))}dz$ is bounded above. This can be seen by applying the Mean Value Theorem, for each $x\geq X$ there exists $\zeta\in$ $\in[X,x]$ such that $\int_X^x\frac{\gamma(z)y'(z)}{k(y(z))}dz=\gamma(X)\int_{y(X)}^{y(\zeta)}\frac{dz}{k(z)}\leq k_1$, where $k_1=$ $=\gamma(X)\int_{y(X)}^{\infty}\frac{dz}{k(z)}$. Hence, we have from (4)

$$\text{(5)}\qquad \int_X^x u(z)(r(z)-s(z))dz\leq -w(x)+k_2-\int_X^x\frac{w^2(z)k'(y(z))}{u(z)p(z)}dz.$$

where $k_2=k_1+w(X)$. Or, by virtue of condition (i)

$$\text{(6)}\qquad \int_X^x u(z)(r(z)-s(z))dz\leq -w(x)+k_2-c\int_X^x\frac{w^2(z)}{u(z)p(z)}dz.$$

Now, we consider the behaviour of x'.

CASE 1. y' is oscillatory. Then, there exists an infinite sequence $\{x_n\}$ such that $x_n\to\infty$ as $n\to\infty$ and $y'(x_n)=0$. Thus, (6) gives

$$\int_X^{x_n} u(z)(r(z)-s(z))dz\leq k_2-c\int_X^{x_n}\frac{w^2(z)}{u(z)p(z)}dz.$$

From (viii), we get $\frac{w^2(z)}{u(z)p(z)}\in L^1[X,\infty)$. Thus, there exists a positive constant N such that $\int\limits_X^{x_n} \frac{w^2(z)}{u(z)p(z)}dz \leq N$ for every $x \geq X$. From this and using the Schwartz inequality we obtain that $-\int\limits_X^{x_n} \frac{w(z)}{u(z)}dz \leq$ $\leq \sqrt{Na}\left(\int\limits_X^x \frac{dz}{u(z)}\right)^{\frac{1}{2}}$. Furthermore, (6) gives $\int\limits_X^{x_n} u(z)(s(z)-r(z))dz \leq$ $\leq -w(x)+k_2$, hence

$$\text{(7)}\quad \int\limits_X^x \frac{1}{u(z)} \int\limits_X^z u(v)(r(v)-s(v))dvdz \leq -\int\limits_X^x \frac{w(z)}{u(z)}dz + k_2 \int\limits_X^x \frac{dz}{u(z)}.$$

Assumptions (vii) implies that $u(x) \leq \gamma + \mu x$ for all large x, where λ and μ are positive constants. This ensures that $\int\limits_{x_0}^{\infty} \frac{dz}{u(z)} = \infty$. This fact and (7) implies that

$$\int\limits_X^x \frac{1}{u(z)} \int\limits_X^z u(v)(r(v)-s(v))dvdz \leq$$

$$\leq \sqrt{Na}\left(\int\limits_X^x \frac{dz}{u(z)}\right)^{\frac{1}{2}} + k_2 \int\limits_X^x \frac{dz}{u(z)} \leq \left(\sqrt{Na}+k_2\right)\int\limits_X^x \frac{dz}{u(z)}.$$

Dividing by $\int\limits_X^x \frac{dz}{u(z)}$ and take the upper limit as $x \to \infty$, we obtain a contradiction to (ix).

CASE 2. $y'(x) > 0$ for $x \geq X_1 > x_0$. Then, it follows from (6) that $\int\limits_X^{x_n} u(z)(r(z)-s(z))dz \leq k_2$ and consequently the desired contradiction to (ix).

CASE 3. $y'(x) < 0$ for $x \geq X_2 > x_0$. If $\frac{w^2(z)}{u(z)p(z)} \in L^1[X,\infty)$, then we can follow the procedure of Case 1 to arrive at a contradiction to (ix). Suppose now that $\frac{w^2(z)}{u(z)p(z)} \notin L^1[X,\infty)$. From (viii) and (5) we get for some constant $\beta > 0$ that

$$-w(x) \geq \beta + \int_{X_2}^{x} \frac{w^2 k'(y(z))}{u(z)p(z)} dz \text{ for every } x \geq X_2. \tag{8}$$

Since $\frac{w^2(z)}{u(z)p(z)} \notin L^1[X, \infty)$, there exists $X_3 \geq X_2$ such that $M = \beta + + \int_{X_2}^{X_3} \frac{w^2(z)k'(y(z))}{u(z)p(z)} dz > 0$. Thus (8) ensures $w(x)$ is negative on $[X_2, \infty)$. By using (8), we have from (5) that

$$-w(x) \left[\beta + \int_{X_2}^{x} \frac{w^2(z)k'(y(z))}{u(z)p(z)} dz \right]^{-1} \geq 1.$$

This inequality yields

$$\log \left[\frac{\left\{ \beta + \int_{X_2}^{x} \frac{w^2(z)k'(y(z))}{u(z)p(z)} \right\}}{M} \right] \geq \log \left[\frac{k(y(X_3))}{k(y)} \right].$$

From here we obtain

$$\beta + \int_{X_2}^{x} \frac{w^2(z)k'(y(z))}{u(z)p(z)} dz \geq \frac{M^*}{k(y)} \text{ for every } x \geq X_3, \text{ with}$$

$$M^* = Mk(y(X_3)) > 0.$$

Hence, from (8) we derive that $-w(x)k(y) \geq M^*$, and therefore

$$y(x) \leq y(X_3) - M^* \int_{X_3}^{x} \frac{dz}{u(z)p(z)} \leq y(M_3) - \frac{M^*}{a} \int_{X_3}^{x} \frac{dz}{u(z)},$$

follows, that $y(x) \to -\infty$ as $x \to \infty$, a contradiction. This completes the proof. $\Diamond$

Corollary 1. *Equation* (1) *is oscillatory if* (ii), (vi) *hold,* $\gamma(x) = = \alpha x^{\alpha-1} p(x) + c x^{\alpha} q(x) \geq 0$ *and* $\gamma'(x) \leq 0$ *for some* $\alpha \in [0, 1]$,

$$\operatorname*{Lim\,inf}_{x\to\infty}\int_{x_0}^{x} z^\alpha(r(z)-s(z))dz > -\infty,$$

$$\operatorname*{Lim}_{x\to\infty}\frac{1}{\log x}\int_{x_0}^{x}\frac{1}{z}\int_{x_0}^{z} v(r(v)-s(v))dvdz > \infty, \quad \textit{if } \alpha = 1,$$

$$\operatorname*{Lim}_{x\to\infty}\frac{1}{x^{1-\alpha}}\int_{x_0}^{x}\frac{1}{z^\alpha}\int_{x_0}^{z} v^\alpha(r(v)-s(v))dvdz > \infty, \quad \textit{if } 0 \le \alpha < 1.$$

Remark 1. The existence of function $k(y)$ is very closet to the oscillatory nature of equation (1). So, if assumptions (i), (ii) and (vi) are not fulfilled, we can exhibit equations that have nonoscillatory solutions. For example, the equation

$$\left(e^{\left(\frac{x^3}{3}+3x\right)}y'\right)' + 2(x^2+1)e^{\left(\frac{x^3}{3}+3x\right)}y = 0,$$

has the nonoscillatory solution $y(x) = e^{-2x}$ (see [19], [20], [21]).

Remark 2. The Th. 1 is consistent with Th. 1 of [19], referent to the oscillatory behaviour of equation $y'' + a(x)f(y)h(y') = 0$, with results of [22] on qualitative nature of equation (3) with $p \equiv 1$, with Atkinson's result (see [1]) on equation

$$y'' + r(x)|y|^\gamma \operatorname{sgn} y = 0 \tag{9}$$

and cover the Th. of Chen [3], refer to the equation (1) with $p \equiv 1$, $q \equiv 0$ and $g(x,y) - R(x,y,y') = yF(x,y^2,y'^2)$.

By the other hand, if we consider the equation with $q(x)$ nonnegative and continuous on $[0,\infty)$ and γ is any real number satisfying $0 < \gamma < 1$, the Th. 1 completes the results of [11], [12] concerning with nonoscillatory solutions of equation (9) and [14], [15] and [16], refer to the oscillatory behaviour of equation (9).

In the following result we do not assume that $k(y)$ satisfies condition (ii). So, it may be applicable for linear, sublinear or superlinear differential equations.

Theorem 2. *Suppose that* (i) *and* (v) *hold. Moreover, assume that*

(x) *there exists a differentiable function* $\phi : [x_0,\infty) \to (0,\infty)$ *and continuous real functions* h, H *on* $D := \{(x,z) : x \ge z \ge x_0\}$ *and* H *has a continuous and nonpositive partial derivative on*

D with respect to the second variable such that $H(x,x)=0$ for $x\ge x_0$, $H(x,z)>0$ for $x>z\ge x_0$, and

(xi) $$-\frac{\partial H(x,z)}{\partial z}=h(x,z)\sqrt{H(x,z)} \text{ for all } (x,z)\in D.$$

Then equation (1) is oscillatory if

(xii) $$\operatorname*{Lim\,sup}_{x\to\infty}\frac{1}{H(x,x_0)}\int_{x_0}^{x}\Bigg\{\phi(z)H(x,z)(r(z)-s(z))-$$
$$-\frac{1}{4k}\left[p(z)\phi(z)\left[h(x,z)-\left(\frac{cq(z)}{p(z)}+\frac{\phi'(z)}{\phi(z)}\right)\sqrt{H(x,z)}\right]^2\right]\Bigg\}dz=\infty.$$

Proof. Let $y(x)$ be a nonoscillatory solution of (1), say $y(x)>0$ for $x\ge x_0$. Taking $w=\frac{\phi(x)p(x)y'(x)}{k(y(x))}$ we obtain from equation (1) that

$$w'(x)\le\phi(x)(r(x)-s(x))+\left(\frac{cq(z)}{p(z)}+\frac{\phi'(x)}{\phi(x)}\right)w(x)-\frac{kw^2(x)}{p(x)\phi(x)}.$$

Hence, for all $x\ge x_0$, we have

$$-\int_{x_0}^{x}\phi(z)H(x,z)(r(z)-s(z))dz\le-\int_{x_0}^{x}H(x,z)w'(z)dz+$$

$$+\int_{x_0}^{x}H(x,z)\left(\frac{cq(z)}{p(z)}+\frac{\phi'(z)}{\phi(z)}\right)w(z)dz-k\int_{x_0}^{x}\frac{H(x,z)w^2(z)}{p(z)\phi(z)}dz=$$

$$=H(x,x_0)w(x_0)-\int_{x_0}^{x}\left(-\frac{\partial H}{\partial z}\right)w(z)dz+$$

$$+\int_{x_0}^{x}H(x,z)\left(\frac{cq(z)}{p(z)}+\frac{\phi'(z)}{\phi(z)}\right)w(z)dz-k\int_{x_0}^{x}\frac{H(x,z)w^2(z)}{p(z)\phi(z)}dz=$$

$$=H(x,x_0)w(x_0)-\int_{x_0}^{x}\Bigg[\Bigg(h(x,z)\sqrt{H(x,z)}-$$

$$-H(x,z)\left(\frac{cq(z)}{p(z)}+\frac{\phi'(z)}{\phi(z)}\right)\Bigg)w(z)+\frac{kH(x,z)w^2(z)}{p(z)\phi(z)}\Bigg]dz=$$

$$= H(x,x_0)w(x_0) - \int_{x_0}^{x} \left[\sqrt{\frac{kH(x,z)}{p(z)\phi(z)}} w(z) - \right.$$

$$\left. - \frac{\sqrt{p(z)\phi(z)}}{2\sqrt{k}} \left(h(x,z) - \sqrt{H(x,z)} \left(\frac{cq(z)}{p(z)} + \frac{\phi'(z)}{\phi(z)} \right) \right) \right]^2 dz +$$

$$+ \frac{1}{4k} \int_{x_0}^{x} p(z)\phi(z) \left(h(x,z) - \sqrt{H(x,z)} \left(\frac{cq(z)}{p(z)} + \frac{\phi'(z)}{\phi(z)} \right) \right)^2 dz \leq$$

$$\leq H(x,x_0)w(x_0) +$$

$$+ \frac{1}{4k} \int_{x_0}^{x} p(z)\phi(z) \left(h(x,z) - \sqrt{H(x,z)} \left(\frac{cq(z)}{p(z)} + \frac{\phi'(z)}{\phi(z)} \right) \right)^2 dz.$$

Hence,

$$\frac{1}{H(x,x_0)} \int_{x_0}^{x} \left\{ \phi(z)H(x,z)(s(z) - r(z)) - \right.$$

$$\left. - \frac{1}{4k} \left[p(z)\phi(z) \left[h(x,z) - \left(\frac{cq(z)}{p(z)} + \frac{\phi'(z)}{\phi(z)} \right) \sqrt{H(x,z)} \right]^2 \right] \right\} dz \leq w(x_0),$$

a contradiction with (xii). This completes the proof. ◊

Corollary 2. *Suppose that condition* (xii) *in* Th. 2 *can be replaced by*

$$\operatorname*{Lim\,sup}_{x \to \infty} \frac{1}{H(x,x_0)} \int_{x_0}^{x} \phi(z)H(x,z)(r(z) - s(z))dz = \infty,$$

and

$$\operatorname*{Lim\,sup}_{x \to \infty} \frac{1}{H(x,x_0)} \int_{x_0}^{x} s(z)\phi(z) \cdot$$

$$\cdot \left(h(x,z) - \sqrt{H(x,z)} \left(\frac{cq(z)}{s(z)} + \frac{\phi'(z)}{\phi(z)} \right) \right)^2 dz \leq \infty.$$

Then the conclusion of Th. 2 *holds.*

Remark 3. Using the continuous functions $H(x,z) = (x-z)^{n-1}$, $x > > z \geq x_0$, where n is an integer with $n > 2$, and $h(x,z) = (n-1)$

$(x-z)^{(n-3)/2}$, $x > z \geq x_0$, the results in [13], [17], [22], [23] (the Th. 1 on equation (3) with $g(u) = u$ and Cor. 1 on equation $(p(x)y')' + r(x)y = 0$, $x \in [0, \infty)$), and [24] can be obtained from Th. 2 as special cases.

Remark 4. Th. 2 can be used to some cases for which some other oscillation criteria can not be used. For, example, consider the Liénard type equation

$$y'' + (1 + y'^2)y' + y^3 + y = 0.$$

If we take $H(x,z)$, $h(x,z)$ as above and $\phi \equiv 1$, then all the hypotheses of Th. 2 are satisfied, whereas to the best of our knowledge no criteria can cover this result, taking into account the unboundedness of f (see, [19] and [20] for further details).

Remark 5. The Ths. 1 and 2 contains, in the case $f(x,y,y') = f(y)$, the results obtained in [5] for a simple case of equation (1).

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