Mathematica Pannonica
13/1 (2002), 103–116

VARIOUS KINDS OF LOCAL CONNECTEDNESS

Janusz J. **Charatonik**

Mathematical Institute, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland and Instituto de Matemáticas, UNAM, Circuito Exterior, Ciudad Universitaria, 04510 México, D. F., México

Alejandro **Illanes**

Instituto de Matemáticas, UNAM, Circuito Exterior, Ciudad Universitaria, 04510 México, D. F., México

Dedicated to Professor Hans Sachs on his 60th birthday

Received: May 2001

MSC 2000: 54 D 05, 54 F 15, 54 F 50

Keywords: Arcwise connected, connected im kleinen, continuum, locally arcwise connected, mapping, open, quasi-locally connected, weakly locally connected.

Abstract: We compare several notions concerning local connectednes (as local arcwise connectedness, strong local arcwise connectedness, weak local connectedness, quasi-local connectedness and others), all considered at a given point of a Hausdorff space. Various relations between them are investigated by constructing examples and establishing implications. We also study conditions under which these properties are preserved either under all (continuous) mappings or under mappings satisfying some openness conditions. A special attention is paid to mappings of continua.

E-mail addresses: jjc@hera.math.uni.wroc.pl and jjc@gauss.matem.unam.mx, illanes@gauss.matem.unam.mx

1. Introduction and preliminaries

A *space* means a topological Hausdorff space, and a *continuum* means a compact, connected space. The symbol $\mathbb{N}$ stands for the set of all positive integers.

Let a point p of a space S be given. The *component* of p in S means the maximal connected subset of S containing p. The *quasicomponent* of p in S means the intersection of all simultaneously closed and open subsets of S containing p. The *arc component* of p in S means the maximal arcwise connected subset of S containing p. Thus, if $K(p, S)$, $Q(p, S)$ and $A(p, S)$ denote the component, quasicomponent and arc component of a point p in a space S, then

$$A(p, S) \subset K(p, S) \subset Q(p, S).$$

A space S is said to be:

— *locally connected at a point* $p \in S$, written LC *at* p, provided that for each open subset A of S such that $p \in A$ there is a connected open subset B of S such that $p \in B \subset A$, [14, p. 89]; equivalently, provided that S has a local base at p composed of connected open sets, [9, p. 105]; in other words, provided that each neighborhood of p contains a connected neighborhood of p which is open in S, [15, 5.22, p. 83];

— *weakly locally connected* (often called *connected im kleinen*) *at a point* $p \in S$, written WLC *at* p, provided that for each open set A of S such that $p \in A$ there exists an open subset B of S such that $p \in B$ and B is contained in a component of A, [14, p. 89] (i.e., if for each open neighborhood A of p the point p is an interior point of the component $K(p, A)$); in other words, provided that for each open set A of S containing p there exists an open subset B of S containing p and lying in A such that for each point $q \in B$ there is a connected set of A containing both p and q, [9, p. 113]; equivalently, provided that each neighborhood of p contains a connected neighborhood of p, [15, 5.10, p. 75];

— *locally arcwise connected at a point* $p \in S$, written LAC *at* p, provided that each neighborhood of p contains an arcwise connected neighborhood of p, [15, 8.24, p. 131] (observe that the arcwise connected neighborhood of p need not be open); equivalently, provided that for each neighborhood U of p the point p is an interior point of the arc component $A(p, U)$;

— *strongly locally arcwise connected at a point* $p \in S$, written SLAC *at* p, provided that each neighborhood of p contains an arcwise connected neighborhood of p which is open in S, [15, 8.43, p. 136];

— *quasilocally connected at a point* $p \in S$, written QLC *at* p, provided that for each neighborhood U of p the quasicomponent $Q(p, U)$ is a neighborhood of p, [17, p. 40]; i.e., provided that the point p is an interior point of the quasicomponent $Q(p, U)$;

— *semi-locally connected* (also called *padded*) *at a point* $p \in$ $\in S$, written SLC *at* p, provided that for each neighborhood U of p there exist open sets W_1 and W_2 such that $p \in W_1 \subset \text{cl}\, W_1 \subset$ $\subset W_2 \subset U$ and $W_2 \setminus \text{cl}\, W_1$ has only finitely many components, [16, p. 19], where connectedness of S is additionally assumed; see also [6, p. 355].

A space is said to have any of the properties defined above provided that it has that property at each of its points.

1.1. Remark. Note that some authors (e.g. K. Kuratowski [12, p. 227] and G. T. Whyburn [16, p. 18]) use the name "locally connected at a point" in the sense of "connected im kleinen at a point". Our term "weakly locally connected" is copied from [18].

The relations between the above recalled concepts are widely known. They can be summarized in the next two theorems. The former is a consequence of the definitions.

1.2. Theorem. *The following implications between the above defined concepts are known for any space* S *and a point* $p \in S$.

$$\begin{array}{ccccccc} & & & & S \text{ is LC at } p & & \\ & & & & \Downarrow & & \\ S \text{ is SLAC at } p & \Rightarrow & S \text{ is LAC at } p & \Rightarrow & S \text{ is WLC at } p & \Rightarrow & S \text{ is QLC at } p \end{array}$$

1.3. Theorem. *For any connected space* S *and a point* $p \in S$ *the following implication holds, and it cannot be reversed.*

$$S \textit{ is } \text{SLC } \textit{at } p \Rightarrow S \textit{ is } \text{LC } \textit{at } p \tag{1.3.1}$$

Proof. See [6, Prop. 2.3, p. 355, and Ex. 5.1, p. 361]. ◊

It will be shown in the next section (Cor. 2.5) that none of the implications of Th. 1.2 can be reversed.

2. Structural results

First we present some examples showing that none of the implications of Th. 1.2 can be reversed in general. Next we will discuss some conditions which are sufficient to prove the inverse implications.

A continuum X is said to be *hereditarily arcwise connected* provided that each subcontinuum of X is arcwise connected. A *dendroid* means an arcwise connected and hereditarily unicoherent (metric) continuum, see e.g. [15, 10.58, p. 192]. Since each subcontinuum of a dendroid is a dendroid, [15, 10.58 (a), p. 192], each dendroid is hereditarily arcwise connected. By the *harmonic fan* we mean the cone over the closure of the harmonic sequence $\{1/n : n \in \mathbb{N}\}$.

The following example is well known. We recall its description for further purposes.

2.1. Example. *There exists a dendroid X containing a point p such that it is* LAC *at p (thus* WLC *at p) while not* LC *at p.*

Proof. In the Euclidean plane put $p = (0,0)$ and, for each $n \in \mathbb{N}$, let H_n be the cone with the vertex $v_n = (\frac{1}{n}, 0)$ over the set $E_n = \{v_{n+1}\} \cup \cup \{(\frac{1}{n+1}, \frac{1}{i}) : i \in \{n+1, n+2, \dots\}\}$. Then each H_n is homeomorphic to the harmonic fan. The union

$$X = \{p\} \cup \bigcup \{H_n : n \in \mathbb{N}\} \tag{2.1.1}$$

is the needed dendroid. It is pictured in [9, Fig. 3-9, p. 113] or [15, Fig. 5.22, p. 84]. ◊

Other continuum which is WLC at a point p but not LC at p is shown in [10, Ex. 1, p. 137].

2.2. Example. *There is an arcwise connected not hereditarily arcwise connected plane continuum X containing a point $p \in X$ such that X is* LC *at p (thus* WLC *at p), while not* LAC *at p.*

Proof. The continuum is a slight modification of the one in [5, Ex. 3.19, p. 215]. We recall its construction here for the reader's convenience. In the plane put $A^0 = \{(0, y) : y \in [-1, 1]\}$, and for each positive integer n let $A^n = \{(\frac{1}{n}, y) : y \in [-1, 1]\}$, and

$$B^n = \begin{cases} \{(x, -1) : x \in [\frac{1}{n+1}, \frac{1}{n}]\} & \text{for } n \text{ odd}, \\ \{(x, 1) : x \in [\frac{1}{n+1}, \frac{1}{n}]\} & \text{for } n \text{ even}. \end{cases}$$

Thus the union $X_0 = A^0 \cup \bigcup \{A^n \cup B^n : n \in \mathbb{N}\}$ is a continuum homeomorphic to the $\sin(1/x)$-curve.

Between every two consecutive vertical segments A^n and A^{n+1} of X_0 insert n disjoint copies $X_{n1}, X_{n2}, \dots, X_{nn}$ properly diminished in such a way that for each $i \in \{1, 2, \dots, n\}$ the limit segment A^0_{ni} of X_{ni} is contained in A^{n+1}, while the first vertical segment A^1_{ni} of the other arc-component of X_{ni} is contained in A^n. We assume also that for each fixed n all segments A^0_{ni} and A^1_{ni} as well as $n+1$ components of $A^{n+1} \setminus \bigcup\{A^0_{ni} : i \in \{1, 2, \dots, n\}\}$ and of $A^n \setminus \bigcup\{A^1_{ni} : i \in \{1, 2, \dots, n\}\}$ have equal lengths. This assumption implies that the inserted copies X_{ni} of X_0 are disjoint with B^n for each n and that the limit segment A^0 of X_0 is contained in the closure of the union of all limit segments A^0_{ni} for all $i \in \{1, 2, \dots, n\}$ and all $n \in \mathbb{N}$. Put

$$X' = X_0 \cup \bigcup \left\{ \bigcup \{X_{ni} : i \in \{1, 2, \dots, n\}\} : n \in \mathbb{N} \right\}.$$

The continuum X' is pictured in [5, p. 216]. Let L be an arc in the plane such that $L \cap X' = \{(0,1), (1,1)\}$. Then $X = X' \cup L$ is an arcwise connected (but not hereditarily arcwise connected) continuum which is LC (thus WLC) at each point of $A^0 \setminus \{(0,1)\}$, while it is not LAC at any of these points. ◊

2.3. Example. *There is a dendroid X containing a point p such that X is* LC *at p (thus* WLC *at p and* LAC *at p), while not* SLAC *at p.*

Proof. In the 3-space put $p = (0,0,0)$ and, for each positive integer n, let H^0_n be the cone with the vertex $v_n = (\frac{1}{n}, 0, 0)$ over the set $E_n = \{v_{n+1}\} \cup \{(\frac{1}{n+1}, \frac{1}{i}, 0) : i \in \{n+1, n+2, \dots\}\}$. Then each H^0_n is homeomorphic to the harmonic fan. Thus the union

$$X^0 = \{p\} \cup \bigcup \{H^0_n : n \in \mathbb{N}\}$$

(lying in the plane $z = 0$) is just the dendroid of Ex. 2.1 (defined by (2.1.1)).

For each $n \in \mathbb{N}$ let m_n be the midpoint of the limit segment $v_n v_{n+1}$ of H^0_n, and let L_n be the straight line segment in the half-space $z \geq 0$ erected at m_n perpendicularly to the plane $z = 0$ and of length $\frac{1}{n}$. Further, for each $n \in \mathbb{N}$, choose a decreasing and tending to 0 sequence of numbers $\{z^k_n : k \in \mathbb{N}\} \subset (0, \frac{1}{n}]$ in such a way that

$$\{z^k_n : k \in \mathbb{N}\} \cap \{z^k_{n+1} : k \in \mathbb{N}\} = \emptyset. \tag{2.3.1}$$

For $k, n \in \mathbb{N}$ define $H^k_n = \{(x, y, z^k_n) : (x, y, 0) \in H^0_n\}$. Thus H^k_n is an isometric copy of H^0_n located above it on the level z^k_n, and we have $\mathrm{Lim}_{\,k\to\infty} H^k_n = H^0_n$ for each $n \in \mathbb{N}$. Condition (2.3.1) guarantees that the copies H^k_n are mutually disjoint for $k, n \in \mathbb{N}$. The union

$$X = X^0 \cup \left(\bigcup\{L_n : n \in \mathbb{N}\}\right) \cup \left(\bigcup\{H_n^k : k, n \in \mathbb{N}\}\right)$$

is a dendroid. Observe that, for each $n \in \mathbb{N}$, the component of $X \setminus \{v_n\}$ containing the point p is an open connected neighborhood of p. Since these neighborhoods form an open local base at p, the continuum X is LC at p. Thus X is LAC at p (compare Cor. 2.14 below). But neither the elements of the open local base nor other (small) open neighborhoods of p are arcwise connected just by the construction. Thus X is not SLAC at p. ◇

2.4. Example. *There is a connected space X containing a point p such that X is* QLC *at p while not* WLC *at p.*

Proof. For each $n \in \mathbb{N}$ let $C_n = \{1+\frac{1}{n+1}\} \times [0,2]$ and $D_n = [0,1] \times \{1+ + \frac{1}{n+1}\}$. Let $A_1 = \bigcup\{C_n : n \in \mathbb{N}\}$ and $B_1 = \bigcup\{D_n : n \in \mathbb{N}\}$. Given a subset Z of the plane and a real number α, put $\alpha Z = \{\alpha z : z \in Z\}$. For each $n \in \mathbb{N}$ let $A_n = \frac{1}{2^{n-1}} A_1$ and $B_n = \frac{1}{2^{n-1}} B_1$. Finally, define $p = (0,0)$ and

$$X = \{p\} \cup \left(\bigcup\{A_n : n \in \mathbb{N}\}\right) \cup \left(\bigcup\{B_n : n \in \mathbb{N}\}\right) \cup ([1, \tfrac{3}{2}] \times \{0\}).$$

Since the set $A_1 \cup ([1, \frac{3}{2}] \times \{0\})$ is connected and its closure intersects each component of B_1, the union $A_1 \cup ([1, \frac{3}{2}] \times \{0\}) \cup B_1$ is a connected set whose closure intersects each component of A_2. Thus $A_1 \cup ([1, \frac{3}{2}] \times \times \{0\}) \cup B_1 \cup A_2$ is connected. Procceding in the way we conclude that X is connected.

For each $n \in \mathbb{N}$ let $X_n = \{p\} \cup (\bigcup\{A_m : m \geq n\}) \cup (\bigcup\{B_m : m \geq \geq n\})$. We claim that

$$Q(p, X_n) = X_{n+1} \cup B_n. \tag{2.4.1}$$

Clearly, $Q(p, X_n) \subset X_{n+1} \cup B_n$. Let G and H be two disjoint open (and closed) subsets of X_n such that $X_n = G \cup H$. Fix a point $q \in B_n$ such that $q \in \operatorname{cl} A_n$. Assume that $q \in G$. Since G is open, almost all components of A_n intersect G. This implies that each component of B_n intersects G. Thus $B_n \subset G$. Consequently, each component of A_{n+1} intersects G, whence it follows that $A_{n+1} \subset G$. Proceeding in this way we conclude that $X_{n+1} \cup B_n \subset G \subset Q(p, X_n)$. So (2.4.1) is proved.

We are ready to prove that X is QLC at p. Let U be an open subset of X such that $p \in U$. Then there exists $n \in \mathbb{N}$ with $X_n \subset U$. Therefore $p \in \operatorname{int}(X_{n+1} \cup B_n) \subset X_{n+1} \cup B_n = Q(p, X_n) \subset Q(p, X)$ according to (2.4.1). This shows that X is QLC at p.

On the other hand, the set $V = X \setminus A_1$ is an open neighborhood of p such that $K(p, V) = \{p\}$. Thus X is not WLC at p. $\Diamond$

Exs. 2.1-2.4 lead to the following corollary.

2.5. Corollary. *None of the four implications of* Th. 1.2 *can be reversed, even for metric connected spaces.*

2.6. Proposition. *Let S be a compact Hausdorff space and $p \in S$. Then S is* QLC *at p if and only if S is* WLC *at p.*

Proof. We only need to show that if S is QLC at p then S is WLC at p. Let U and V be open subsets of S such that $p \in V \subset \text{cl}\, V \subset U$. Then $p \in \text{int}\, Q(p, V) \subset \text{int}\, Q(p, \text{cl}\, V)$. Since $\text{cl}\, V$ is a compact Hausdorff space, $Q(p, \text{cl}\, V) = K(p, \text{cl}\, V)$, see [7, Th. 6.1.23, p. 357]. Thus $p \in \text{int}\, K(p, \text{cl}\, V) \subset \text{int}\, K(p, U)$. Therefore S is WLC at p. $\Diamond$

2.7. Remark. Ex. 2.4 shows that compactness of the space is essential in Prop. 2.6. On the other hand, according to Prop. 2.6, Ex. 2.4 cannot be strengthened to get a continuum with the considered property.

2.8. Remark. In [13, Fig. 1] an example is presented of a plane arcwise connected continuum which is LC at a point p while is not LAC at p. However, this example is not a dendroid.

2.9. Remark. Connectedness of S is an essential assumption in Th. 1.3. In fact, if $S = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ with the topology inherited from the real line, then S is SLC at 0 while is not LC at this point.

The implications in Ths. 1.2 and 1.3 can be reversed under some additional assumptions. For sake of completeness and to begin with, recall a known result (see [14, Th. 10, p. 90]).

2.10. Theorem. *If a space is* WLC *at each point of some open set that contains a point p, then it is* LC *at p.*

As a consequence of Th. 2.10 one obtains the following (see [9, Th. 3-11, p. 114] or [15, 5.22 (b), p. 84]).

2.11. Corollary. *If a space is* WLC *(at each of its points), then it is* LC.

We will show the following result.

2.12. Theorem. *If a hereditarily arcwise connected continuum is* WLC *at a point, then it is* LAC *at this point.*

Proof. Let a hereditarily arcwise connected continuum X be WLC at a point $p \in X$, and let U be an arbitrary neighborhood of p. Since X is a regular space, there is a neighborhood V of p such that $\text{cl}\, V \subset U$. By connectedness im kleinen of X at p there is a connected neighborhood W of p with $W \subset V$. The closure $C = \text{cl}\, W$ is closed and connected subset of X, so it is a subcontinuum of X, thus it is arcwise connected. Since $p \in W \subset V$, the set C is a neighborhood of p, and since $C \subset \text{cl}\, V \subset U$, we see that W contains an arcwise connected neighborhood of p, as needed. ◊

2.13. Corollary. *For hereditarily arcwise connected continua weak local connectedness (i.e., connectedness im kleinen) at a point is equivalent to local arcwise connectedness at this point.*

2.14. Corollary. *If a hereditarily arcwise connected continuum is* LC *at a point, then it is* LAC *at this point.*

2.15. Remarks (a) The converse implication to that of Cor. 2.14 does not hold by Ex. 2.1.

(b) Ex. 2.2 shows that the assumption of hereditary arcwise connectedness of X is essential in Th. 2.12 and it cannot be weakened to arcwise connectedness of X.

(c) Ex. 2.3 shows that Th. 2.12 cannot be strengthened so that strong local arcwise connectedness is obtained in the conclusion under the same assumptions.

3. Mapping results

Let LC(X), WLC(X), LAC(X), SLAC(X), QLC(X), SLC(X) denote the sets of points of a space X at which X is locally connected, weakly locally connected, locally arcwise connected, strongly locally arcwise connected, quasilocally connected and semi-locally connected, respectively.

In [8, (2) and (3), p. 28] the following is shown.

3.1. Proposition. *Let a space X be compact. If $f : X \to Y = f(X)$ is a surjective mapping, then*

(3.1.1) $f^{-1}(y) \subset \text{WLC}(X) \Longrightarrow y \in \text{WLC}(f(X))$ *for each* $y \in f(X)$,

(3.1.2) $$f(X) \setminus \text{WLC}(f(X)) \subset f(X \setminus \text{WLC}(X)).$$

Implication (3.1.1) and inclusion (3.1.2) have many applications, in particular to study invariant mapping properties of compact spaces. Some of them were shown already in the same paper [8, Th., p. 28, and Cor., p. 29]. Thus it is worth to consider similar results for other concepts of local connectedness.

It is shown in [8, p. 28] that (3.1.1) implies (3.1.2). In fact, they are equivalent, and the equivalence holds in a much more general setting.

Namely we have the following easy assertion.

3.2. Assertion. *For every sets X and Y, their subsets* $\mathrm{P}_1(X) \subset X$ *and* $\mathrm{P}_2(Y) \subset Y$ *and a function* $f : X \to Y = f(X)$ *the following two conditions are equivalent.*

(3.2.1) $f^{-1}(y) \subset \mathrm{P}_1(X) \Longrightarrow y \in \mathrm{P}_2(f(X))$ *for each* $y \in f(X)$,

(3.2.2) $$f(X) \setminus \mathrm{P}_2(f(X)) \subset f(X \setminus \mathrm{P}_1(X)).$$

Proof. Assume (3.2.1). Note that inclusion (3.2.2) is equivalent to the inclusion $f(X) \setminus f(X \setminus \mathrm{P}_1(X)) \subset \mathrm{P}_2(f(X))$. So, let $y \in f(X) \setminus f(X \setminus \mathrm{P}_1(X))$. Then

$$f^{-1}(y) \subset f^{-1}(f(X) \setminus f(X \setminus \mathrm{P}_1(X))) = f^{-1}(f(X)) \setminus f^{-1}(f(X \setminus \mathrm{P}_1(X))) \subset \subset X \setminus (X \setminus \mathrm{P}_1(X)) = \mathrm{P}_1(X),$$

and applying (3.2.1) we get $y \in \mathrm{P}_2(f(X))$, as needed.

Assume (3.2.2), and suppose that $y \notin \mathrm{P}_2(f(X))$. Then $y \in f(X) \setminus \mathrm{P}_2(f(X))$, whence $y \in f(X \setminus \mathrm{P}_1(X))$ by (3.2.2). Therefore there is a point $x \in X \setminus \mathrm{P}_1(X)$ such that $f(x) = y$. Thus $x \in f^{-1}(y) \setminus \mathrm{P}_1(X)$, so the proof is complete. ◊

In the light of the above mentioned results of [8] and of many other applications of Prop. 3.1 the following problem is of a special importance.

3.3. Problem. Determine conditions (concerning the domain space X, or the mapping $f : X \to Y = f(X)$, or both) under which for some

$$\mathrm{P}_1, \mathrm{P}_2 \in \{\mathrm{LC}, \mathrm{WLC}, \mathrm{LAC}, \mathrm{SLAC}, \mathrm{QLC}, \mathrm{SLC}\}$$

one of the following two implications holds:

(3.3.1) $x \in \mathrm{P}_1(X) \Longrightarrow f(x) \in \mathrm{P}_2(f(X))$ for each $x \in X$,

(3.2.1) $f^{-1}(y) \subset \mathrm{P}_1(X) \Longrightarrow y \in \mathrm{P}_2(f(X))$ for each $y \in f(X)$.

Note that (3.3.1) is a stronger condition than (3.2.1).

A particular case of a special importance of Problem 3.3 is when $\mathrm{P}_1 = \mathrm{P}_2$. So we have the following problem.

3.4. Problem. Determine conditions (concerning the domain space X, or the mapping $f : X \to Y = f(X)$, or both) under which for

(3.4.0) $\mathrm{P} \in \{\mathrm{LC}, \mathrm{WLC}, \mathrm{LAC}, \mathrm{SLAC}, \mathrm{QLC}, \mathrm{SLC}\}$

one of the following two implications holds:

(3.4.1) $x \in \mathrm{P}(X) \Longrightarrow f(x) \in \mathrm{P}(f(X))$ for each $x \in X$,

(3.4.2) $f^{-1}(y) \subset \mathrm{P}(X) \Longrightarrow y \in \mathrm{P}(f(X))$ for each $y \in f(X)$.

To formulate a result that concerns Problem 3.3 we need an auxiliary concept. A metric space X is said to be *uniformly arcwise connected* provided that it is arcwise connected and for each positive number ε there is a positive integer n such that any arc in X contains n points that cut the arc into subarcs of diameter less than ε. The cone over the standard Cantor ternary set is called the *Cantor fan*. Recall the following result (see [11, Th. 3.5 and Cor. 3.6, p. 322]).

3.5. Statement. *A dendroid is uniformly arcwise connected if and only if it is a continuous image of the Cantor fan.*

3.6. Example. *There are uniformly arcwise connected dendroids X and Y such that there are mappings from X onto Y and for no mapping $f : X \to Y = f(X)$, for no point $x \in X$ and no two* $\mathrm{P}_1, \mathrm{P}_2$ *with*

$$\mathrm{P}_1 \in \{\mathrm{LC}, \mathrm{WLC}, \mathrm{LAC}, \mathrm{SLAC}, \mathrm{QLC}\} \quad \textit{and} \quad \mathrm{P}_2 \in \{\mathrm{LC}, \mathrm{WLC}, \mathrm{LAC}, \mathrm{SLAC}, \mathrm{QLC}, \mathrm{SLC}\}$$

the implications (3.3.1) *and* (3.2.1) *hold.*

Proof. Let $\mathcal{C}$ stand for the standard Cantor ternary set of numbers in $[0, 1]$. In the Euclidean plane let X be the cone with the vertex $v = (0, 1)$ over the set $\{(c, 0) : c \in \mathcal{C}\}$. Thus X is the Cantor fan. Denote by X' the image of X under the central symmetry with respect to the point $(0, \frac{1}{2})$. In other words, X' is the cone with the vertex $v' = (0, 0)$ over the set $\{(-c, 1) : c \in \mathcal{C}\}$. So again X' is the Cantor fan, and the common part of the two fans is the segment vv'. Hence the union $Y = X \cup X'$ is a dendroid. Note that each arc in Y is the union of at most three straight line segments, whence it follows by the construction that the length of any arc in Y is bounded by $1+2\sqrt{2} < 4$. Consequently, the dendroid Y is uniformly arcwise connected, see [1, C2, p. 193]. Thus, according to Statement 3.5, there is a mapping from X onto Y.

Observe that if $\mathrm{P}_1 \in \{\mathrm{LC}, \mathrm{WLC}, \mathrm{LAC}, \mathrm{SLAC}, \mathrm{QLC}\}$ then $\mathrm{P}_1(X) = \{v\}$, and if $\mathrm{P}_2 \in \{\mathrm{LC}, \mathrm{WLC}, \mathrm{LAC}, \mathrm{SLAC}, \mathrm{QLC}, \mathrm{SLC}\}$ then $\mathrm{P}_2(Y) = \emptyset$. Hence for any mapping $f : X \to Y = f(X)$ implications (3.3.1) and (3.2.1) cannot be satisfied. $\Diamond$

3.7. Remarks (a) It follows from Ex. 3.6 that in Prop. 3.1 the inclusion $f^{-1}(y) \subset \mathrm{WLC}(X)$ in (3.1.1) is essential and it cannot be reduced to $x \in \mathrm{WLC}(X)$ to obtain the conclusion $f(x) \in \mathrm{WLC}(f(X))$. In other words, (3.4.1) does not hold for $\mathrm{P} = \mathrm{WLC}$ (equivalently, for $\mathrm{P} = \mathrm{QLC}$, see Prop. 2.6).

(b) Ex. 3.6 shows also that to get implication (3.3.1) some extra conditions on either X or f (or both) are necessary.

A mapping $f : X \to Y$ is said to be:

— interior at a point $p \in X$ provided that for each open neighborhood U of p in X the point $f(p)$ is in the interior of $f(U)$;

— open if it maps open subsets of X to open subsets of Y; thus f is open if and only if it is interior at each point of the domain.

The following is known (see [2, Prop. 5 and Cor. 6, p. 271]).

3.8. Proposition. *Let X be any space, and let a mapping f defined on X be interior at a point $x \in X$. Then*

$$x \in \mathrm{WLC}(X) \implies f(x) \in \mathrm{WLC}(f(X)). \tag{3.8.1}$$

3.9. Corollary. *Let X be any space, and let a mapping f defined on X be interior at a point $x \in X$. If* $\mathrm{P} \in \{\mathrm{LC}, \mathrm{LAC}, \mathrm{SLAC}\}$, *then*

$$x \in \mathrm{P}(X) \implies f(x) \in \mathrm{WLC}(f(X)) \tag{3.9.1}$$

(and thus $x \in \mathrm{P}(X) \implies f(x) \in \mathrm{QLC}(f(X))$).

3.10. Corollary. *Let X be any space, and let a mapping f defined on X be open. If* $\mathrm{P} \in \{\mathrm{LC}, \mathrm{LAC}, \mathrm{SLAC}\}$, *then implication* (3.9.1) *holds.*

Conclusion of Cor. 3.9 cannot be strengthened to get $f(x) \in \in \mathrm{LC}(f(X))$ in (3.9.1). An example showing this is presented in [2, Ex. 7, p. 271]. We redo its construction here not only for the reader convenience, but also to obtain a stronger conclusion and to use it in another example.

Recall that an arc ab contained in a space S is said to be *free* provided that $ab \setminus \{a, b\}$ is an open subset of S. A mapping with connected point-inverses is said to be *monotone*.

3.11. Example. *There exist plane dendroids X and Y, a point $p \in X$ and a mapping $f : X \to Y = f(X)$ such that*

(3.11.1) *f is monotone and interior at p,*

(3.11.2) $f^{-1}(f(p)) = \{p\}$,

(3.11.3) $p \in \mathrm{LC}(X) \cap \mathrm{SLAC}(X) \cap \mathrm{SLC}(X)$ *(and thus $p \in \mathrm{LAC}(X) \cap \cap \mathrm{WLC}(X)$),*

(3.11.4) $f(p) \notin \mathrm{LC}(f(X))$ *(and thus $f(p) \notin \mathrm{SLC}(f(X))$).*

Proof. In the plane put $p = (0,0)$ and, for each $n \in \mathbb{N}$, let $v_n = (\frac{1}{n}, 0)$. Further, let m_n be the midpoint of the segment $v_{n+1}v_n$, and let $m_{n,i}$ for $i \in \{n+1, n+2, \dots\}$, be the harmonic sequence of points lying just above m_n, i.e., so that the first coordinate of each $m_{n,i}$ equals that of m_n and the second coordinate is $\frac{1}{i}$. Define G_n as the cone with the vertex v_n over the set $\{m_n\} \cup \{m_{n,i} : i \in \{n+1, n+2, \ldots\}\}$. Then each G_n is homeomorphic to the harmonic fan, and it has $v_n m_n$ as its limit segment.

The union $X = pv_1 \cup (\bigcup\{G_n : n \in \mathbb{N}\})$ is a dendroid. Observe that for each n the segment $v_{n+1}m_n$ is a free arc in X, and there are no other maximal free arcs contained in the segment pv_1. Thus (3.11.3) is satisfied by the construction.

Shrink each of these free arcs (lying in the segment pv_1) to a point, and let $f : X \to f(X) = Y$ be the quotient mapping. Note that conditions (3.11.1) and (3.11.2) hold by the definition of f. As a monotone image of a dendroid, Y is a dendroid, see e.g. [15, Cor. 13.41, p. 297]. Observe that Y is homeomorphic to the dendroid of Ex. 2.1 (i.e., of [9, Fig. 3-9, p. 113] or [15, Fig. 5.22, p. 84]), whence $f(p) \notin$ $\notin \mathrm{LC}(Y)$ by construction; thus $f(p) \notin \mathrm{SLC}(Y)$ according to (1.3.1). Therefore (3.11.4) follows. $\Diamond$

3.12. Observation. It follows from (3.11.2) of Ex. 3.11 that neither (3.4.1) nor (3.4.2) hold if $\mathrm{P} \in \{\mathrm{LC}, \mathrm{SLC}\}$. In other words, WLC in Prop. 3.1 cannot be replaced by LC or by SLC.

Let us come back to dendroids X and Y of Ex. 3.11. Observe that the dendroid X can be obtained from the dendroid Y (that has been previously described in Ex. 2.1 by (2.1.1)) by blowing each point v_n (the vertex of H_n) to a free arc such that the diameters of the inserted arcs tend to zero when n tends to infinity. Do the same operation with the dendroid X of Ex. 2.3, name the obtained dendroid again by X, and define $f : X \to f(X) = Y$ to be (again) a monotone and interior at p mapping that shrinks each added free arc back to a point. Thus Y is homeomorphic to the dendroid X of Ex. 2.3. As a consequence we get the following.

3.13. Example. *There exist dendroids X and Y, a point $p \in X$ and a mapping $f : X \to Y = f(X)$ such that*

(3.13.1) *f is monotone and interior at p,*
(3.13.2) $f^{-1}(f(p)) = \{p\}$,
(3.13.3) $p \in \mathrm{SLAC}(X)$,
(3.13.4) $f(p) \notin \mathrm{SLAC}(f(X))$.

3.14. Observation. It follows from (3.13.2) of Ex. 3.13 that neither (3.4.1) nor (3.4.2) hold if $\mathrm{P} = \mathrm{SLAC}$. In other words, WLC in Prop. 3.1 cannot be replaced by SLAC.

Ex. 3.6 shows that implication (3.4.1) does not hold for $\mathrm{P} = \mathrm{LAC}$.

However, the following questions remain open.

3.15. Question. Does (3.4.2) hold with P = LAC?

3.16. Questions. Does (3.4.2) hold with P = QLC? Note that, by Props. 3.1 and 2.6, the space X in a possible counterexample (if any) must not be compact.

A mapping $f : X \to Y$ between spaces X and Y is said to be *locally open at a point* $p \in X$ provided that there exists a closed neighborhood U of p such that $f(U)$ is a closed neighborhood of $f(p)$ and the partial mapping $f|U : U \to f(U)$ is open. It is known, [4, Statement 13, p. 360], that if a mapping is locally open at a point of its domain, then it is interior at this point. Ex. 3.11 shows that the inverse implication is not true. Obviously a mapping is open if and only if it is locally open at each point of its domain.

Ex. 3.11 shows that the concepts of local connectedness at a point and of semi-local connectedness at a point are not preserved under mappings which are interior at the considered point. But if the mapping is assumed to satisfy a stronger condition, namely to be locally open at the point, then the invariance takes place, not only for LC and SLC, but also for other considered concepts. Namely we have the following proposition which generalizes [2, Prop. 10 and Cor. 11, p. 273]. Its easy proof is left to the reader.

3.17. Proposition. *Let a mapping* $f : X \to Y = f(X)$ *defined on a space* X *be locally open at a point* $x \in X$*, and let*

(3.4.0) $\qquad$ P $\in$ {LC, WLC, LAC, SLAC, QLC, SLC}.

Then implication (3.4.1) *holds.*

3.18. Corollary. *If a mapping* $f : X \to Y = f(X)$ *is open, then implication* (3.4.1) *holds for each* P *listed in* (3.4.0).

Acknowledgement. The authors thank Adrián Soto for asking some questions that motivated this paper.

References

[1] CHARATONIK, J. J.: Two invariants under continuity and the incomparability of fans, *Fund. Math.* **53** (1964), 187–204.

[2] CHARATONIK, J. J.: Local connectivity, open homogeneity and hyperspaces, *Rev. Mat. Complut. Madrid* **6** (1993), 269–276.

[3] CHARATONIK, J. J.: Local connectedness and connected open functions, *Portugal. Math.* **53** (1996), 503–514.

[4] CHARATONIK, J. J. and OMILJANOWSKI, K.: On the set of interiority of a mapping, *Glasnik Mat.* **17**/37 (1982), 341–361.

[5] CHARATONIK, J. J., OMILJANOWSKI, K., RICCERI, B. and VILLANI, A.: Openness properties of real multifunctions on some connected spaces, *Rend. Circ. Mat. Palermo (2)* **37** (1988), 201–245.

[6] GROOT, J. De and McDOWELL, R. H.: Locally connected spaces and their compacifications, *Illinois J. Math.* **11** (1967), 353–364.

[7] ENGELKING, R.: General topology, Heldermann Verlag, Berlin, 1989.

[8] ENGELKING, R. and LELEK, A.: Cartesian products and continuous images, *Colloq. Math.* **8** (1961), 27–29.

[9] HOCKING, J. G. and YOUNG, G. S.: Topology, Addison-Wesley, Reading, Mass., 1961, (more recent edition by Dover Publ., New York, 1988).

[10] JONES, F. B.: Concerning aposyndetic and non-aposyndetic continua, *Bull. Amer. Math. Soc.* **58** (1952), 137–151.

[11] KUPERBERG, W.: Uniformly pathwise connected continua, Studies in Topology (Proceedings of the Charlotte Topology Conference, Univ. North Carolina, Charlotte, N. C. 1974), Academic Press, New York, 1975, 315–324.

[12] KURATOWSKI, K.: Topology, vol. 2, Academic Press and PWN, New York, London and Warszawa, 1968.

[13] MAKUCHOWSKI, W.: On local connectedness in hyperspaces, *Bull. Polish Acad. Sci. Math.* **47** (1999), 119–126.

[14] MOORE, R. L.: Foundations of point set theory, Amer. Math. Soc. Colloq. Publ. 13, New York, 1932, revised edition 1962.

[15] Nadler, Jr. S. B.: Continuum theory, M. Dekker, New York, Basel and Hong Kong, 1992.

[16] WHYBURN, G. T.: Analytic topology, Amer. Math. Soc. Colloq. Publ. 28, Providence, 1942, reprinted with corrections 1971.

[17] WILDER, R. L.: Topology of manifolds, Amer. Math. Soc. Colloq. Publ. 42, Providence, 1949.

[18] ZHOU, Y.: On local connectedness, *Math. Japonica* **35** (1990), 65–67.