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ON THE CLASS NUMBER OF BINARY QUADRATIC FORMS: AN OMEGA ESTIMATE FOR THE ERROR TERM

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Abstract: This paper deals with a lower estimate for the error term in the asymptotic expansion for the average order of the class number for negative discriminants. Our analysis is based on a functional equation due to Shintani and a technique introduced in a work of Tang on the sphere problem.

1. Introduction

For each positive integer n, we consider the set $\mathcal{Q}_n$ of positive definite, binary quadratic forms with integral coefficients of discriminant $-n$, i.e.,

$$\mathcal{Q}_n = \{aX^2 + bXY + cY^2 : b^2 - 4ac = -n \text{ and } a > 0\}.$$

Two forms $AX^2+BXY+CY^2$, $aX^2+bXY+cY^2$ are called equivalent, if and only if there is a matrix $S \in SL_2(\mathbb{Z})$, such that

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$$AX^2 + BXY + CY^2 = (X,Y)S^t \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} S \begin{pmatrix} X \\ Y \end{pmatrix}.$$

For a given discriminant $-n$, the number $N(n)$ of equivalence classes is finite. To study the average order of this arithmetic function, we consider the Dirichlet summatory function

$$A(t) = \sum_{n \le t} N(n), \tag{1.1}$$

where t is a large real variable. In his masterwork Disquisitiones Arithmeticae, C.F. Gauss stated an approximate formula for $A(t)$. In this century I. M. Vinogradov [12], [13] proved several upper bounds for the error term

$$E(t) := A(t) - \frac{\pi}{18} t^{3/2} + \frac{1}{4} t \tag{1.2}$$

culminating in $E(t) \ll t^{2/3+\epsilon}$. Quite recently, Chamizo and Iwaniec [3] improved this classical upper bound to

$$E(t) \ll t^{21/32+\epsilon},$$

where $21/32 = 0.65625$. (This was stated by Chamizo and Iwaniec at the end of their paper.) The main object of the present paper is to prove a two-sided Omega estimate for the error term $E(t)$.

Theorem. *For real $t \to \infty$ we have*

$$E(t) = \Omega_{\pm}\left(\sqrt{t \log t}\right).$$

Remark.1. The study of the average order of the class number $A(t)$ is closely related to a three dimensional lattice point problem. Indeed

$$A(t) = \#\{(a,b,c) \in \mathbb{Z}^3 : 4ac - b^2 \le t \text{ and either}$$
$$-a < b \le a < c \text{ or } 0 \le b < a = c\}.$$

(see e.g. Hlawka and Schoissengeier [7], Satz 3, p. 91.) It is therefore interesting to compare the upper and lower bounds for $E(t)$ with the corresponding estimates for the lattice rest of the three dimensional sphere problem: Let $r_3(n)$ be the number of representations of n as a sum of three squares, and let $P(t)$ be defined by

$$P(t) = \sum_{n \le t} r_3(n) - \frac{4\pi}{3} t^{3/2}. \tag{1.5}$$

Then each upper bound of $E(t)$ proved by Vinogradov [13,14] holds for $P(t)$, too. Heath-Brown [6] improved the upper bound for $P(t)$ to

$$P(t) \ll t^{21/32+\epsilon},$$

refining a previous result of Chamizo and Iwaniec [2]. Concerning lower estimates Tsang [11] recently proved by an ingenious new method

$$P(t) = \Omega_{\pm}\left(\sqrt{t \log t}\right).$$

Comparing the lower estimates for $E(t)$ and $P(t)$, we see that the two-sided Omega estimate for $E(t)$ in (1.2) turns out as sharp as the lower bounds for the error term $P(t)$ in the classical lattice point problem for the three dimensional sphere.

2. Instead of considering all positive definite, binary quadratic forms, it is also customary to specialize on the set of primitive positive quadratic forms. A binary quadratic form $aX^2 + bXY + cY^2$ is called *primitive*, if there is no nontrivial common divisor of a, b and c. Primitive positive definite quadratic forms of a given negative discriminant $-n$ fall into equivalence classes, too. Their class number is denoted by $h(-n)$. It is easily seen that

$$N(n) = \sum_{k^2 \mid n} h(-n/k^2). \tag{1.3}$$

By (1.3) and the Möbius inversion formula, we get

$$\sum_{n \le t} h(-n) = \sum_{k \le \sqrt{t}} \mu(k) \sum_{n \le t/k^2} N\left(\frac{n}{k^2}\right). \tag{1.4}$$

By (1.2), we get an asymptotic formula for (1.4) with an error term

$$E_*(t) := \sum_{n \le t} h(-n) - \frac{\pi}{18\zeta(3)} t^{3/2} + \frac{3}{2\pi^2} t.$$

In the paper cited above, Chamizo and Iwaniec improved the classical bounds of Chen [4] and Vinogradov [12] for $E_*(t)$ to

$$E_*(t) \ll t^{21/32+\epsilon}.$$

Unfortunately, in view of the presence of the Möbius function in (1.4), there is little hope to get omega estimates for $E_*(t)$ which are as sharp as those for $E(t)$. We remark that the same problem appears, when counting primitive lattice points in a sphere. (A lattice point $(a, b, c) \in \mathbb{Z}^3$ is called *primitive* if the greatest common divisor of a, b and c equals one.)

2. Some Lemmata

We start by introducing a new arithmetic function $N^*(n)$ which is closely related to our function $N(n)$ and whose generating Dirichlet series satisfies a functional equation, established by Shintani [9].

Lemma 1. *Let $N^*(n)$ be defined by*

$$N^*(n) = 2 \sum_{k^2 \mid n} h(-n/k^2)\omega_{-n/k^2}^{-1} ,$$

and

$$\omega_d = \begin{cases} 4 & \text{if } d = -4 \\ 6 & \text{if } d = -3 \\ 2 & \text{otherwise.} \end{cases}$$

Then

$$\sum_{n \le t} N(n) = \sum_{n \le t} N^*(n) + O(\sqrt{t}) . \tag{2.0}$$

Proof. We note that $N^*(n)$ equals $N(n)$ except when n has a divisor d with $n = 3d^2$ or $n = 4d^2$. It follows that in these cases $N^*(n) - N(n)$ equals $-2/3$ or $-1/2$, respectively. Therefore, the error is less than

$$\sum_{d^2 \ll t} 1 \ll \sqrt{t} . \quad \Diamond$$

Lemma 2. *For* $\operatorname{Re}(s) > 3/2$, *each of the two Dirichlet series*

$$\zeta_1(s) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{N^*(n)}{n^s} , \qquad \zeta_2(s) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{N^*(4n)}{(4n)^s}$$

has an analytic continuation over the whole complex plane to a meromorphic function and they satisfy the following functional equation:

$$\zeta_1(s) =$$
$$= -4\pi^{-3/2}\pi^{2s}4^{-s}G(s)\zeta_2(3/2 - s) - (2\pi)^{-2}(4\pi^2)^s\Gamma(-2s + 2)\zeta(2 - 2s) ,$$

where $\zeta(s)$ denotes the Riemann zeta function and

$$G(s) = \frac{\Gamma(3/2 - s)}{\Gamma(s)} .$$

Furthermore, the analytic continuations are holomorphic except for $s = 3/2$ and $s = 1$. The principal parts of their Laurent expansion *at* $s = 3/2$, *respectively at* $s = 1$ *are* $12^{-1}\pi(s-3/2)^{-1}$, *respectively* $-4^{-1}(s-1)^{-1}$.

Proof. After some elementary calculations, involving the duplication and reflection formula for the gamma function, this is Th. 2 of Shintani [9]. Note that $\zeta_-(s) = \zeta_1(s)$ and $\zeta_-{}^*(s) = \zeta_2(s)$ with the notation used there. $\Diamond$

In what follows we denote by δ and ϵ small positive quantities, which need not be the same at each occurence. By $C(\lambda,\mu)$ (λ, μ real numbers) we mean the oriented polygonal line which joins the points $\lambda - i\infty$, $\lambda - i$, $\mu - i$, $\mu + i$, $\lambda + i\infty$, in that order.

Lemma 3. *For each integer* $m \geq 2$ *and* $x > 0$

$$E_m(x) := \frac{1}{\Gamma(m)} \int_0^x (x-u)^{m-1} E(u)\, du$$

satisfies

$$E_m(x) = E_m^{(1)}(x) + E_m^{(2)}(x)\,,$$

where

$$E_m^{(1)}(x) = -\frac{4}{\pi^{3/2}} \sum_{n=1}^{\infty} \frac{N^*(4n)}{(4n)^{3/2}} (\pi^2 n)^{-m} I_m^*(\pi^2 n x)\,,$$

and

$$E_m^{(2)}(x) = -(2\pi)^2 \sum_{n=1}^{\infty} n^{-2} (4\pi^2 n^2)^{-m} \cdot$$

$$\cdot \frac{1}{2\pi i} \int_{-\delta - i\infty}^{-\delta + i\infty} \Gamma(-2s+2) s^{-m-1} (4\pi^2 n^2 x)^{s+m}\, ds\,.$$

with $\delta > 0$. *The function* $I_m^*(y)$ *are defined, for each integer* $m \geq 0$, *by*

$$I_m^*(y) = \sum_{k=-1,\ldots,-m} \mathrm{Res}_{s=k} \left(G(s) \frac{\Gamma(s)}{\Gamma(s+m+1)} y^{s+m} \right) + I_m(y)$$

where $I_m(y)$ *is given by an absolutely convergent integral*

$$I_m(y) = \frac{1}{2\pi i} \int_{C(\lambda,\mu)} G(s) \frac{\Gamma(s)}{\Gamma(s+m+1)} y^{s+m}\, ds.$$

Here λ, μ *are real numbers satisfying* $\lambda > 3/4$ *and* $\mu < -m$. *The function* $I_m(y)$ *possesses an asymptotic expansion*

(2.1) $$I_m(y) =$$
$$= \pi^{-1/2} \sum_{j=0}^{3} y^{(1+m-j)/2} \cos\left(2\sqrt{y} - \frac{\pi}{2}(2+m-j)\right) + O\left(y^{(m-3)/2}\right).$$

Proof. A version of Perron's formula yields for $m \geq 2$

(2.2)
$$\frac{1}{\Gamma(m)} \int_0^x (x-u)^{m-1} A(u)\, du = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta_1(s) \frac{\Gamma(s)}{\Gamma(s+m+1)} x^{s+m}\, ds.$$

Now we shift the line of integration to the left hand side of zero, observing that

$$\zeta_1(\sigma + it) \ll (1+|t|)^{\frac{3}{2}+2\delta+\epsilon}$$

in $|t| \geq 1$, $\sigma \geq -\delta$ (this is a consequence of the Phragmén-Lindelöf principle). For the gamma functions involved, we recall Stirling's formula in the weak form

$$|\Gamma(\sigma+it)| \asymp |t|^{\sigma-\frac{1}{2}} \exp(-\frac{\pi}{2}|t|)$$

uniformly in $|t| \geq 1, \sigma_1 \leq \sigma \leq \sigma_2(\sigma_1, \sigma_2$ arbitrary). From this it immediately follows that the integrand in (2.2) is $\ll |t|^{-m-1+\frac{3}{2}+\epsilon}$ where ϵ can be made arbitraryly small by the choice of δ. Thus we obtain

$$E_m(x) = \frac{1}{2\pi i} \int_{-\delta-i\infty}^{-\delta+i\infty} \zeta_1(s) \frac{\Gamma(s)}{\Gamma(s+m+1)} x^{m+s}\, ds$$

for the new integral is absolutely convergent, since $m > \frac{3}{2}$.

Using the functional equation of $\zeta_1(s)$ and inserting the Dirichlet series we conclude that

$$E_m(x) = -\frac{4}{\pi^{3/2}} \sum_{n=1}^{\infty} \frac{N^*(4n)}{(4n)^{3/2}} (\pi^2 n)^{-m} I_m^{**}(\pi^2 nx) + E_m^{(2)}(x),$$

with

$$I_m^{**}(y) = \frac{1}{2\pi i} \int_{-\delta-i\infty}^{-\delta+i\infty} G(s) \frac{\Gamma(s)}{\Gamma(s+m+1)} y^{s+m}\, ds.$$

It is evident from Lemma 2 that all the singularities of $G(s)$ are on the positive real axis. Observig this, we can deform the line of integration such that $I_m^{**}(y) = I_m^*(y)$ as defined in Lemma 3, provided that $\lambda \geq 0$ and $\mu < -m$. In order to get absolutely convergent integrals $I_m(y)$ for every $m \geq 0$ we choose λ greater than $\frac{3}{4}$. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}y}(I^*{}_{m+1}(y)) = I^*{}_m(y)\,.$$

Notice that this is also valid for $I_m(y)$ since this differs from $I^*_m(y)$ only by a finite sum of differentiable functions.

To complete the proof of Lemma 3, it remains to establish the asymptotic expansion of

$$G(s)\frac{\Gamma(s)}{\Gamma(s+m+1)}\,.$$

In what follows we write $R_k(s)$ for expressions of the form

$$R_k(s) = \sum_{j=1}^{4} c_{k,j}s^{-j}$$

where $c_{k,j}$ are any complex coefficients. We use Stirling's formula in the form

$$\log\Gamma(s+c) = (s+c-\frac{1}{2})\log s - s + \frac{1}{2}\log 2\pi + R_1(s) + O(|s|^{-5})$$

with $c \in \mathbb{C}$ arbitrary, which holds uniformly for $|\arg(s+c)| \leq \beta_0 < \pi$. (The coefficients $c_{1,j}$ and the O-constant may depend on c.) Employing this, we find that[1]

$$\log\left(G(s)\frac{\Gamma(s)}{\Gamma(s+m+1)}\right) \equiv \log F_0(s) + R_1(s) + O(|s|^{-5})\,,$$

where

$$F_0(s) = \pi^{-1/2}2^m 4^s\Gamma(-2s+1-m)\cos\left(\pi(s+m+1/2)\right).$$

Thus, on any set avoiding the poles of the terms involved,

$$G(s)\frac{\Gamma(s)}{\Gamma(s+m+1)} = F_0(s)(1+R_2(s)+O(|s|^{-5})) =$$

$$= F_0(s)\left(1+\sum_{j=1}^{2}c^*{}_j\prod_{l=1}^{j}(-2s+1-m-l)+O((1+|s|)^{-5})\right) =$$

$$= F_0(s) + \sum_{j=1}^{2}c^*{}_jF_j(s) + \Delta(s)$$

with

[1]Throughout the paper, log denotes the principal branch of the complex logarithm and $\equiv$ means congruence modulo $2\pi i$.

$$F_j(s) = \pi^{-1/2} 2^m 4^s \Gamma(-2s+1-m-j) \cos(\pi(s+m+1/2)).$$

We estimate the remainder $\Delta(s)$ by

$$\Delta(s) \ll |t|^{-5}|F_0(s)| \ll |t|^{-6-m}|G(s)| \ll |t|^{-6-m+3/2-2\sigma},$$

uniformly in $|t| \geq 1$, $\sigma_1 \leq \sigma \leq \sigma_2$ $(\sigma_1, \sigma_2$ arbitrary). We can therefore bound the contribution of $\Delta(s)$ to the integral $I_m(y)$,

$$\int_{C(\Lambda,\mu)} \Delta(s) y^{s+m}\, ds \ll y^{\mu+m} + y^{\Lambda+m} \ll 1 + y^{(m-3)/2},$$

by the choice of $\Lambda = -(m+3)/2$ (notice that μ is only restricted by $\mu \leq$ $\leq -m$ and may therefore be assumed to be less than Λ). Consequently,

$$I_m(y) = J_0(y) + \sum_{j=1}^{3} c^*{}_j J_j(y) + O(y^{(m-3)/2})$$

where, for $0 \leq j \leq 3$,

$$J_j(y) = \frac{1}{2\pi i} \int_{C(\lambda,\mu)} F_j(s) y^{s+m}\, ds.$$

To evaluate the remaining integrals, we use the following identity

$$\frac{1}{2\pi i} \int_{C(\lambda_1,\mu_1)} \Gamma(-s_1) \cos\left(\frac{\pi}{2}s_1 + \gamma\right) z^{s_1}\, ds_1 = \cos(z-\gamma),$$

where λ_1, μ_1 are real numbers satisfying $\lambda_1 > \frac{1}{2}, \mu_1 < 0$ and $z \in \mathbb{R}^+$ $+$. Comparing the arguments of the different functions involved this completes the proof of Lemma 3. $\diamond$

Our last task in this section is to establish an asymptotic expansion of the Borel mean-value of $\mathcal{B}(t)$ of the error term $E(t)$, i.e.,

$$\mathcal{B}(t) = \frac{1}{\Gamma(k+1)} \int_0^\infty \mathrm{e}^{-u} u^k E(Xu)\, du.$$

Proposition 1. *For a large positive real number t we have*

$$\mathcal{B}(t) \stackrel{\text{def}}{=} \frac{1}{\Gamma(k+1)} \int_0^\infty \mathrm{e}^{-u} u^k\, E(Xu)\, du = \frac{t}{\pi} S(t) + O(t), \tag{2.3}$$

where

$$S(t) := \sum_{0<n\leq t^\epsilon/\sqrt{X}} \frac{N^*(4n)}{n} \cos(2\pi\sqrt{n}\, t) \exp(-\pi^2 nX/2),$$

with $X = X(t)$, $k = k(t)$ two parameters depending on t, such that $X \to 0^+$, $k \to \infty$ and $t^\epsilon X^{-1/2} \to \infty$ (for any fixed $\epsilon > 0$), as $t \to \infty$.

We choose in particular $X = X(t) = t^{-2\delta}$ for some $\delta > 0$ (sufficiently small) and

$$k = k(t) = t^2 X^{-1}.$$

Proof. The proof is based on a classical method which has been developed in Szegő and Walsfisz [10], Berndt [1], Hafner [5] and Nowak [8].

We substitute $h(u) = \mathrm{e}^{-u}u^k$ in $\mathcal{B}(t)$, integrate by parts twice, apply Lemma 3 and conclude that

$$\mathcal{B}(t) = \mathcal{B}_1(t) + \mathcal{B}_2(t),$$

where

$$\mathcal{B}_1(t)\frac{X^{-2}}{\Gamma(k+1)}\int_0^\infty h''(u)E_2^{(1)}(Xu)\,du,$$

$$\mathcal{B}_2(t) = \frac{X^{-2}}{\Gamma(k+1)}\int_0^\infty h''(u)E_2^{(2)}(Xu)\,du.$$

To deal with $\mathcal{B}_2(t)$ we invert our repeated integration by parts to get

$$\mathcal{B}_2(t) = -\frac{2\pi^2}{\Gamma(k+1)}\int_0^\infty \mathrm{e}^{-u}u^k\frac{1}{2\pi i}\sum_{n=1}^\infty \frac{1}{n^2}I(4\pi^2n^2Xu),$$

where

$$I(y) = \frac{1}{2\pi i}\int_{-\delta-i\infty}^{-\delta+i\infty}\Gamma(-2s+2)s^{-1}y^s\,ds.$$

Since $\Gamma(-2s+2)s^{-1} = (4s-2)\Gamma(-2s)$, this integral evaluates to

$$I(y) = \mathrm{e}^{-\sqrt{y}}(-\sqrt{y}+1).$$

Since the series involved is absolutely convergent, we conclude by Stirling's formula that $\mathcal{B}_2(t) \ll \sqrt{Xk} = t$, the last equality by the choice of X and k.

For $\mathcal{B}_1(t)$, we substitute the series representation for $E_2^{(1)}(x)$ of Lemma 3, interchange the order of summation and integration and apply iterated integration by parts one more time. This leads to (2.4)

$$\mathcal{B}_1(t) = -4\pi^{-3/2}\sum_{n=1}^\infty \frac{N^*(4n)}{(4n)^{3/2}}\frac{1}{\Gamma(k+1)}\int_0^\infty \mathrm{e}^{-u}u^kI^*{}_0(\pi^2nXu)\,du.$$

Now we substitute (2.1) for the integrals $I^*{}_0(y) = I_0(y)$ and remark that $\frac{N^*(4n)}{n^{3/2}} \ll n^\epsilon$ for each $\epsilon > 0$. Therefore, the exponent of n arising from the O-term of the asymptotic expansion (2.1) is less than -1. We conclude that the contribution of the O-term to the asymptotic expansion of $\mathcal{B}_1(t)$ is bounded by

$$\frac{X^{-3/2}}{\Gamma(k+1)}\int_0^\infty \mathrm{e}^{-u}u^{k-3/2}\,du \ll (Xk)^{-3/2} = t^{-3/2},$$

in view of Stirling's formula.

To deal with the main terms of (2.1), we make use of a result from classical analysis dating back to Szegő and Walfisz [10].

Lemma 4. *Let α be a real constant and c, c' positive quantities. Then for $k \to \infty$,*

$$J(k,T) = \frac{1}{\Gamma(k+1)}\int_0^\infty \mathrm{e}^{-u}u^{k+\alpha}\exp(iT\sqrt{u})\,du =$$
$$= k^\alpha \exp(-\tfrac{1}{8}T^2)\exp(iT\sqrt{k}) + O(k^{\alpha-\frac{1}{2}+\epsilon})$$

if $ck^{-\epsilon} \le T \le c'k^\epsilon$. Furthermore, if $T \ge c'k^\epsilon$,

$$J(k,T) \ll T^{-C}$$

for every real constant C.

Applying this lemma to the integrals which arise if we substitute the asymptotic expansion (2.1) of $I_0(y)$, we conclude that for $0 \le j \le 3$,

$$\frac{1}{\Gamma(k+1)}\int_0^\infty \mathrm{e}^{-u}u^k I_0^*(\pi^2 nXu)\,du =$$
$$= \pi^{-1/2}(\pi^2 nX)^{(1-j)/2}\frac{1}{\Gamma(k+1)}\cdot$$
$$\cdot\int_0^\infty \mathrm{e}^{-u}u^{k+(1-j)/2}\cos(2\pi\sqrt{nXu} - \pi(2-j)/2)\,du =$$
$$= \begin{cases} H_j + O(k^\epsilon) & \text{if } ck^{-\epsilon} \le c_1(nX)^{1/2} \le c'k^\epsilon \\ O((nX)^{-2}) & \text{if } c_1(nX)^{1/2} \ge c'k^\epsilon, \end{cases}$$

where

$$H_j = \pi^{-1/2}(\pi^2 nXk)^{(1-j)/2}\cos(2\pi\sqrt{nXk} - \pi(2-j)/2).$$

The final step is to estimate the different contributions to (2.4). The terms corresponding to n which satisfy $c_1(nX)^{1/2} \le c'k^\epsilon$ contribute for $1 \le j \le 3$

$$\ll \sum_{n \le c_3 X^{-1}k^{2\epsilon}} \frac{N^*(4n)}{n^{3/2}} \ll k^\epsilon,$$

whereas the terms corresponding to $c_1(nX)^{1/2} \ge c'k^\epsilon$ contribute only

$O(1)$. Clearly the term $j = 0$ yields the main term, which completes the proof of Prop. 1. ◊

3. The Omega estimate

In this section we use a new technique introduced by Tsang [11], on the lattice rest of the sphere problem, to establish two sided omega estimates for the error term $E(t)$.

Lemma 5. *For $M \to \infty$, we have*

$$\sum_{n\le M} \frac{(N^*(4n))^2}{n^2} \gg \log M\,.$$

Proof. Since

$$\sum_{n\le t} N^*(4n) = Ct^{3/2} + O(t)\,,$$

we conclude by summation by parts that

$$\sum_{n\le M} \frac{N^*(4n)}{n^{3/2}} = \frac{3C}{2}\log M + O(1)\,.$$

Therefore, applying Cauchy's inequality, we have

$$(\log M)^2 \ll \left(\sum_{n\le M} \frac{N^*(4n)}{n^{3/2}}\right)^2 \le \sum_{n\le M} \frac{N^*(4n)^2}{n^2} \sum_{n\le M} \frac{1}{n} \ll$$
$$\ll \sum_{n\le M} \frac{N^*(4n)^2}{n^2} \log M\,,$$

which completes the proof of Lemma 5. ◊

To establish our Theorem we need Lemma 1 of Tsang [11]. Since this paper is not yet published, we state this Lemma and the proof for completeness.

Lemma 6 (Tsang [11], Lemma 1). *Let h be a real-valued integrable function defined on an interval I. If*

$$|I|^{-1}\left|\int_I h^3\right| \le \theta\left(|I|^{-1}\int_I h^2\right)^{3/2}$$

for some $\theta < 1$, then

$$\sup_I(\pm h) \geq \left(\frac{1-\theta}{2}\right)^{1/3} (|I|^{-1}\int_I h^2)^{1/2}.$$

Proof. Let h^+ and h^- denote the positive and negative parts of h respectively, that is, $h^{\pm} = \frac{1}{2}(|h| \pm h)$. Then $(h^{\pm})^3 = \frac{1}{2}(|h|^3 \pm h)$ and hence

$$(*) \qquad |I|^{-1}\int_I (h^\pm)^3 = \frac{1}{2}|I|^{-1}\int_I |h|^3 \pm \frac{1}{2}|I|^{-1}\int_I h^3 .$$

On the other hand, by Cauchy-Schwarz's inequality, we have

$$|I|^{-1}\int_I |h|^3 \geq \left(|I|^{-1}\int_I h^2\right)^{3/2} .$$

Thus, by $(*)$

$$\sup_I (h^\pm)^3 \geq |I|^{-1}\int_I (h^\pm)^3 \geq \frac{1}{2}(1-\theta)\left(|I|^{-1}\int_I h^2\right)^{3/2} ,$$

and Lemma 6 follows. ◇

Proposition 2. *For sufficiently large U and $L \asymp U^{6(\epsilon+\delta)} \ll U$ (provided that ϵ and δ are sufficiently small), we have*

$$\text{(i)} \qquad \frac{1}{L}\int_U^{U+L} (\mathcal{B}(t))^2\, dt \gg U^2 \log U ,$$

$$\text{(ii)} \qquad \frac{1}{L}\int_U^{U+L} (\mathcal{B}(t))^3\, dt \ll U^3 .$$

First we show how our **Theorem follows from this proposition**. From Lemma 6 we immediately get

$$(3.1) \qquad \sup_{U\leq t\leq U+L} (\pm\mathcal{B}(t)) \gg U\sqrt{\log U} .$$

We now suppose that

$$(3.2) \qquad E(t) \leq K\sqrt{t\log t} ,$$

and show that K cannot be arbitrarily small. In fact, the definition of $\mathcal{B}(t)$ and (3.2) imply that

$$\mathcal{B}(t) \leq \frac{KX^{1/2}}{\Gamma(k+1)} \int_0^\infty \mathrm{e}^{-u} u^{k+1/2} \sqrt{\log(Xu)}\, du + c \leq$$

$$\leq cK(Xk)^{1/2} \sqrt{\log(Xk)} = cKt\sqrt{\log t}$$

where we have used Hafner's Lemma 2.3.6 from [5], p. 51, to evaluate the integral. Together with (3.1), this yields a positive lower bound for K which establishes our Theorem. ◊

It remains to **prove the Prop. 2.** To this end we follow the proof of Tsang [11].

Proof of (i). We start with the asymptotic expansion for the Borel mean-value of the error term, established in Prop. 1. Since $U + L \asymp U$, equation (2.3) implies $(\mathcal{B}(t))^2 \asymp t^2(S(t))^2 \asymp U^2(S(t))^2$. Therefore,

$$\frac{1}{L}\int_U^{U+L} (\mathcal{B}(t)^2\, dt \asymp U^2 \frac{1}{L}\int_U^{U+L} (S(t))^2\, dt.$$

Squaring out $(S(t))^2$ and using the elementary formula

$$\cos A \cos B = \frac{1}{2}\left(\cos(A-B) + \cos(A+B)\right) \tag{3.3}$$

we can write

$$\frac{1}{L}\int_U^{U+L} (S(t))^2\, dt := \frac{1}{2L}\int_U^{U+L} \left(S_0(t) + S_1(t) + S_2(t)\right)\, dt \tag{3.4}$$

where

$$S_0(t) := \sum_{0<n\leq\kappa} \frac{(N^*(4n))^2}{n} \mathrm{e}^{-\pi^2 Xn},$$

$$S_1(t) := \sum_{\substack{0<m,n\leq\kappa \\ m\neq n}} \frac{N^*(4m)N^*(4n)}{mn} \cos\left(2\pi t(\sqrt{m} - \sqrt{n})\right)\mathrm{e}^{-\pi^2 X(m+n)/2},$$

$$S_2(t) := \sum_{0<m,n\leq\kappa} \frac{N^*(4m)N^*(4n)}{mn} \cos\left(2\pi t(\sqrt{m} + \sqrt{n})\right)\mathrm{e}^{-\pi^2 X(m+n)/2},$$

with $\kappa = t^{\epsilon+\delta}$ throughout.

We will show that the main term on the left-hand side of (3.4) comes from S_0. Indeed, the contribution of $S_1(t)$ is

$$\ll \frac{1}{L} \sum_{0<m<n\ll\kappa^2} \frac{N^*(4m)N^*(4n)}{mn} \frac{1}{\sqrt{n}-\sqrt{m}} \ll$$

$$\ll \frac{1}{L} \sum_{n<\kappa^2} \frac{N^*(4n)}{\sqrt{n}} \sum_{m<n} \frac{N^*(4m)}{m} \ll \frac{\kappa^3}{L} \ll 1$$

since

$$\frac{1}{L}\int_U^{U+L} \cos\left(2\pi t\left(\sqrt{m}-\sqrt{n}\right)\right) dt \ll \frac{1}{L}\frac{1}{\sqrt{m}-\sqrt{n}}.$$

The contribution of $S_2(t)$ is clearly not more than this. Finally consider the contribution of $S_0(t)$:

$$\frac{1}{L}\int_U^{U+L} S_0(t)\,dt \gg \sum_{n\ll t^{2\delta}} \frac{N^*(4n)^2}{n^2} \gg \log t \gg \log U,$$

by Lemma 5 and the fact that $t \asymp U$. This completes the proof of part (i) of Prop. 2.

Proof of (ii). For simplicity put

$$r = r(k,m,n) := \frac{N^*(4k)}{k}\frac{N^*(4m)}{m}\frac{N^*(4n)}{n}.$$

Taking the third power of both sides of (2.3) and using formula (3.3) repeatedly yields $(\mathcal{B}(t))^3 \asymp U^3(S(t))^3$ and

$$(S(t))^3 = \frac{1}{4}\left(S_0(t)+S_1(t)+S_2(t)\right),$$

where

$$S_0(t) := 3 \sum_{\sqrt{k}+\sqrt{m}=\sqrt{n}} r\,\mathrm{e}^{-\pi^2 Xm}$$

$$S_1(t) := 3 \sum_{\sqrt{k}+\sqrt{m}\neq\sqrt{n}} r\,\cos\left(2\pi t(\sqrt{k}+\sqrt{m}-\sqrt{n})\right)\mathrm{e}^{-\pi^2 X(k+m+n)/2}$$

$$S_2(t) := \sum r\,\cos\left(2\pi t\left(\sqrt{k}+\sqrt{m}+\sqrt{n}\right)\right)\mathrm{e}^{-\pi^2 X(k+m+n)/2}$$

where all the sums are taken over $0 < k,m,n \leq \kappa$, with the appropriate restriction specified by each sum. We integrate term by term and note that the main term of part (ii) of Prop. 2 comes from $S_0(t)$, since the contribution of $S_1(t)$ can be analogously estimated as the corresponding sum in part (i), noting that

$$|\sqrt{k}+\sqrt{m}-\sqrt{n}| \gg (\max(k,m,n))^{-3/2} \qquad \text{for} \qquad \sqrt{k}+\sqrt{m}-\sqrt{n}\neq 0.$$

Therefore,

$$\frac{1}{L}\int_U^{U+L} (S(t))^3 \ll \sum_{\substack{0<k\le m\le n\ll \kappa^2 \\ \sqrt{k}+\sqrt{m}=\sqrt{n}}} \frac{N^*(4k)}{k}\frac{N^*(4m)}{m}\frac{N^*(4n)}{n}.$$

For positive integers k, m, n the condition $\sqrt{k}+\sqrt{m}=\sqrt{n}$ holds if and only if k, m and n all have the same maximal square-free divisor q, say, i.e.,

$$k = a^2 q,\ m = b^2 q,\ n = c^2 q,$$

with $a, b, c \in \mathbb{N}$ satisfying $a+b=c$. Since $\frac{N^*(4n)}{\sqrt{n}} \ll n^\epsilon$, we can estimate the contribution of this sum as in Tsang [11], by

$$\ll \sum_{q\le\kappa^2} \sum_{a+b\le\frac{\kappa^2}{\sqrt{q}}} \frac{N^*(4a^2q)}{a^2q}\frac{N^*(4b^2q)}{b^2q}\frac{N^*(4(a+b)^2q)}{(a+b)^2q}$$

$$\ll \sum_{q\le\kappa^2} q^{-3/2} \sum_{a+b\le\frac{\kappa^2}{\sqrt{q}}} \frac{(a^2q)^\epsilon}{a}\frac{(b^2q)^\epsilon}{b}\frac{((a+b)^2q)^\epsilon}{(a+b)}$$

$$\ll \sum_{q\le\kappa^2} q^{-3/2+3\epsilon} \sum_{a+b\le\frac{\kappa^2}{\sqrt{q}}} a^{-3/2+\epsilon}b^{-3/2+\epsilon} \ll 1\,.$$

This completes the proof of part (ii) of Prop. 2 and therefore the proof of our Theorem.

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