Mathematica Pannonica
12/2 (2001), 269 – 289

NECKLACE RINGS AND THEIR RADICALS

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Received: October 2000

MSC 2000: 16 N 80

Keywords: Necklace, necklace ring, f-ring, radical,hypernilpotent, supernilpotent, hypoidempotent, subidempotent radicals.

Abstract: We investigate the algebraic properties of the necklace ring — in particular to describe the radical of such rings for various radical properties.

1. Introduction

A necklace is obtained by placing α colored beads, chosen from a set of n colors, around a circle. If a necklace is asymmetric under rotation, it is said to be primitive. It is known that the number of primitive necklaces $M(\alpha, n)$ is given by the necklace polynomial

$$M(\alpha, n) = \frac{1}{n}\sum_{d/n} \mu\left(\frac{n}{d}\right)\alpha^d.$$

Here μ denotes the classical Möbius function, i.e. if n is a positive integer, then

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$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n = p^2 m \text{ for a prime } p \\ (-1)^k & \text{if } n = p_1 p_2 \ldots p_k \text{ for distinct primes } p_i. \end{cases}$$

It has been observed by Metropolis and Rota in 1983 [4] that the identities connecting $M(\alpha\beta, n)$ with $M(\alpha, i)$ and $M(\beta, j)$ and $M(\beta^r, n)$ with $M(\beta, j)$ can be expressed in ring-theoretic terms by means of the so-called necklace ring. In addition, they showed that the cyclotomic identity provides an isomorphism between the necklace ring and a ring structure on certain rational functions, the latter being isomorphic to the ring of Witt vectors.

Subsequently these relationships have been studied by various authors, but mainly as combinatorial entities. Only recently, these rings became the object of purely algebraic investigations. Necklace rings can be defined over arbitrary commutative rings with identity. Parmenter and Spiegel [5] determined the Jacobson radical of such a necklace ring and also showed that the necklace ring over a field F algebraically determines F.

Here we continue these algebraic investigations, concentrating mainly on the radical theoretic aspects of necklace rings over arbitrary rings. Apart from their intrinsic algebraic value, the radical theory of necklace rings is interesting from another view point as well. A necklace ring has as underlying group a complete direct sum of countable copies of a ring A, and as a ring it has a homomorphic image which is a subdirect sum of countable copies of the ring A. In general, not much is known about the relationship between the radical of a ring and that of an arbitrary direct product of the ring with itself. So, even though not directly related, knowledge about the relationship between the radical of a ring and the radical of the associated necklace ring, may provide more insight into the radicals of direct products.

The analogy of these investigations to that of incidence algebras, as pointed out by Parmenter and Spiegel [5] continues: Incidence algebras over partially ordered sets were first introduced because of their combinatorial applications (cf. Gian-Carlo Rota [6]) and later studied because of their intrinsic interest as algebraic structures. In particular, Spiegel [7] determined the Jacobson radical of an incidence algebra over a commutative ring with identity and subsequently an investigation into the general radical theory of incidence algebras over arbitrary rings followed [10].

For positive integers i and j, we use $[i,j]$ and (i,j) to denote the least common multiple and the greatest common divisor of i and j respectively. Useful to remember is that $ij = [i,j](i,j)$ and that $(i,j) = ri + sj$ for some integers s and r. By $\mathbb{N}_\omega$ we denote the set $\mathbb{N}_\omega = \{1,2,3,\cdots\} \cup \{\omega\}$ where ω denotes the first limit ordinal. A ring usually means an associative ring, not necessarily commutative and not necessarily with an identity. For $k \in \mathbb{N}_\omega$, let $P_k(A)$ be the direct sum of k-copies of the ring A (if $k = \omega$, $P_k(A)$ is the complete direct sum of infinitely countable copies of A). For $a \in P_k(A)$, we use a_n to denote the n-*th* component of a, i.e. $a = (a_1, a_2, a_3, \dots)$ where it is understood that $a = (a_1, a_2, a_3, \dots, a_k)$ whenever k is finite. We often use the phrase "for suitable n" meaning $n \in \{1,2,\dots,k\}$ if k is finite and $n \in \{1,2,3,\dots\}$ if $k = \omega$.

On $(P_k(A),+)$, the underlying group of the ring $P_k(A)$, define a multiplication of $a,b \in P_k(A)$ by $(ab)_n = \sum_{[i,j]=n}(i,j)a_ib_j$ for all suitable n.

For a commutative ring A with identity, Metropolis and Rota [4] have shown that for $k = \omega$, $P_k(A) = (N_\omega(A),+,.)$ is a ring, called the *necklace ring over* A. This can be extended to arbitrary rings A and indices k:

Proposition 1. *For each* $k \in \mathbb{N}_\omega$, $N_k(A)$ *is a ring where* $(N_k(A),+) = (P_k(A),+)$ *and the multiplication is as defined above.*

Proof. The verification of the distributivity is routine — we only remark on the associativity. For $a,b,c, \in N_k(A)$ and any suitable n, the n-th components of $(ab)c$ and $a(bc)$ are given by

$$((ab)c)_n = \sum_{[i,j]=n} \sum_{[s,t]=i} (i,j)(s,t)(a_sb_t)c_j \quad \text{and}$$

$$(a(bc))_n = \sum_{[p,q]=n} \sum_{[u,v]=q} (p,q)(u,v)a_p(b_uc_v), \quad \text{respectively.}$$

Since $(i,j)(s,t) = \frac{ijst}{[i,j][s,t]} = \frac{ijst}{ni} = \frac{stj}{n}$ and $[[s,t],j] = [s,t,j]$ (the latter is the least common multiple of s,t and j), we get $((ab)c)_n = \sum_{[s,t,j]=n} \frac{stj}{n} a_sb_tc_j$. Likewise $(a(bc))_n = \sum_{[p,u,v]=n} \frac{puv}{n} a_pb_uc_v$ from which the equality $(ab)c = a(bc)$ follows. ◊

If A is commutative, then so is $N_k(A)$ and if $1 \in A$, then $(1,0,0,\dots)$ is the identity for $N_k(A)$. In order to determine the relationships between the radical of A and that of $N_k(A)$, knowledge about the relationship between the ideals of these two rings will be required.

This is done in the next section.

2. Ideals and homomorphisms of the necklace rings

For any suitable $n, \{a \in N_k(A) | a_i = 0$ for all $i \neq n\}$ is a subring of $N_k(A)$ denoted by $(0,0,\ldots 0,A,0,\ldots)$. This subring is isomorphic to the ring $(A,+,*_n)$ where $x *_n y = nxy$ for all $x,y \in A$. If k is finite and $n = k$, we have $(0,0,\ldots,0,A) \triangleleft N_k(A)$.

Let $I \triangleleft A$. Then it is routine to verify that $N_k(I) \triangleleft N_k(A)$ with

$$\frac{N_k(A)}{N_k(I)} \cong N_k\left(\frac{A}{I}\right).$$

From this it follows that if A is a subdirect sum of rings $A_\tau, \tau \in \Lambda$ for some index set Λ, then $N_k(A)$ is a subdirect sum of the rings $N_k(A_\tau)$. In particular, if A is a direct sum of rings A_τ, then $N_k(A)$ is a direct sum of the rings $N_k(A_\tau)$.

Define a function $\varphi_{k,I} : N_k(A) \to P_k(A/I)$ by

$$\varphi_{k,I}(a_1,a_2,a_3,\ldots) = (\widehat{a}_1 + I, \widehat{a}_2 + I, \widehat{a}_3 + I, \ldots)$$

where $\widehat{a}_n = \sum_{d|n} da_d$ for all suitable n.

We denote the image of $\varphi_{k,I}$ by $\Phi_k(A/I)$, i.e. $\Phi_k(A/I) := \{(a_1 + I, a_1 + 2a_2 + I, a_1 + 3a_3 + I, a_1 + 2a_2 + 4a_4 + I \ldots) \mid a_n \in A$ for all suitable $n\}$. Let $D_k(I) = \{(a_1,a_2,a_3,\ldots) \in N_k(A) \mid na_n \in I$ for all suitable $n\}$.

Proposition 2. *$\varphi_{k,I}$ is a ring homomorphism with* $\ker \varphi_{k,I} = D_k(I)$ *and* $\frac{N_k(A)}{D_k(I)} \cong \Phi_k(A/I)$ *is a subdirect sum of k-copies of A/I.*

Proof. Let us write φ for $\varphi_{k,I}$. It is clear that φ is a group homomorphism with kernel $D_k(I)$. To show that φ is a ring homomorphism, we need to show that

$$\sum_{d|n} \sum_{[i,j]=d} d(i,j)a_ib_j = \left(\sum_{d|n} da_d\right)\left(\sum_{d|n} db_d\right)$$

for all $a,b \in N_k(A)$ and all suitable n. The required equality will follow if we can show that the two sets $\{d(i,j)a_ib_j \mid [i,j] = d$ and $d|n\}$ and $\{dd'a_db_{d'} \mid$ with $d|n$and $d'|n\}$ coincide. Let $d|n$ and choose i and j such that $[i,j] = d$. Then $i|n$ and $j|n$ and $d(i,j)a_ib_j = [i,j](i,j)a_ib_j = ija_ib_j$. Conversely, suppose $d|n$ and $d'|n$. Let $u = [d,d']$. Then $u|n$ and $dd'a_db_{d'} = [d,d'](d,d')a_db_{d'} = u(d,d')a_db_{d'}$. Hence we conclude that φ is a ring homomorphism.

For each suitable n, let $\pi_n : P_k(A/I) \to A/I$ be the n-th projection and let $\gamma_n := \pi_n \circ \varphi$. For any $x \in A$, let $a = (x, 0, 0, \ldots)$. Then $a \in N_k(A)$ and $\gamma_n(a) = x + I$. Thus, γ_n is a surjective homomorphism with $\ker \gamma_n = \{a \in N_k(A) | \sum_{d|n} da_d \in I\} \supseteq D_k(I)$. Since $\bigcap_n \ker \gamma_n = D_k(I)$, we have $\Phi_k(A/I) \cong \frac{N_k(A)}{D_k(I)}$ and the latter is isomorphic to a subdirect sum of the k rings $(\frac{N_k(A)}{D_k(I)})/(\frac{\ker \gamma_n}{D_k(I)}) \cong A/I$ for all suitable n. ◊

Proposition 3.

1. *$\varphi_{k,I}$ is injective if and only if the following condition is satisfied: If $x \in A$ and $nx \in I$ for a suitable n, then $x = 0$.*
2. *$\varphi_{k,I}$ is surjective if and only if the following condition is satisfied: For every $x \in A$ and every suitable n, there is a $y = y(x, n) \in A$ such that $x - ny \in I$.*

Proof. We write φ for $\varphi_{k,I}$.

1. If φ is injective, then $D_k(I) = \ker \varphi = 0$. Let $x \in A$ with $nx \in I$. For each suitable i, let $a_i = \begin{cases} x \text{ if } i = n \\ 0 \text{ if } i \neq n \end{cases}$. Then $a = (a_1, a_2, a_3, \ldots) \in D_k(I) = 0$, i.e. $x = 0$. The converse is clear.

2. Suppose φ is surjective. Let $x \in A$ and choose any suitable n. Let $a_i = \begin{cases} x \text{ if } i = n \\ 0 \text{ if } i \neq n \end{cases}$ for all suitable i. Then $(a_1 + I, a_2 + I, a_3 + I, \ldots) \in P_k(A/I)$ and by the assumption there is a $b \in N_k(A)$ with

$$\varphi(b) = (a_1 + I, a_2 + I, a_3 + I, \ldots).$$

This means $b_1 \in I$, $\quad b_1 + 2b_2 \in I, \ldots, \sum_{d|n-1} db_d \in I$ and $\sum_{\substack{d|n \\ d\neq n}} db_d + nb_n + I = \sum_{d|n} db_d + I = x + I$. Since $\sum_{\substack{d|n \\ d\neq n}} db_d \in I$, we conclude that $x - nb_n \in I$.

Conversely, suppose the condition is satisfied. Let

$$(c_1 + I, c_2 + I, c_3 + I, \ldots) \in P_k(A/I).$$

We choose $a_1, a_2, a_3, \ldots \in A$ inductively as follows: Let $a_1 := c_1$. Since $c_2 - c_1 \in A$, by our assumption we have an $a_2 \in A$ with $c_2 - c_1 - 2a_2 \in I$. Thus, $a_1 + 2a_2 + I = c_2 + I$. Suppose that for all $m < n$ we have found $a_1, a_2, \ldots, a_m \in A$ with $\sum_{d|m} da_d + I = c_m + I$. By the assumption there is an $a_n \in A$ with $(c_n - \sum_{\substack{d|n \\ d\neq n}} da_d) - na_n \in I$.

Thus, $c_n + I = \sum_{d|n} da_d + I$. Then $a = (a_1, a_2, a_3, \ldots) \in N_k(A)$ and $\varphi(a) = (c_1 + I, c_2 + I, c_3 + I, \ldots)$. ◊

As a special case of the above, we have the results of Varadarajan and Wehrhahn [9] and Parmenter and Spiegel [5] (who only considered commutative rings with identity and $k = \omega$):

Corollary 4 ([9], [5]). *Let A be an arbitrary ring and let $k = \omega$. The homomorphism $\varphi_{\omega,0} : N_\omega(A) \to P_\omega(A)$ is:*

1. *injective if and only if $(A, +)$ is torsion-free and*
2. *surjective if and only if $(A, +)$ is divisible.* $\Diamond$

If $\operatorname{char} A = p$, p a prime, then

$$D_k(0) = \{a \in N_k(A) | \quad a_n = 0 \quad \text{if } (n,p) = 1\} = \\ = \{(0, \ldots 0, a_p, 0, \ldots, 0, a_{2p}, 0, \ldots) \mid a_{ip} \in A\}.$$

For $k = p + 1$, we then have $D_k(0) \cap (0, 0, \ldots, 0, A) = 0$ and both the ideals $D_k(0)$ and $(0, 0, \ldots, 0, A)$ are non-zero. This means $D_k(0)$ is not an essential ideal of $N_k(A)$. However, when k is infinite, then $D_\omega(0)$ is an essential ideal of $N_\omega(A)$ as we will show below. We use $(0 : A)_A$ to denote the left annihilator of A, i.e. $(0 : A)_A = \{x \in A \mid xA = 0\}$.

Proposition 5. *Let A be a ring with $(0 : A)_A = 0$ and with* $\operatorname{char} A = = p$, *$p$ a prime. Then $D_w(0)$ is an essential ideal of $N_\omega(A)$.*

Proof. Let $I \lhd N_\omega(A)$ with $D_\omega(0) \cap I = 0$. Let $a \in I$ and $x \in A$. Define $b \in D_\omega(0)$ by

$$b_n = \begin{cases} x & \text{if } n = p \\ 0 & \text{if } n \neq p. \end{cases}$$

Then $ab = 0$. Choose any m with $(m, p) = 1$ and let $n := pm$. Then $a_m x = \sum_{[i,j]=n} (i, j) a_i b_j = (ab)_n = 0$. Indeed, in the summation we only have to look at those cases when $j = p$ (for all other j we have $b_j = 0$). For such j, we need not consider the $i's$ for which p/i since for such values of $i, (i, p)a_i = pa_i = 0$. We thus assume $(i, p) = 1$. But then $pm = n = [i, p] = \frac{ip}{(i,p)} = ip$, i.e. $i = m$. Thus $a_m x = = \sum_{[i,j]=n} (i, j) a_i b_j = 0$ for all $x \in A$. Since $(0 : A)_A = 0$, we get $a_m = 0$ for all m with $(m, p) = 1$. Hence $I \subseteq D_\omega(0) \cap I = 0$. $\Diamond$

We have seen in Prop. 3 that in general $\varphi_{k,0} : N_k(A) \to P_k(A)$ need not be surjective (i.e. $\Phi_k(A) \subset P_k(A)$). However, when k is infinite and A has characteristic a prime, then $\Phi_\omega(A)$ is isomorphic to $P_\omega(A)$. Parmenter and Spiegel have shown this for commutative rings with identity ([5], Prop. 4.2), but the proof easily extends to the general case which we record as:

Proposition 6. *Let A be a ring with prime characteristic. Then $\Phi_\omega(A)$ is isomorphic to $P_\omega(A)$.* $\Diamond$

Proposition 7. *Let $I \lhd A$ and $m \geq 1$. Then $(D_k(I))^{2m} \subseteq N_k(I^m)$.*

Proof. (by induction on m). Let $m = 1$ and choose $a, b \in D_k(I)$. Then $na_n, nb_n \in I$ for all suitable n and

$$(ab)_n = \sum_{[i,j]=n} (i,j)a_ib_j = \sum_{[i,j]=n} (r_{ij}i + s_{ij}j)a_ib_j =$$
$$= \sum_{[i,j]=n} (r_{ij}(ia_i)b_j + s_{ij}a_i(jb_j)) \in I$$

for some integers r_{ij} and s_{ij}. Thus $(D_k(I))^2 \subseteq N_k(I)$.

Suppose $(D_k(I))^{2m} \subseteq N_k(I^m)$ for $m \geq 1$. Let $a_1, a_2, \dots, a_{2(m+1)} \in \in D_k(I)$ with $a_i = (a_{i1}, a_{i2}, a_{i3}, \dots)$ for $i = 1, 2, \dots, 2(m+1)$. Then $na_{in} \in I$ for all i and n.

Now

$$(a_1a_2 \dots a_{2(m+1)})_n = \sum_{[i,j]=n} (i,j)(a_1a_2 \dots a_{2m})_i(a_{2m+1}a_{2m+2})_j =$$

$$= \left(\sum_{[i,j]=n} (i,j)(a_1a_2 \dots a_{2m})_i \right) \left(\sum_{[s,t]=j} (s,t)(a_{2m+1})_s(a_{2m+2})_t \right).$$

By the induction assumption $(a_1a_2 \dots a_{2m})_i \in I^m$ and from the first step, $\sum_{[s,t]=j}(s,t)(a_{2m+1})_s(a_{2m+2})_t \in I$. Thus $(a_1a_2 \dots a_{2(m+1)})_n \in \in I^{m+1}$ for all n; hence $(D_k(I))^{2(m+1)} \subseteq D_k(I^{m+1})$. ◊

For any $I \lhd A$, we always have $N_k(I) \subseteq D_k(I)$. Next we look at the requirements for the equality.

Proposition 8. *Let $I \lhd A$. Then $D_k(I) = N_k(I)$ if and only if for any suitable n and $x \in A$, if $nx \in I$ then $x \in I$.*

Proof. Suppose $D_k(I) = N_k(I)$ and let $x \in A$ with $nx \in I$. Let $a_i = \begin{cases} x & \text{if } i = n \\ 0 & \text{if } i \neq n. \end{cases}$ Then $a \in D_k(I) = N_k(I)$, i.e. $x \in I$.

Conversely, suppose the condition holds. Let $a \in D_k(I)$. Then $na_n \in I$ for all suitable n and by the assumption $a_n \in I$ follows for all such n. Thus $a \in N_k(I)$. ◊

We conclude this section with an auxiliary result. For $u \geq 1$, let $M_u(A)$ denote the complete $u \times u$ matrix ring over A.

Proposition 9. *For any ring A, $u \geq 1$ and $k \in \mathbb{N}_\omega$, the rings $M_u(N_k(A))$ and $N_k(M_u(A))$ are isomorphic.*

Proof. Define $\psi : M_u(N_k(A)) \to N_k(M_u(A))$ as follows:

Let

$$X = [X_{rs}]_{u\times u} = \begin{bmatrix} X_{11} & X_{12} & \dots & X_{1u} \\ X_{21} & X_{22} & \dots & X_{2u} \\ \vdots & & & \vdots \\ X_{u1} & X_{u2} & \dots & X_{uu} \end{bmatrix}$$

where $X_{rs} \in N_k(A)$, say $X_{rs} = (x_1^{rs}, x_2^{rs}, \dots, x_n^{rs}, \dots)$ for all $r, s = = 1, 2, 3, \dots, u$.

For each suitable n, let $\overline{X}_n$ be the matrix

$$\overline{X}_n = [x_n^{rs}]_{u\times u} = \begin{bmatrix} x_n^{11} & x_n^{12} & \dots & x_n^{1u} \\ x_n^{21} & x_n^{22} & \dots & x_n^{2u} \\ \vdots & & & \vdots \\ x_n^{u1} & x_n^{u2} & \dots & x_n^{uu} \end{bmatrix}.$$

Let $\psi(X) = (\overline{X}_1, \overline{X}_2, \overline{X}_3, \dots)$, i.e.

$\psi([X_{rs}]) = \psi([(x_1^{rs}, x_2^{rs}, x_3^{rs}, \dots)]) = ([x_1^{rs}], [x_2^{rs}], [x_3^{rs}], \dots)$.

The injectivity, surjectivity and preservation of addition of ψ follows without any undue strain, apart from the cumbersome notation. We only remark on the preservation of the multiplication.

Let $X = [X_{rs}]_{u\times u}$, $\quad Y = [Y_{rs}]_{u\times u} \in M_u(N_k(A))$. Then

$$XY = [(XY)_{rs}]_{u\times u} = \left[\sum_{t=1}^{u} X_{rt}Y_{ts}\right]_{u\times u} =$$

$$= \left[\sum_{t=1}^{u} (x_1^{rt}, x_2^{rt}, x_3^{rt}, \dots)(y_1^{ts}, y_2^{ts}, y_3^{ts}, \dots)\right]_{u\times u}$$

which yields the n-th component of the (r, s)-entry in the matrix XY as

$$\sum_{t=1}^{u} \left(\sum_{[i,j]=n} (i, j) x_i^{rt} y_j^{ts} \right) \qquad \text{for all suitable } n.$$

Thus the (r, s)-entry of the matrix in the n-th component of $\psi(XY)$ is given by

$$\sum_{t=1}^{u} \sum_{[i,j]=n} (i, j) x_i^{rt} y_j^{ts}.$$

On the other hand, the n-th component of $\psi(X)\psi(Y)$ is given by

$$\sum_{[i,j]=n} (i,j)[x_i^{rs}]_{u\times u}[y_j^{rs}]_{u\times u} = \sum_{[i,j]=n} (i,j)\left[\sum_{t=1}^{u} x_i^{rt} y_j^{ts}\right]_{u\times u} =$$

$$= \left[\sum_{t=1}^{u} \sum_{[i,j]=n} (i,j) x_i^{rt} y_j^{ts}\right]_{u\times u} . \Diamond$$

3. Radical theory: general results

We shall see in the sequel that there is no canonical way to describe the radical of the necklace ring in terms of the radical of the base ring. This relationship not only depends on the type of radical (hypernilpotent or hypoidempotent), but also on the type of ring as well as the choice of k.

By radical we always mean a radical in the sense of Kurosh and Amitsur, see for example Divinsky [2], Wiegandt [11] or Szász [8]. For a radical α, we use $\mathcal{S}\alpha$ to denote the corresponding semisimple class, i.e. $\mathcal{S}\alpha = \{A| \quad \alpha(A) = 0\}$. A radical α is called hypernilpotent (respt. hypoidempotent) if all the nilpotent rings are radical (respt. semisimple). A hereditary hypernilpotent radical (respt. hypoidempotent radical) is called supernilpotent (respt. subidempotent). Heredity of a radical α means $I \triangleleft A \in \alpha$ implies $I \in \alpha$; or equivalently, $\alpha(I) = I \cap \alpha(A)$ for all $I \triangleleft A$. The supernilpotent radicals include the prime, nil, Levitzky (= locally nilpotent), Jacobson and Brown-McCay radicals. The von Neumann regular radical and the Blair radical (= f-regular) are examples of subidempotent radical classes.

We start with a general result which shows that for most of the well-known radicals, it is sufficient to consider only rings with identity. For a ring A, let A^* denote the Dorroh extension of A, i.e. the canonical embedding of A into a ring A^* with identity. We always have $A \triangleleft A^*$ and $A^*/A \cong \mathbb{Z}$ where $\mathbb{Z}$ denotes the ring of integers. De la Rosa and Heyman [1] have shown that $\alpha(A^*) = \alpha(A)$ for all rings A if and only if the radical α satisfies $\alpha(\mathbb{Z}) = 0$.

Proposition 10. *Let α be a radical for which $\alpha(\mathbb{Z}) = 0$. For any ring A and $k \in \mathbb{N}_\omega$,*

$$\alpha(N_k(A^*)) = \alpha(N_k(A)).$$

Proof. Since $(\mathbb{Z}, +)$ is torsion-free, $N_k(\mathbb{Z}) \cong \Phi_k\ (\mathbb{Z})$ (cf. Cor. 4).

By Prop. 2 we have $N_k(\mathbb{Z}) \in \mathcal{S}\alpha$. Since $N_k(A) \lhd N_k(A^*)$, we have $\alpha(N_k(A)) \subseteq \alpha(N_k(A^*))$. But $N_k(A^*)/N_k(A) \cong N_k(A/A^*) \cong N_k(\mathbb{Z}) \in \mathcal{S}\alpha$; hence $\alpha(N_k(A^*)) \subseteq N_k(A)$. Thus $\alpha(N_k(A^*)) \subseteq \alpha(N_k(A))$ and the equality follows. ◊

A radical α has the matrix extension property if $\alpha(M_u(A)) = M_u(\alpha(A))$ for all rings A and $u \geq 1$. The next result follows directly from Prop. 9:

Proposition 11. *Let α be a radical with the matrix extension property. For any ring A, $k \in \mathbb{N}_\omega$ and $u \geq 1$, we have $\alpha(N_k(M_u(A))) \cong M_u(\alpha(N_k(A)))$.* ◊

There are two natural ideals of $N_k(A)$, namely $N_k(\alpha(A))$ and $D_k(\alpha(A))$ which could serve as a link between the radical $\alpha(N_k(A))$ of $N_k(A)$ and the radical $\alpha(A)$ of A. We next investigate the relationship between these two ideals. We always have $N_k(\alpha(A)) \subseteq D_k(\alpha(A))$ and the requirement for the converse follows from Prop. 8 which we record in:

Proposition 12. *Let α be any radical class. Then the following are equivalent:*

1. $N_k(\alpha(A)) = D_k(\alpha(A))$.
2. *For any suitable n and $x \in A$, if $nx \in \alpha(A)$, then $x \in \alpha(A)$.*
3. $(A/\alpha(A), +)$ *is torsion-free.*

A last general result is:

Proposition 13. *For any radical class α and $k \in \mathbb{N}_\omega, \alpha(N_k(A)) \subseteq D_k(\alpha(A))$.*

Proof. By Prop. 2, and since semisimple classes are subdirectly closed, we have $\frac{N_k(A)}{D_k(\alpha(A))} \cong \Phi_k(A/\alpha(A)) \in \mathcal{S}\alpha$. Hence $\alpha(N_k(A)) \subseteq D_k(\alpha(A))$. ◊

4. Hypernilpotent radicals

We shall see below that for $k = \omega$, the equality $\alpha(N_k(A)) = D_k(\alpha(A))$ does not hold in general; not even if the radical is supernilpotent. An instance when it does is given by

Proposition 14. *Let α be a hypernilpotent radical and suppose $\alpha(A)$ is nilpotent. Then $\alpha(N_k(A)) = D_k(\alpha(A))$ for any $k \in \mathbb{N}_\omega$.*

Proof. Suppose $(\alpha(A))^m = 0$. By Prop. 7 we have $(D_k(\alpha(A)))^{2m} \subseteq N_k((\alpha(A))^m) = 0$. Since α is hypernilpotent, $D_k(\alpha(A)) \subseteq \alpha(N_k(A))$. The equality then follows from Prop. 13. ◊

We remark that the equality $\alpha(N_k(A)) = D_k(\alpha(A))$ implies $N_k(\alpha(A)) \subseteq \alpha(N_k(A))$. In general this inclusion is sharp, even for supernilpotent radicals: Let $A = \mathbb{Z}_4$ be the ring of integers mod4 and let $\mathcal{J}$ denote the Jacobson radical. Since A is commutative with identity, it follows from Parmenter and Spiegel [5] (or Prop. 14) that

$$\mathcal{J}(N_\omega(\mathbb{Z}_4)) = D_\omega(\mathcal{J}(\mathbb{Z}_4)) = D_\omega(\{0,2\}) =$$
$$= \{a \in N_\omega(\mathbb{Z}_4) \mid \quad a_{2n-1} \in \{0,2\} \text{ for all } n\}.$$

Thus $b = (0,1,0,1,0,1,0,1,\dots) \in \mathcal{J}(N_\omega(\mathbb{Z}_4))$, but $b \notin N_\omega(\mathcal{J}(\mathbb{Z}_4))$.

For any hypernilpotent radical α, we have $D_k(0) \subseteq \alpha(N_k(A))$ (using Prop. 7). The next result gives the requirements for the equality. At the outset, it is worthwhile to remember that $A \in \mathcal{S}\alpha$ implies $\Phi_k(A) \in \mathcal{S}\alpha$ (Prop. 2).

Proposition 15. *Let α be a hypernilpotent radical. Then:*

1. $\alpha(N_k(A)) = D_k(0)$ *if and only if* $\Phi_k(A) \in \mathcal{S}\alpha$.
2. $\alpha(N_k(A)) = 0$ *if and only if* $\Phi_k(A) \in \mathcal{S}\alpha$ *and whenever* $nx = 0$ *for a suitable* n *and* $x \in A$, *then* $x = 0$.
3. $\alpha(N_\omega(A)) = 0$ *if and only if* $\Phi_\omega(A) \in \mathcal{S}\alpha$ *and* $(A,+)$ *is torsion-free.*

Proof. From Prop. 2 we know $\alpha(N_k(A))/D_k(0) \lhd N_k(A)/D_k(0) \cong \cong \Phi_k(A)$.

Thus the equivalence in 1. follows. The last two statements, read in conjunction with Prop. 3, are just special cases of 1. ◊

We should point out that the radical does its job fairly well: Let α be a hypernilpotent radical with $\alpha(A)$ nilpotent. Then $(N_k(A))/ / \alpha(N_k(A)) = N_k(A)/D_k(\alpha(A)) \cong \Phi_k(A/\alpha(A))$ and the latter is a subdirect sum of k copies of the semisimple ring $A/\alpha(A)$.

Proposition 16. *Let α be a hypernilpotent radical. Then $N_k(A) \in \in \alpha$ if and only if $\Phi_k(A) \in \alpha$. In particular, if $\Phi_k(\alpha(A)) \in \alpha$, then $N_k(\alpha(A)) \subseteq \alpha(N_k(A))$.*

Proof. Since $D_k(0) \in \alpha$, the equivalence follows from the isomorphism $\frac{N_k(A)}{D_k(0)} \cong \Phi_k(A)$ and, on the one hand, the homomorphic closure of α and on the other hand, the extension closure of α. ◊

Let us remark that if k is infinite and $\operatorname{char} A = p, p$ a prime, then $N_\omega(A) \in \alpha$ if and only if $P_\omega(A) \in \alpha$ for any hypernilpotent radical α (cf. Prop. 6). In addition, for such radicals α and any field F, we have $N_\omega(F) \in \alpha$ if and only if $P_\omega(F) \in \alpha$. Indeed, if $\operatorname{char} F$ is a prime, it follows from the proceeding remark. If $\operatorname{char} F$ is 0, then $(F,+)$ is divisible and by Cor. 4 we have $\Phi_\omega(F) \cong P_\omega(F)$.

Proposition 17. *Let α be a hypernilpotent radical. Then the following are equivalent:*

1. $\alpha(N_k(A)) = D_k(\alpha(A))$.
2. $(D_k(\alpha(A)))^m \subseteq \alpha(N_k(A))$ *for some* $m \geq 1$.
3. $N_k(\alpha(A)) \subseteq \alpha(N_k(A))$.

If, in addition, α is also hereditary (i.e. α is supernilpotent), then we may add the following to the list of equivalences:

4. $N_k(\alpha(A)) \in \alpha$.
5. $\Phi_k(\alpha(A)) \in \alpha$.

Proof. 1. $\Rightarrow$ 3.: $N_k(\alpha(A)) \subseteq D_k(\alpha(A)) = \alpha(N_k(A))$.

3. $\Rightarrow$ 2.: $(D_k(\alpha(A)))^2 \subseteq N_k(\alpha(A)) \subseteq \alpha(N_k(A))$ (cf. Prop. 7).

2. $\Rightarrow$ 1.: Suppose $(D_k(\alpha(A)))^m \subseteq \alpha(N_k(A))$ for some $m \geq 1$. Then $\frac{D_k(\alpha(A))}{\alpha(N_k(A))}$ is a nilpotent ideal of $\frac{N_k(A)}{\alpha(N_k(A))} \in \mathcal{S}\alpha$. Thus $D_k(\alpha(A)) =$ $= \alpha(N_k(A))$.

4. $\Rightarrow$ 3.: This implication holds in general and, if α is hereditary, the converse implication is clear.

The last equivalence follows from Prop. 16. ◊

Worthwhile noting is that if α is supernilpotent and A is a ring with char A prime or if A is a field, then $\alpha(N_\omega(A)) = D_\omega(\alpha(A))$ if and only if $N_\omega(\alpha(A)) \in \alpha$ if and only if $P_\omega(A) \in \alpha$. But, at least for the well-known supernilpotent radicals α, we have $\alpha(F) = 0$ for any field F. Hence, $\alpha(N_\omega(F)) = D_\omega(0)$ as we have seen in Prop. 15.

Corollary 18. *Let α be a hypernilpotent radical. Then the following are equivalent:*

1. $\alpha(N_k(A)) = D_k(\alpha(A))$ *for all rings* A.
2. $A \in \alpha$ *implies* $N_k(A) \in \alpha$.
3. $A \in \alpha$ *implies* $\Phi_k(A) \in \alpha$.◊

When k is finite, things are much simpler:

Proposition 19. *Let α be a supernilpotent radical and suppose k is finite. Then $\alpha(N_k(A)) = D_k(\alpha(A))$ for all rings A. In particular, $N_k(A) \in \alpha$ if and only $A \in \alpha$.*

Proof. Let $A \in \alpha$. By a result of Heinicke [3] it is known that any hereditary radical is closed under finite subdirect sums. Hence $\Phi_k(A) \in$ $\in \alpha$ (cf Prop. 2) and the result follows Cor. 18. ◊

Next we will determine the Jacobson radical of $N_k(A)$. For the special case when A is commutative with identity, Parmenter and Spiegel [5] have shown that $\mathcal{J}(N_\omega(A)) = D_\omega(\mathcal{J}(A))$. We will extend this result to arbitrary rings and any $k \in \mathbb{N}_\omega$. For k finite, we may use Prop. 19 above; so suppose k is infinite. In this case, the required result will follow from a more general procedure.

Suppose that for all rings A there is a function $f_A : A \times A \to A$ which satisfies:

(1) $f_A(a, 0) = a$ for all $a \in A$ and

(2) If $\psi : A \to B$ is a surjective homomorphism, then $\psi(f_A(a, b)) = f_B(\psi(a), \psi(b))$ for all $a, b \in A$.

An element $a \in A$ is called an *f-element* of A if there is a $b \in A$ such that $f_A(a, b) = 0$. The ring A is called an *f-ring* if a is an f-element of A for all $a \in A$.

Let A be an f-ring. We want to know when $\Phi_k(A)$ is an f-ring. Let $a = (a_1, a_1 + 2a_2, \ldots, \sum_{d|n} da_d, \ldots) \in \Phi_k(A)$. We need to find a $b = (b_1, b_1 + 2b_2, b_1 + 3b_3, \ldots, \sum_{d|n} db_d, \ldots) \in \Phi_k(A)$ such that $f_{\Phi_k(A)}(a, b) = 0$. We will construct the components $b_1, b_1 + 2b_2, b_1 + 3b_3, \ldots$ of b inductively. For $a \in A$, since A is an f-ring, there is a $b_1 \in A$ such that $f_A(a_1, b_1) = 0$. Suppose we have found $b_1, b_2, \ldots b_n \in A$ for a suitable n with $n \geq 1$, such that

$$f_A\left(\sum_{d|m} da_d, \quad \sum_{d|m} db_d\right) = 0 \text{ for all } m = 1, 2, \ldots, n.$$

We need to find $b_{n+1} \in A$ such that

$$f_A\left(\sum_{d|n+1} da_d, \quad \sum_{d|n+1} db_d\right) = 0.$$

If such a b_{n+1} can be found, we may conclude by induction that there is a $b \in \Phi_k(A)$ such that

$$f_A\left(\sum_{d|n} da_d, \quad \sum_{d|n} db_d\right) = 0$$

for all suitable n.

Then $b = (b_1, b_1 + 2b_2, \ldots \sum_{d|n} db_d, \ldots) \in \Phi_k(A)$ and for all suitable n and if $\pi_n : \Phi_k(A) \to A$ denotes the n-th projection, we have

$$\pi_n(f_{\Phi_k(A)}(a,b)) = f_{\pi_n(\Phi_k(A))}(\pi_n(a),\pi_n(b)) =$$
$$= f_A\left(\sum_{d|n} da_d, \quad \sum_{d|n} db_d\right) = 0.$$

Hence, $f_{\Phi_k(A)}(a,b) = 0$.

In order to find the above mentioned $b_{n+1} \in A$, we need to consider two conditions that the property f may satisfy:

(F1) Let R be a ring, $x \in R$ and $n \geq 1$. If $f_R(nx, y) = 0$ for some $y \in R$, then there is a $z \in R$ such that $f_R(nx, nz) = 0$.

(F2) Let R be a ring, $I \lhd R$ and $x, y \in R$ such that $f_R(x,y)$ is an f-element of I. Then there is a $z \in R$ with $y - z \in I$ and $f_R(x,z) = 0$.

Proposition 20. *Suppose the property f satisfies conditions* (F1) *and* (F2). *If A is an f-ring, then $\Phi_k(A)$ is an f-ring.*

Proof. In view of the proceeding discussion, suppose we have found $b_1, b_2, \ldots b_n \in A$ such that

$$f_A\left(\sum_{d|m} da_d, \quad \sum_{d|m} db_d\right) = 0 \text{ for all } m = 1, 2, \ldots, n.$$

We complete the proof by showing how to choose b_{n+1}.

Now both $\overline{a} := (a_1, a_1 + 2a_2, \ldots, \sum_{d|n+1} da_d)$ and $b' := (b_1, b_1 + 2b_2, \ldots, \sum_{d|n} db_d, \sum_{\substack{d|n+1 \\ d\neq n+1}} db_d)$ are elements of $\Phi_{n+1}(A)$. Let $\gamma : \Phi_{n+1}(A) \to \Phi_n(A)$ be the function that "forgets" the $(n+1)$-th component. Clearly γ is a surjective ring homomorphism. For each $m = 1, 2, \ldots, n$ let $\pi_m : \Phi_n(A) \to A$ be the m-th projection.

Then for each m,

$$\pi_m(f_{\Phi_n(A)}(\gamma(\overline{a}), \gamma(b'))) = f_A\left(\sum_{d|m} da_d, \quad \sum_{d|m} db_d\right) = 0,$$

hence

$$\gamma(f_{\Phi_{n+1}(A)}(\overline{a}, b')) = f_{\Phi_n(A)}(\gamma(\overline{a}), \gamma(b')) = 0.$$

Thus

$$e := f_{\Phi_{n+1}}(\overline{a}, b') \in \ker\gamma = (0, 0, \ldots, 0, (n+1)A),$$

say $e = (0, 0, \ldots, 0, (n+1)x)$ for some $x \in A$. Since A is an f-ring, there is a $y \in A$ such that $f_A((n+1)x, y) = 0$. By condition (F1) there is an $s \in A$ with

$$f_A((n+1)x,(n+1)s) = 0.$$

Then $c := (0,0,\dots,0,(n+1)s) \in \ker\gamma$ and $f_{\Phi_{n+1}(A)}(e,c) = 0$. Indeed, for all $m = 1,2,\dots,n,n+1$ if we also use π_m to denote the m-projection $\pi_m : \Phi_{n+1}(A) \to A$, we have

$$\begin{aligned}\pi_m(f_{\Phi_{n+1}(A)}(e,c)) &= f_A(\pi_m(e),\pi_m(c))\\ &= \begin{cases} f_A(0,0) & \text{if } m \in \{1,2,\dots,n\}\\ f_A((n+1)x,(n+1)s) & \text{if } m = n+1\end{cases}\\ &= 0.\end{aligned}$$

Thus $e = f_{\Phi_{n+1}(A)}(\overline{a},b')$ is an f-element of $\ker\gamma$. By (F2) there is a $\overline{b} \in \Phi_{n+1}(A)$ such that $b' - \overline{b} \in \ker$ and $f_{\Phi_{n+1}(A)}(\overline{a},\overline{b}) = 0$. Thus $\gamma(\overline{b}) = \gamma(b') = (b_1, b_1 + 2b_2, \dots, \sum_{d|n} db_d)$ which means that the $(n+1)$-th component of $\overline{b}$ is of the form $\sum_{\substack{d|n+1\\ d\neq n+1}} db_d + (n+1)t$ for some $t \in A$. Let $b_{n+1} := t$. Then

$$\begin{aligned} f_A\left(\sum_{d|n+1} da_d, \sum_{d|n+1} db_d\right) &= f_{\pi_{n+1}(\Phi_{n+1}(A))}(\pi_{n+1}(\overline{a}),\pi_{n+1}(\overline{b}))\\ &= \pi_{n+1}(f_{\Phi_{n+1}(A)}(\overline{a},\overline{b})) = 0. \ \Diamond\end{aligned}$$

Example 21. For any ring A, let $f_A(a,b) := a+b-ab$. Then $f(a,0) = a$ and if ψ is a homomorphism of A, then $\psi(f_A(a,b)) = f_{\psi(A)}(\psi(a),\psi(b))$. Moreover, (F1) and (F2) are also satisfied:

(F1) Suppose $0 = f_A(ka,b) = ka + b - kab$. Let $c := ab - a$. Then $b = kab - ka = kc$ and $f_A(ka,kc) = 0$.

(F2) Let $I \lhd A$ with $x,y \in A$ such that $f_A(x,y)$ is an f-element of I. Then there is an $t \in I$ with $f_A(f_A(x,y),t) = 0$. Let $z := f_A(y,t)$. Then $f_A(x,z) = f_A(f_A(x,y),t) = 0$ and $y - z = y - (y + t - yt) = yt - t \in I$.

It is clear that A is an f-ring if and only if A is Jacobson radical. From Props. 19 and 20 we then have

Proposition 22. *For any ring A and $k \in \mathbb{N}_\omega$,*

$$\mathcal{J}(N_k(A)) = D_k(\mathcal{J}(A))$$

where J denotes the Jacobson radical.$\Diamond$

Proposition 23. *Let α be any radical with $\mathcal{J} \subseteq \alpha$ and let A be a ring with $\mathcal{J}(A) = \alpha(A)$. Then $\alpha(N_k(A)) = \mathcal{J}(N_k(A)) = D_k(\alpha(A))$.*

Proof. From Prop. 13 and the assumptions, we have $\alpha(N_k(A)) \subseteq D_k(\alpha(A)) = D_k(\mathcal{J}(A)) = \mathcal{J}(N_k(A)) \subseteq \alpha(N_k(A))$. $\Diamond$

For example, if $\mathcal{G}$ denotes the Brown-McCoy radical and A is a commutative ring or A has the dcc on left ideals, then $\mathcal{G}(A) = \mathcal{J}(A)$. Then Prop. 23 is applicable which yields $\mathcal{G}(N_k(A)) = \mathcal{J}(N_k(A))$. However, one should not be tempted to think that $\mathcal{J}(A) = \alpha(A)$ will always imply $\mathcal{J}(N_k(A)) = \alpha(N_k(A))$ for a radical α as the next example shows.

Example 24. Let α be the nil radical. We know α is supernilpotent and $\alpha \subseteq \mathcal{J}$. Let B be the Zassenhaus ring given in Ex. 3 of Divinsky [2] which we reconstruct here for completeness: For each real number τ with $0 < \tau < 1$, consider the set of all symbols x_τ. Let F be a field with char $F = 0$. Then B is the commutative algebra over F with $\{x_\tau | \ 0 < < \tau < 1\}$ as basis. This means the elements of B are finite sums of the form $\sum_\tau b_\tau x_\tau (b_\tau \in F)$ with formal addition and multiplication - the latter determined by the following products of the base elements:

$$x_\tau x_\upsilon = \begin{cases} x_{\tau+\upsilon} & \text{if } \tau+\upsilon < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then it is known that B is nil (i.e. $B \in \alpha$), but B is not nilpotent.

Let $b = (b_1, b_2, b_3, \dots)$ be the element of $N_\omega(B)$ with $b_n = x_{\tau_n}$ where $\tau_n = \frac{1}{2^{n+1}}$ for all $n = 1, 2, 3, \dots$. Then $b^m \neq 0$ for all $m \geq 1$. Indeed, it is easy to see that the m-th component of b^m contains the term $m^{m-1}(b_m)^m = m^{m-1} x_{\frac{m}{2^{m+1}}} \neq 0$. Thus $\alpha(N_\omega(B)) \neq N_\omega(B)$. Since $\alpha \subseteq \subseteq \mathcal{J}$, we have $B = \alpha(B) = \mathcal{J}(B)$, but $\alpha(N_\omega(B)) \subset N_\omega(B) = D_\omega(B) = = D_\omega(\mathcal{J}(B)) = \mathcal{J}(N_\omega(B))$. One may also verify that $\alpha(N_\omega(B)) \neq 0$.

This example also shows that in general $\alpha(N_\omega(A))$ need not coincide with $N_\omega(\alpha(A))$ nor with $D_\omega(\alpha(A))$, even though $\alpha(A) = A$ and α is a supernilpotent radical. $\Diamond$

In addition to Prop. 23, there are more cases when $\alpha(A) = \mathcal{J}(A)$ does imply $\alpha(N_k(A)) = \mathcal{J}(N_k(A))$:

Proposition 25. *Let α be a hypernilpotent radical and let A be a ring with $\alpha(A) = \mathcal{J}(A)$ nilpotent. Then $\alpha(N_k(A)) = \mathcal{J}(N_k(A))$.*

Proof. From Props. 14 and 22 we get $\alpha(N_k(A)) = D_k(\alpha(A)) = = D_k(\mathcal{J}(A)) = \mathcal{J}(N_k(A))$. $\Diamond$

5. Hypoidempotent radicals

Lastly we will investigate the hypoidempotent radicals of $N_k(A)$. Here we shall see that the description of the radical is even less orderly than for the hypernilpotent radicals (depending very much on k and the characteristic of A). If α is a hypoidempotent radical, then $\frac{D_k(\alpha(A))}{N_k(\alpha(A))} \in$ $\in \mathcal{S}\alpha$ (by Prop. 7). Since $\mathcal{S}\alpha$ is closed under extensions, we conclude from Prop. 2 and

$$\frac{D_k(\alpha(A))}{N_k(\alpha(A))} \lhd \frac{N_k(A)}{N_k(\alpha(A))} \to \frac{N_k(A)}{D_k(\alpha(A))} \cong \Phi_k(A/\alpha(A)) \in \mathcal{S}\alpha,$$

that $\frac{N_k(A)}{N_k(\alpha(A))} \in \mathcal{S}\alpha$. Thus we have

Proposition 26. *Let α be a hypoidempotent radical. For any ring A and $k \in \mathbb{N}_\omega$, we have $\alpha(N_k(A)) \subseteq N_k(\alpha(A))$. The equality holds if and only if $N_k(\alpha(A)) \in \alpha$.*

Corollary 27. *Let α be a hypoidempotent radical. Then the following are equivalent:*

1. *$\alpha(N_k(A)) = N_k(\alpha(A))$ for all rings A.*
2. *$A \in \alpha$ implies $N_k(A) \in \alpha$.* $\diamondsuit$

However, we shall see below that $N_k(A) \in \alpha$ is not very likely for arbitrary rings, even if $A \in \alpha$.

Proposition 28. *Let α be a subidempotent radical.*

1. *For finite $k, N_k(A) \in \alpha$ if and only if $A \in \alpha$ and whenever $nx = 0$ for $x \in A$ and $n \in \{1, 2, 3, \ldots, k\}$, then $x = 0$. In particular, $N_k(\alpha(A)) \in \alpha$ if and only if whenever $nx = 0$ for $x \in \alpha(A)$ and $n \in \{1, 2, 3, \ldots, k\}$ then $x = 0$.*
2. *Suppose $k = \omega$ and α satisfies: $B \in \alpha$ implies $\Phi_k(B) \in \alpha$. Then $N_\omega(A) \in \alpha$ if and only if $A \in \alpha$ and $(A, +)$ is torsion-free. In particular, $N_k(\alpha(A)) \in \alpha$ if and only if $(\alpha(A), +)$ is torsion-free.*

Proof. 1. Suppose $N_k(A) \in \alpha$. Since α is hereditary, $D_k(0) \in \mathcal{S}\alpha \cap$ $\cap\, \alpha = 0$ (cf. Prop. 7). Hence $N_k(A) \cong \Phi_k(A)$ (cf. Prop. 2). Since A is a homomorphic image of $\Phi_k(A)$, we have $A \in \alpha$. Conversely, suppose $A \in \alpha$ and $nx = 0$ implies $x = 0$, $x \in A$ and $1 \leq n \leq k$. Then $N_k(A) \cong$ $\cong \Phi_k(A) \in \alpha$ since $D_k(0) = 0$ and α is closed under finite subdirect sums.

2. Similar to (1) above. $\diamondsuit$

Example 29. Let υ denote the von Neumann regular radical. If $A \in$ $\in \upsilon$, then $\Phi_k(A) \in \upsilon$ for all $k \in \mathbb{N}_\omega$. Indeed, for finite k it follows since υ is hereditary (and thus closed under finite subdirect sums). For $k = \omega$, we will use Prop. 20. For a ring A, let $f_A(a, b) = a - aba$. Then $f_A(a, 0) = a$ and for any homomorphism ψ of $A, \psi(f_A(a, b)) =$ $= f_{\psi(A)}(\psi(a), \psi(b))$. Moreover, the conditions (F1) and (F2) are also satisfied:

(F1) Suppose $f_A(na, b) = 0$. Then $na = nabna$ and for $c := := nbabab$,

$$f_A(na, nc) = na - nancna = na - nannbababna = \\ = na - (nabna)b(nabna) = na - nabna = 0.$$

(F2) Let $I \lhd A$, $a, b \in A$ such that $f_A(a, b) \in I$ and $f_A(f_A(a, b), i) = = 0$ for some $i \in I$. Then

$$a = aba + aia - aiaba - abaia + abaiaba = \\ = a(b + i - iab - bai + baiab)a = aca$$

where

$$c = b + i - iab - bai + baiab.$$

Thus

$$f_A(a, c) = 0 \text{ and } b - c \in I.$$

It is clear that $A \in \upsilon$ if and only if A is an f-ring; hence $\Phi_\omega(A) \in \upsilon$ if $A \in \upsilon$ follows from Prop. 20. ◊

Proposition 30. *Let α be a subidempotent radical. Then $\alpha(N_\omega(A)) = = 0$ for any ring A with* $\operatorname{char} A$ *prime and* $(0 : A)_A = 0$.

Proof. By Prop. 5 we know $D_\omega(0)$ is an essential ideal of A. Since α is subidempotent,

$$D_\omega(0) \cap \alpha(N_\omega(A)) = \alpha(D_\omega(0)) = 0;$$

hence $\alpha(N_\omega(A)) = 0$. ◊

For finite k, the following is often a useful tool when calculating the radical of $N_k(A)$:

Lemma 31. *Let α be a hereditary radical with k finite. If $A \in \alpha$ and $I \lhd N_k(A)$ with $D_k(0) \cap I = 0$, then $I \subseteq \alpha(N_k(A))$.*

Proof.

$$I \cong \frac{I}{I \cap D_k(0)} \cong \frac{I + D_k(0)}{D_k(0)} \lhd \frac{N_k(A)}{D_k(0)} \cong \Phi_k(A) \in \alpha;$$

hence $I \subseteq \alpha(N_k(A))$. ◊

Corollary 32. *Suppose α is a subidempotent radical, k is finite and $A \in \alpha$. Then $\alpha(N_k(A))$ is the (unique) maximal element of the set $\{I \lhd N_k(A) | D_k(0) \cap I = 0\}$.*

Proof. Since α is subidempotent, $\alpha(N_k(A)) \cap D_k(0) = 0$. Suppose $I \lhd \lhd N_k(A)$ with $D_k(0) \cap I = 0$ and $\alpha(N_k(A)) \subseteq I$. By the above lemma we have $\alpha(N_k(A)) = I$. ◊

Proposition 33. *Let α be a subidempotent radical and let k be finite. Let A be a ring with* char $A = p$, *p prime, and $A \in \alpha$. Then $\alpha(N_k(A)) = (V_1, V_2, \dots, V_k)$ where* $V_n = \begin{cases} 0 & \text{if } pn \leq k \text{or if } p|n \\ A & \text{otherwise} \end{cases}$ *for all $n = 1, 2, 3, \dots, k$.*

Proof. Note firstly that $A \in \alpha$ implies $(0 : A)_A = 0$ (because α is hereditary and hypoidempotent).

Let $I := (V_1, V_2, V_3, \dots, V_k)$ where the V_n's are defined as above. We show $I \lhd N_k(A)$. That I is a subgroup of $N_k(A)$ is clear. Let $a \in I$ and $b \in N_k(A)$. We show $ab \in I$, the other case $ba \in I$ follows similarly. If $pn \leq k$, then $(ab)_n = \sum_{[i,j]=n}(i,j)a_i b_j = 0$ since for each i with $[i,j] = n$, we have $ip \leq np \leq k$. From the definition of $I = (V_1, V_2, V_3, \dots, V_k)$ we have $a_i = 0$.

If $n = pu$ for some $u \geq 1$, choose any i with $[i,j] = n = pu$. Then $i|pu$ implies $i = pd$ or $i = d$ for some d with $d|u$. If $i = pd$, then $a_i = 0$ and if $i = d$, then $pi = pd \leq pu = n < k$ also giving $a_i = 0$. Hence $ab \in I$. Note that $D_k(0) \cap I = 0$; thus Prop. 31 yields $I \subseteq \alpha(N_k(A))$.

Since α is subidempotent, we have $D_k(0) \cap \alpha(N_k(A)) = 0$. This means $\alpha(N_k(A)) \subseteq W = (W_1, W_2, \dots, W_k)$ where

$$W_n = \begin{cases} 0 & \text{if } pn \leq k \text{ and } p \nmid n \\ A & \text{otherwise.} \end{cases}$$

Indeed, let $a \in \alpha(N_k(A))$ and choose $x \in A$ arbitrary. Let

$$b = (0, 0, \dots, 0, x, 0, \dots, 0)$$

where $b_p = x$. Then $b \in D_k(0)$ and so $ab = 0$.

Let $n \in \{1, 2, \dots, k\}$ with $pn \leq k$ and $p \nmid n$. We determine the pn-th component of ab. For this we only have to look at the terms of $\sum_{[i,j]=n}(i,j)a_i b_j$ for which $j = p$. For these we consider two cases. If $(i,p) = 1$, then $pn = [i,p] = ip$, i.e. $i = n$. If $(i,p) \neq 1$, then $(i,p) = p$ and thus $i = pu$ for some $u \geq 1$. Then $pn = [i,p] = i = pn$, i.e. $n = u$ and $0 = \sum_{[i,j]=pn}(i,j)a_i b_j = \sum_{[i,p]=pn}(i,p)a_i b_p = a_n x + pa_{pn}x = a_n x$ for all $x \in A$. Thus $a_n = 0$ and so $\alpha(N_k(A)) \subseteq W$. Since α is hypoidempotent, we have $\alpha(N_k(A)) = (\alpha(N_k(A)))^2 \subseteq W^2$ and to complete the proof we show $W^2 \subseteq I$. Let $a, b \in W$. We show $(ab)_n = 0$ whenever $n \in \{1, 2, 3, \dots, k\}$ with $pn \leq k$ or $p|n$.

Suppose firstly $pn \leq k$. Then $(ab)_n = \sum_{[i,j]=n}(i,j)a_i b_j$ and for any i, j with $[i,j] = n$, we have $pi \leq pn \leq k$ and $pj \leq pn \leq k$. If any one of i or j is not a multiple of p, then $a_i b_j = 0$.

Suppose that both i and j are multiples of p. Then $p|(i,j)$ and so $(i,j)a_ib_j = 0$.

Suppose $p|n$, say $n = pu$ for some $u \geq 1$. Consider any i and j with $[i,j] = n = pu$. If both i and j are multiples of p, then $p|(i,j)$and $(i,j)a_ib_j = 0$. Suppose thus one of them, say j, is not a multiple of p. Then $(p,j) = 1$.

Since $j|pn$ and $(j,p) = 1$, we have $j|u$. Thus $jp \leq up = n \leq k$ and $p \nmid j$ which makes b_j and also $(i,j)a_ib_j$ zero. We conclude that also for this case $(ab)_n = 0$. Thus $W^2 \subseteq I$ as claimed. ◊

Corollary 34. *Let α be a subidempotent radical and let k be finite. If A is a ring with prime characteristic p, then*

$$\alpha(N_k(A)) = (V_1, V_2, V_3, \ldots, V_k) \qquad \textit{where}$$

$$V_n = \begin{cases} 0 & \textit{if } pn \leq k \textit{ or } p|n \\ \alpha(A) & \textit{otherwise.} \end{cases}$$

Proof. If $\alpha(A) = 0$, the result follows from Prop. 26. Suppose thus $\alpha(A) \neq 0$. Then, again from Prop. 26, we know $\alpha(N_k(A)) \subseteq N_k(\alpha(A))$. Thus $\alpha(N_k(A)) \subseteq \alpha(N_k(\alpha(A)))$. On the other hand, since $N_k(\alpha(A)) \triangleleft \triangleleft N_k(A)$, we have $\alpha(N_k(\alpha(A))) \subseteq \alpha(N_k(A))$. Thus $\alpha(N_k(A)) = \alpha(N_k(\alpha(A)))$ and the result follows from the previous proposition since the characteristic of A is prime. ◊

Example 35. Let α be a subidempotent radical for which $\mathbb{Z}_p \in \alpha$ for all primes p ($\mathbb{Z}_p$ is the ring of integers mod p). From Props. 30 and 33 we have, for example:

$$\alpha(N_\omega(\mathbb{Z}_p)) = 0 \text{ for all } p$$

$$\alpha(N_k(\mathbb{Z}_p)) = \begin{cases} N_k(\mathbb{Z}_p) & \text{if } k < p \\ (0, \mathbb{Z}_p, \mathbb{Z}_p, \ldots, \mathbb{Z}_p, 0) & \text{if } k = p, \end{cases}$$

$$\alpha(N_4(\mathbb{Z}_2)) = (0, 0, \mathbb{Z}_2, 0),$$

$$\alpha(N_7(\mathbb{Z}_2)) = (0, 0, 0, 0, \mathbb{Z}_2, 0, \mathbb{Z}_2) \text{ and}$$

$$\alpha(N_7(\mathbb{Z}_5)) = (0, \mathbb{Z}_5, \mathbb{Z}_5, \mathbb{Z}_5, 0, \mathbb{Z}_5, \mathbb{Z}_5).$$

Using the fact that the necklace ring construction is well-behaved with respect to the formation of direct sums (cf. the beginning of Section 2), as is the radical of finite direct sums, we may exploit the above result further. For example, if $A = \mathbb{Z}_6 = \mathbb{Z}_2 \oplus \mathbb{Z}_3$, then $2A \cong \mathbb{Z}_3$ and $3A \cong \mathbb{Z}_2$. Thus

$$\begin{aligned}\alpha(N_8(A)) &= \alpha(N_8(\mathbb{Z}_2)) \oplus \alpha(N_8(\mathbb{Z}_3))\\ &= (0,0,0,0,\mathbb{Z}_2,0,\mathbb{Z}_2,0) \oplus (0,0,0,\mathbb{Z}_3,\mathbb{Z}_3,0,\mathbb{Z}_3,\mathbb{Z}_3)\\ &= (0,0,0,\mathbb{Z}_3,\mathbb{Z}_2\oplus\mathbb{Z}_3,0,\mathbb{Z}_2\oplus\mathbb{Z}_3,\mathbb{Z}_3)\\ &= (0,0,0,2A,A,0,A,2A)\end{aligned}$$

and likewise

$$\alpha(N_3(A)) = (0,2A,3A). \quad \diamond$$

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