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APPROXIMATION BY TIME DISCRETIZATION OF SPECIAL STOCHASTIC EVOLUTION EQUATIONS

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Abstract: An approximation method by time discretization is investigated for an abstract stochastic evolution equation, involving Lipschitz continuous functions and a nonlinear maximal monotone operator, which satisfies a growth condition. A sequence of step functions is constructed and proved that it converges strongly to the solution of the evolution equation.

1. Introduction

Functional analytical formulations with generalized solution concepts are very useful in the investigation of stochastic partial differential equations. Important results were obtained by I. Gyöngy [2], [3], [4], W. Grecksch, C. Tudor [1], N. V. Krylov, B. L. Rozovskij [7], E. Pardoux [9] and illustrate different methods in the research of stochastic

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partial differential equations; for example, one can consider infinitesimal operators of a semigroup or monotone operators on Sobolev spaces to obtain assertions about the existence and uniqueness of the solutions of such equations.

Methods such as Galerkin's method or Rothe's method, inspired from the research of deterministic partial differential equations (see, for instance, [12], [10]) have been extended to the stochastic case. The Galerkin method consists in approximating the solution of the equation by a sequence of solutions of finite dimensional equations. Rothe's method involves the approximation of the solution by a sequence of solutions of time discretized equations. In the book [1] W. Grecksch and C. Tudor present the Rothe method for an abstract stochastic parabolic evolution equation. The "classic" linearity and continuity condition of the operator under consideration is replaced by monotonicity, hemicontinuity and growth condition. We also mention the paper of I. Gyöngy, D. Nualart [5], where the authors give an implicit approximation scheme similar to the Rothe method. In [8] H. Liske and E. Platen give numerical results for several time discrete approximation methods. Numerical methods for approximation of stochastic differential equations are subject of the book of P. Kloeden and E. Platen [6].

The purpose of our paper is to present Rothe's method (time discretization) for an abstract stochastic parabolic evolution equation (in the sense of Ito), where the operator is maximal monotone and satisfies an abstract Garding inequality (weak coerciveness) and a growth condition (local boundedness). We establish the existence and uniqueness (with probability 1) of the solution of the following abstract stochastic evolution equation:

$$dX(\omega,t) = -\mathcal{A}X(\omega,t)dt + f(t,X(\omega,t))dt + g(t,X(\omega,t))dw(\omega,t) \tag{1}$$

with the initial condition

$$X(\omega,0) = X_0(\omega), \tag{2}$$

where $\omega \in \Omega$ (a.e.) and $t \in [0,T] \subset \mathbb{R}$. We also prove that the sequence of the solutions of the discretized equations converge in mean square to the solution of the given equation. We want to point out that our assumptions on the operator $\mathcal{A}$ are more general than in the papers [9], [1], [11], [7] (no linearity or continuity or hemicontinuity condition, but a maximal monotonicity condition). Our assumptions will be illustrated in the next section.

Notations

$(\Omega, \mathcal{F}, P)$	complete probability space
EX	mathematical expectation of the random variable X
$\mathcal{B}_{[0,T]}$	the σ-algebra of all Borel sets of the interval $[0,T] \subset \mathbb{R}$
$\mathcal{L}^2_S(\Omega)$	space of all $\mathcal{F}$-measurable random variables $v : \Omega \times [0,T] \to S$ with $E\|v\|^2_S < \infty$
$\mathcal{L}^2_S(\Omega \times [0,T])$	space of all $\mathcal{F} \times \mathcal{B}_{[0,T]}$-measurable processes $v : \Omega \times [0,T] \to S$ that are adapted to $(\mathcal{F}_t)_{t\in[0,T]}$ and $E\int_0^T \|v(t)\|^2_S dt < \infty$
$\rightharpoonup$	weak convergence (in the sense of functional analysis)

2. Assumptions

We assume that the following hypotheses are fulfilled:

(H$_1$) $(\Omega, \mathcal{F}, P)$ is a complete probability space, $\{\mathcal{F}_t \mid t \in [0,T]\}$ is a filtration (contained in $\mathcal{F}$) with respect to a given real Wiener process $\Big(w(t)\Big)_{t\in[0,T]}$;

(H$_2$) (V, H, V^*) is an evolution triple (see [12], p. 416), where $(V, \|\cdot\|_V)$ is a real separable, reflexive Banach space and $(H, (\cdot,\cdot))$ is a real separable Hilbert space;

(H$_3$) $\mathcal{A} : V \to V^*$ is a maximal monotone operator (i.e. $\mathcal{A}$ is monotone and has no proper extension) satisfying the following conditions:

1) There exists two constants $a_1 > 0$, $a_2 \in \mathbb{R}$ such that for each $v \in V$ it holds:

$$\langle \mathcal{A}v, v\rangle \geq a_1\|v\|^2_V - a_2\|v\|^2$$

(abstract Garding inequality);

2) For each $r > 0$ there exists a $c_r > 0$ such that for any function $v : \Omega \times [0,T] \to V$ with $E\int_0^T\|v(t)\|^2_V dt < r$ it follows that $E\int_0^T\|\mathcal{A}v(t)\|^2_{V^*} dt < c_r$ (growth condition);

(H$_4$) $f, g : [0,T] \times H \to H$ are mappings satisfying the following inequalities for all $t_1, t_2 \in [0,T]$ and for all $v_1, v_2 \in H$

$$\|f(t_1, v_1) - f(t_2, v_2)\|^2 \leq \alpha |t_1 - t_2|^2 + \beta \|v_1 - v_2\|^2,$$

$$\|g(t_1, v_1) - g(t_2, v_2)\|^2 \leq \alpha |t_1 - t_2|^2 + \beta \|v_1 - v_2\|^2,$$

where $\alpha, \beta > 0$ are given constants.

($\mathbf{H}_5$) $X_0 : \Omega \to H$ is $\mathcal{F}_0$-measurable and $E\|X_0\|^2 < \infty$.

An adapted process $\Big(X(t)\Big)_{t\in[0,T]}$ from the space $\mathcal{L}^2_V(\Omega \times [0,T])$ with $E\|X(t)\|^2 < \infty$ for all $t \in [0,T]$ is called **solution of problem ($\mathbf{P}_1$)** if it satisfies equation (1) and the initial condition (2) in the following sense:

$$(X(t) - X_0, v) =$$

$$= -\int_0^t \langle \mathcal{A}X(s), v\rangle ds + \int_0^t (f(s, X(s)), v) ds + \int_0^t (g(s, X(s)), v) dw(s)$$

for a.e. $\omega \in \Omega$ and for all $v \in V, t \in [0,T]$.

Remark 1.1. In every reflexive Banach space S, an equivalent norm can be introduced so that S and S^* are locally uniformly convex and thus also strictly convex with respect to the new norms on S and S^* ([12] Prop. 32.23, p. 862). Without any loss of generality we can consider in our hypothesis that the reflexive Banach space V is also locally uniformly convex. In [1] the space V was assumed to be a Hilbert space. In our paper the space V must not be a Hilbert space.

Remark 1.2. An classical example of an oparator that satisfies ($\mathbf{H}_3$) is the Laplace operator. Another example is the following (see [12], p. 427): let $V = \dot{W}^m_2(G)$, $H = \mathcal{L}^2(G)$, where G is a bounded region in $\mathbb{R}^n$ (with $n \geq 1$) and $\mathcal{A} : V \to V^*$

$$\langle \mathcal{A}u, v\rangle = \int_G \Big(\sum_{i,j=1}^n D_i u D_j v + \sum_{i=1}^n u D_i v + uv \Big) dx.$$

Remark 1.3. A slightly more general assumption is ($\mathbf{H}^*{}_4$) instead of ($\mathbf{H}_4$):

($\mathbf{H}^*{}_4$) $f, g : \Omega \times [0,T] \times H \to H$ are mappings such that $f(\cdot, h), g(\cdot, h)$ are predictable in (t, ω) for each $h \in H$ and they satisfy the following inequalities for all $t_1, t_2 \in [0,T]$ and for all $v_1, v_2 \in H$

$$\|f(\omega, t_1, v_1) - f(\omega, t_2, v_2)\|^2 \leq \alpha |t_1 - t_2|^2 + \beta \|v_1 - v_2\|^2,$$

$$\|g(\omega, t_1, v_1) - g(\omega, t_2, v_2)\|^2 \leq \alpha |t_1 - t_2|^2 + \beta \|v_1 - v_2\|^2,$$

where $\alpha, \beta > 0$ are given constants. All the results of this paper hold also in this case.

Remark 1.4. The abstract Garding inequality assumed for $\mathcal{A}$ and the assumptions on f, g are closely related to the coerciveness condition on the oparator considered in the papers [1], [3], [7], [11].

3. Main Result

Let a be a real positive number and consider the following stochastic evolution equation

$$(3)\quad d\Big(e^{-at}X(t)\Big) = -e^{-at}\Big(\mathcal{A}X(t) + aX(t)\Big)dt + \\ + e^{-at}f(t, X(t))dt + e^{-at}g(t, X(t))dw(t)$$

with $\omega \in \Omega$ and $t \in [0, T]$, and the initial condition (2).

We denote by $(\mathbf{P_2})$ the problem of finding in the space $\mathcal{L}^2_V(\Omega \times \times [0, T])$ an adapted process $\Big(X(t)\Big)_{t \in [0,T]}$ that satisfies equation (3) and the initial condition (2). Using the Ito formula, it can be shown that X is a solution of $(\mathbf{P_1})$ if and only if X is a solution of $(\mathbf{P_2})$. The advantage of problem $(\mathbf{P_2})$ is the possibility of a favorable choice of the constant a (see relation (31)), that will give us some useful properties. Let $0 = t_0 < t_1 < \cdots < t_N = T$ be an equidistant partition of $[0, T]$ with $h_N = t_n - t_{n-1} = \frac{T}{N}$, $n \in \{1, 2, \ldots, N\}$ and set $w_n = w(t_n)$, for each $n \in \{0, 1, \ldots, N\}$. We **discretize** problem $(\mathbf{P_2})$ in the following way:

$$(4)\quad e^{-at_n}x_n + e^{-at_n}(\mathcal{A}x_n + ax_n)h_N = \\ = e^{-at_{n-1}}y_{n-1} + e^{-at_{n-1}}f(t_{n-1}, x_{n-1})h_N, \; n = 1, 2, \ldots, N,$$

$$(5)\quad y_n = x_n + g(t_{n-1}, x_n)(w_n - w_{n-1}), \; n = 1, 2, \ldots, N,$$

$$(6)\quad x_0 = y_0 = X_0,$$

where $x_n \in V$ and $y_n \in H$ $(n = 1, 2, \ldots, N)$.

Lemma 3.1. *For each $n \in \{1, 2, \ldots, N\}$ equation (4) has with probability 1 a unique $\mathcal{F}_{t_{n-1}}$-measurable V-valued solution $x_n \in \mathcal{L}^2_V(\Omega)$.*

In order to prove Lemma 3.1 one can apply a corollary of the theorem by Browder and Minty about monotone operators (see [12], Cor. 32.26, p. 868) and the method of induction. For a.e. $\omega \in \Omega$ we introduce the step process

$$\hat{x}_N(t) = \sum_{n=1}^{N} I_{[t_{n-1},t_n[}(t) x_n,$$

for all $t \in [0,T]$ and set $\hat{x}_N(T) = x_N$ for $t = T$. The main result of this paper is the following theorem:

Theorem 3.2. *There exists a unique (with probability* 1*) solution* $X = \left(X(t)\right)_{t\in[0,T]}$ *of problem* ($\mathbf{P_1}$) *such that*

(i) *the process* $\left(X(t)\right)_{t\in[0,T]}$ *has in* H *continuous trajectories;*

(ii) *the sequence* $(\hat{x}_N)$ *converges strongly to* X *in the space* $\mathcal{L}^2_V(\Omega \times [0,T])$*;*

(iii) $(\hat{x}_N(T))$ *converges strongly to* $X(T)$ *in the space* l.

4. Proof of the Main result

To begin we introduce some more notation. For each $n \in \{1, 2, \ldots, N\}$ we denote

$$v_n = e^{-at_n} x_n \quad \text{and} \quad u_n = e^{-at_n} y_n,$$

and for all $t \in [0,T]$ we set

$$I_n(t) = I_{[t_{n-1},t_n[}(t), \quad n \neq N$$

$$I_N(t) = I_{[t_{N-1},t_N]}(t), \quad n = N,$$

and consider the following step processes

$$\hat{v}_N(t) = \sum_{n=1}^{N} I_n(t) v_n, \qquad \hat{u}_N(t) = \sum_{n=1}^{N} I_n(t) u_n. \tag{7}$$

Other notations are the following:

$$F_N(t) = \sum_{n=1}^{N} I_n(t) e^{-at_{n-1}} f(t_{n-1}, x_{n-1}),$$

$$G_N(t) = \sum_{n=1}^{N} I_n(t) e^{-at_n} g(t_{n-1}, x_n),$$

$$\mathcal{B}_N(t) = \sum_{n=1}^{N} I_n(t) e^{-at_n} \mathcal{A} x_n.$$

The following lemma contains some a-priori estimates, that will be used to prove weak convergence properties in Lemma 4.2 and Lemma 4.3.

Lemma 4.1. *There exist two constants $c_1, c_2 > 0$ (that do not depend on N) such that*

$$E\|v_n\|^2 \le c_1, \tag{8}$$

$$E\|u_n\|^2 \le c_1 \tag{9}$$

for all $n \in \{1, 2, \ldots, N\}$ and

$$E\int_0^T \|\hat{v}_N(t)\|_V^2 dt \le c_2, \tag{10}$$

$$E\int_0^T \|\hat{x}_N(t)\|_V^2 dt \le c_2 \tag{11}$$

for all natural numbers N.

Proof. Let $n \in \{1, 2, \ldots, N\}$. Using (4) we obtain

$$(1 + ah_N)\|v_n\|^2 + \langle e^{-at_n} \mathcal{A} x_n, v_n \rangle h_N = \\ = (u_{n-1}, v_n) + (e^{-at_{n-1}} f(t_{n-1}, x_{n-1}), v_n) h_N$$

for a.e. $\omega \in \Omega$. Using hypothesis $(\mathbf{H}_3)$, we then get

$$\langle e^{-at_n} \mathcal{A} x_n, v_n \rangle + a\|v_n\|^2 \ \ge a_1 \|v_n\|_V^2 + (a - a_2)\|v_n\|^2,$$

(we choose $a > a_2$), so (12) becomes

$$\|v_n\|^2 + a_1\|v_n\|_V^2 h_N \le (u_{n-1}, v_n) + e^{-at_{n-1}} (f(t_{n-1}, x_{n-1}), v_n) h_N.$$

Then by the Schwarz inequality and elementary calculus we obtain

$$\frac{1}{2}\|v_n\|^2 + a_1\|v_n\|_V^2 h_N \le \\ \le \frac{1}{2}\|u_{n-1}\|^2 + \frac{1}{2} e^{-at_{n-1}} \|f(t_{n-1}, x_{n-1})\|^2 h_N + \frac{1}{2}\|v_n\|^2 h_N. \tag{12}$$

In addition, from (5) we get

$$E\|y_{n-1}\|^2 = E\|x_{n-1}\|^2 + E\|g(t_{n-2}, x_{n-1})(w_{n-1} - w_{n-2})\|^2 + \\ +2E\Big(\big(g(t_{n-2}, x_{n-1}), x_{n-1}\big)(w_{n-1} - w_{n-2})\Big)$$

for $n \geq 2$. Taking into consideration the properties of the conditional expectation and of a Wiener process, on the one hand, and the $\mathcal{F}_{t_{n-2}}$-measurability of x_{n-1}, on the other hand, we obtain

$$E\|u_{n-1}\|^2 = E\|v_{n-1}\|^2 + h_N e^{-2at_{n-1}} E\|g(t_{n-2}, x_{n-1})\|^2. \tag{13}$$

Since f and g are Lipschitz continuous, one can show that

$$e^{-2at_{n-1}} E\|f(t_{n-1}, x_{n-1})\|^2 \leq k_1(1 + E\|v_{n-1}\|^2) \tag{14}$$

$$e^{-2at_{n-1}} E\|g(t_{n-2}, x_{n-1})\|^2 \leq k_1(1 + E\|v_{n-1}\|^2), \tag{15}$$

where k_1 is a positive constant that does not depend on N.

From (12), (13), (14) and (15) we obtain

$$\frac{1}{2}E\|v_n\|^2 + a_1 h_n E\|v_n\|_V^2 \leq \\ \leq \frac{1}{2}E\|v_{n-1}\|^2 + k_1 h_N + k_1 h_N E\|v_{n-1}\|^2 + \frac{h_N}{2}E\|v_n\|^2.$$

Summing from $n = 1$ to $n = p$, where $p \in \{1, 2, \ldots, N\}$, we thus have

$$\frac{1}{2}E\|v_p\|^2 + a_1 h_N \sum_{n=1}^{p} E\|v_n\|_V^2 \leq \\ \leq \left(\frac{1}{2} + k_1 h_N\right) E\|v_0\|^2 + k_1 T + \left(\frac{1}{2} + k_1\right) h_N \sum_{n=1}^{p} E\|v_n\|^2. \tag{16}$$

Hence

$$\frac{1}{2}E\|v_p\|^2 \leq \left(\frac{1}{2} + k_1 T\right) E\|X_0\|^2 + k_1 T + \left(\frac{1}{2} + k_1\right) h_N \sum_{n=1}^{p} E\|v_n\|^2.$$

Applying Gronwall's Lemma we obtain

$$E\|v_p\|^2 \leq \left(\left(\frac{1}{2} + k_1 T\right) E\|X_0\|^2 + k_1 T\right) e^{k_2 T}, \text{ for all } p \in \{1, \ldots, N\},$$

where $k_2 := \frac{1}{2} + k_1$. These inequalities, (13) and (15) imply the existence of a positive constant c_1 (that does not depend on N) such that (8) and (9) hold.

From (16) (for $p = N$) and (8) it follows that

$$a_1 h_n \sum_{n=1}^{N} E\|v_n\|_V^2 \leq \left(\frac{1}{2} + k_1 T\right) E\|X_0\|^2 + k_1 T + c_1 \left(\frac{1}{2} + k_1\right) T.$$

Hence there exists a positive constant k_3 (which does not depend on N) such that

$$E\int_0^T \|\hat{v}_N(t)\|_V^2 dt \leq k_3,$$

for all natural numbers N and so by the definition of $\hat{v}_N$ we have

$$E\int_0^T \|\hat{x}_N(t)\|_V^2 dt \leq e^{2aT} E\int_0^T \|\hat{v}_N(t)\|_V^2 dt \leq e^{2aT} k_3.$$

Hence, (10) and (11) are proved. ◊

Lemma 4.2. *There exist $\hat{v} \in \mathcal{L}_V^2(\Omega \times [0,T])$, $\hat{u} \in l$ and a subsequence (N') of (N) such that*

$$\hat{v}_{N'} \rightharpoonup \hat{v}, \qquad \hat{x}_{N'} \rightharpoonup \hat{x} \quad in \quad \mathcal{L}_V^2(\Omega \times [0,T]),$$

where $\hat{x}(t) = e^{at}\hat{v}(t)$ for a.e. $(\omega, t) \in \Omega \times [0,T]$, as soon as

$$\hat{u}_{N'} \rightharpoonup \hat{v} \quad in \quad \mathcal{L}_H^2(\Omega \times [0,T]),$$

$$\hat{u}_{N'}(T) \rightharpoonup \hat{u}, \qquad \hat{v}_{N'}(T) \rightharpoonup \hat{u} \quad in \quad \mathcal{L}_H^2(\Omega).$$

Proof. By Lemma 4.1 and by Prop. 5.1 from the Appendix there exist $\hat{v} \in \mathcal{L}_V^2(\Omega \times [0,T])$ and a subsequence (N') of (N) such that $\hat{v}_{N'} \rightharpoonup \hat{v}$ in $\mathcal{L}_V^2(\Omega \times [0,T])$. As for (13) it can be shown that

(17)
$$E\|u_n - v_n\|^2 = h_N e^{-2at_n} E\|g(t_{n-1}, x_n)\|^2, \quad \text{for all} \quad n \in \{1, 2, \ldots, N\},$$

from which with (15) we get

$$E\int_0^T \|\hat{u}_N(t) - \hat{v}_N(t)\|^2 dt \leq k_1 h_N^2 \sum_{n=1}^{N} (1 + E\|v_n\|^2).$$

Together with (8) this inequality yields

$$E\int_0^T \|\hat{u}_N(t) - \hat{v}_N(t)\|^2 dt \to 0,$$

so $\hat{u}_{N'} \rightharpoonup \hat{v}$ in the space $\mathcal{L}_H^2(\Omega \times [0,T])$. Note that the inequality

$$0 \le \left(1 - e^{-\delta s}\right)^2 \le \delta^2 s^2. \tag{18}$$

holds for all $s, \delta \ge 0$. Using this inequality we can write

$$E\int_0^T \|\hat{x}_N(t) - e^{at}\hat{v}_N(t)\|_V^2 dt \le a^2 e^{2aT} h_N^2 E\int_0^T \|\hat{v}_N(t)\|_V^2 dt,$$

so in view of (10) we have

$$E\int_0^T \|\hat{x}_N(t) - e^{at}\hat{v}_N(t)\|_V^2 dt \to 0. \tag{19}$$

Hence $\hat{x}_{N'} \rightharpoonup \hat{x}$. In view of (9) we have

$$E\|\hat{u}_N(T)\|^2 = E\|u_N\|^2 \le c_1$$

for each natural number N. By Prop. 5.1 there exist $\hat{u} \in l$ and a subsequence (N'') of (N') such that $(\hat{u}_{N''}(T))$ converges weakly to $\hat{u}$ in the space l. From (17) it follows that $E\|\hat{u}_N(T) - \hat{v}_N(T)\|^2 \to 0$. As a consequence we have

$$\hat{v}_{N''}(T) \rightharpoonup \hat{u} \qquad \text{in} \quad \mathcal{L}_H^2(\Omega).$$

For simplicity we will denote the subsequence of indices (N'') obtained above also by (N'). $\Diamond$

Lemma 4.3. *There exist $\mathcal{B} \in \mathcal{L}_{V^*}^2(\Omega \times [0,T])$, $F, G \in \mathcal{L}_H^2(\Omega \times [0,T])$ and a subsequence (N'') of (N') such that*

(i) $\mathcal{B}_{N''}(t) \rightharpoonup \mathcal{B}(t)$, $\quad e^{-at}\mathcal{A}\hat{x}_{N''}(t) \rightharpoonup \mathcal{B}(t)$ *in* $\mathcal{L}_{V^*}^2(\Omega \times [0,T])$;
(ii) $F_{N''}(t) \rightharpoonup F(t)$, $\quad e^{-at}f(t, \hat{x}_{N''}(t)) \rightharpoonup F(t)$ *in* $\mathcal{L}_H^2(\Omega \times [0,T])$;
(iii) $G_{N''}(t) \rightharpoonup G(t)$, $\quad e^{-at}g(t, \hat{x}_{N''}(t)) \rightharpoonup G(t)$ *in* $\mathcal{L}_H^2(\Omega \times [0,T])$.

Proof. We can write

$$E\int_0^T \|\mathcal{B}_N(t)\|_*^2 dt \le E\int_0^T \|\mathcal{A}\hat{x}_N(t)\|_*^2 dt.$$

From hypothesis $(\mathbf{H}_3)$ and from (11) it follows that $(\mathcal{B}_N)$ is a bounded sequence in the space $\mathcal{L}_{V^*}^2(\Omega \times [0,T])$. Since V is continuously embedded in H, there exists a positive constant c such that

$$\|v\| \le c\|v\|_V \quad \text{for all} \quad v \in V. \tag{20}$$

By using (14), (10) and (20) we obtain

$$E\int_0^T \|F_N(t)\|^2 dt \le k_1(T + E\|X_0\|^2) + ck_1 E\int_0^T \|\hat{v}_N(t)\|_V^2 dt \le k_4,$$

where k_4 is a positive constant that does not depend on N. By using (15) and (10) we see that

$$E\int_0^T \|G_N(t)\|^2 dt \le k_1 T + k_1 E\int_0^T \|\hat{v}_N(t)\|^2 dt \le k_1 T + cc_2 k_1.$$

In view of Prop. 5.1 there exist $\mathcal{B} \in \mathcal{L}^2_{V^*}(\Omega \times [0,T])$, $F, G \in \mathcal{L}^2_H(\Omega \times \times [0,T])$ and a subsequence (N'') of (N') such that $(\mathcal{B}_{N''})$ converges weakly to $\mathcal{B}$ in $\mathcal{L}^2_{V^*}(\Omega\times[0,T])$, $(F_{N''})$ converges weakly to F and $(G_{N''})$ converges weakly to G in $\mathcal{L}^2_H(\Omega \times [0,T])$. We also have

$$E\int_0^T \|\mathcal{B}_N(t) - e^{-at}\mathcal{A}\hat{x}_N(t)\|_*^2 dt \le a^2 e^{2aT} h_N^2 E\int_0^T \|\mathcal{A}\hat{x}_N(t)\|_*^2 dt,$$

so from hypothesis $(\mathbf{H}_3)$ and from (11) we obtain

$$E\int_0^T \|\mathcal{B}_N(s) - e^{-at}\mathcal{A}\hat{x}_N(t)\|_*^2 dt \to 0.$$

Thus

$$e^{-at}\mathcal{A}\hat{x}_{N''}(t) \rightharpoonup \mathcal{B}(t) \quad \text{in} \quad \mathcal{L}^2_{V^*}(\Omega \times [0,T]).$$

From (4) and (5) we obtain

$$\begin{aligned} x_n - x_{n-1} =& (1 - e^{-ah_N} - ah_N e^{-ah_N})x_n - h_N e^{-ah_N}\mathcal{A}x_n + \\ &+ h_N f(t_{n-1}, x_{n-1}) + g(t_{n-2}, x_{n-1})(w_{n-1} - w_{n-2}) \end{aligned}$$

for all $n \in \{2, 3, \dots N\}$. Taking into account Lemma 4.1 and (8),(14), (15) and (18), it then follows that

$$E\|x_n - x_{n-1}\|^2 \le h_N k_5 + h_N\|\mathcal{A}x_n\|_*^2, \qquad \text{for all} \quad n \in \{1, 2, \dots, N\}$$

(k_5 is a positive constant that does not depend on N). Now by using (14), (8), (18) and $(\mathbf{H}_4)$ we obtain

$$E\int_0^T \|F_N(t) - e^{-at}f(t, \hat{x}_N(t)\|^2 \le h_N k_6 + 2h_N\beta E\int_0^T \|\mathcal{A}\hat{x}_N(t)\|_*^2 dt$$

(k_6 is a positive constant that does not depend on N). Taking into consideration (11) and hypothesis $(\mathbf{H}_3)$ we conclude that

$$E\int_0^T \|F_N(t) - e^{-at}f(t,\hat{x}_N(t)\|^2 dt \to 0$$

and hence

$$e^{-at}f(t,\hat{x}_{N''}(t)) \rightharpoonup F(t).$$

Taking into consideration (15), (10), (18) and $(\mathbf{H}_4)$, we have

$$E\int_0^T \|G_N(t) - e^{-at}g(t,\hat{x}_N(t)\|^2 dt \le 2a^2 h_N^2 k_1 E\int_0^T \|\hat{v}_N(t)\|^2 dt + h_N^2 k_7$$

for each N (k_7 is a positive constant that does not depend on N). So we get

$$E\int_0^T \|G_N(t) - e^{-at}g(t,\hat{x}_N(t)\|^2 dt \to 0$$

and hence

$$e^{-at}g(t,\hat{x}_{N''}(t)) \rightharpoonup G(t). \quad \diamond$$

The results from Th. 3.2 are contained in the following theorem, that gives us the proof of the **main result** of the paper.

Theorem 4.4. *There exists a $\mathcal{F}_t$-measurable process $\Big(X(t)\Big)_{t\in[0,T]}$ with*

$$X = \hat{x} \qquad in \qquad \mathcal{L}_V^2(\Omega\times[0,T]),$$

such that the following assertions are true:

(i) *For all $t\in[0,T], v\in V$ and a.e. $\omega\in\Omega$ it holds*

$$(21)\quad (e^{-at}X(t),v) = (X_0,v) - \int_0^t \langle\mathcal{B}(s),v\rangle ds - a\int_0^t (\hat{v}(s),v)ds +$$
$$+\int_0^t (F(s),v)ds + \int_0^t (G(s),v)dw(s)$$

and $\Big(X(t)\Big)_{t\in[0,T]}$ has in H continuous trajectories.

(ii) *There exists a subsequence of $(\hat{x}_N)$ that converges strongly to X in $\mathcal{L}_V^2(\Omega\times[0,T])$.*

(iii) *There exists a subsequence of $(\hat{x}_N(T))$ that converges strongly to $X(T)$ in $\mathcal{L}_H^2(\Omega)$.*

(iv) *The process X is with probability 1 the unique solution of problem ($\mathbf{P}_1$).*

(v) *The whole sequence $(\hat{x}_N)$ converges strongly to X in $\mathcal{L}^2_V(\Omega \times \times [0,T])$ and the whole sequence $(\hat{x}_N(T))$ converges strongly to $X(T)$ in $\mathcal{L}^2_H(\Omega)$.*

Proof. (i) We will now denote the subsequence of indices (N'') obtained from the previous lemmas also by (N). From (4) and (5) we conclude that

$$u_n + e^{-at_n}(\mathcal{A}x_n + ax_n)h_N = u_{n-1} + e^{-at_{n-1}} f(t_{n-1}, x_{n-1})h_N + \\ + e^{-at_n} g(t_{n-1}, x_n)(w_n - w_{n-1})$$

for each $n \in \{1, 2, \dots, N\}$. Let $t \in [0, T[$ be arbitrarily chosen. Then there exists a $q \in \{1, 2, \dots, N\}$ such that $t \in [t_{q-1}, t_q[$. Summing up from $n = 1$ to $n = q$ we get

$$\hat{u}_N(t_q) + \int_0^{t_q} \mathcal{B}_N(s)ds + a\int_0^{t_q} \hat{v}_N(s)ds = X_0 + \int_0^{t_q} F_N(s)ds + \int_0^{t_q} G_N(s)dw(s).$$

Using notations (7) this equation can be rewritten as

$$\hat{u}_N(t) = X_0 - \int_0^t \mathcal{B}_N(s)ds - a\int_0^t \hat{v}_N(s)ds + \int_0^t F_N(s)ds + \\ + \int_0^t G_N(s)dw(s) + r^1_N(t) + r^2_N(t) + r^3_N(t) + r^4_N(t), \tag{22}$$

where $r^1_N : \Omega \times [0,T] \to V^*$, $r^2_N, r^3_N, r^4_N : \Omega \times [0,T] \to H$ are defined by

$$r^1_N(t) = -\sum_{n=1}^N I_n(t) \int_t^{t_n} \mathcal{B}_N(s)ds, \qquad r^2_N(t) = \sum_{n=1}^N I_n(t) \int_t^{t_n} F_N(s)ds,$$

$$r^3_N(t) = \sum_{n=1}^N I_n(t) \int_t^{t_n} G_N(s)dw(s), \qquad r^4_N(t) = \sum_{n=1}^N I_n(t) \int_t^{t_n} \hat{v}_N(s)ds.$$

We note that

$$E\int_0^T \|r^1_N(t)\|^2_* dt \le h_N T E\int_0^T \|\mathcal{A}\hat{x}_N(t)\|^2_* dt,$$

$$E\int_0^T \|r_N^2(t)\|^2 dt \leq h_N T E\int_0^T \|F_N(t)\|^2 dt,$$

$$E\int_0^T \|r_N^3(t)\|^2 dt \leq h_N T E\int_0^T \|G_N(t)\|^2 dt,$$

$$E\int_0^T \|r_N^4(t)\|^2 dt \leq h_N T E\int_0^T \|\hat{v}_N(t)\|^2 dt.$$

Therefore we have especially $r_N^1 \rightharpoonup 0$ in $\mathcal{L}^2_{V^*}(\Omega \times [0,T])$, as well as $r_N^2 \rightharpoonup 0$, $r_N^3 \rightharpoonup 0$ and $r_N^4 \rightharpoonup 0$ in $\mathcal{L}^2_H(\Omega \times [0,T])$.

We take into consideration the weak convergences in Lemma 4.2 and Lemma 4.3, use Cor. 5.4 and pass to the limit in (22) when $N \to \infty$ to obtain

$$e^{-at}\hat{x}(t) = X_0 - \int_0^t \mathcal{B}(s)ds - a\int_0^t \hat{v}(s)ds + \int_0^t F(s)ds + \int_0^t G(s)dw(s)$$

for a.e. $(\omega, t) \in \Omega \times [0,T]$.

There exists a $\{\mathcal{F}_t | t \in [0,T]\}$-measurable process $\Big(X(t)\Big)_{t\in[0,T]}$ such that

(23) $e^{-at}X(t) =$

$$= X_0 - \int_0^t \mathcal{B}(s)ds - a\int_0^t \hat{v}(s)ds + \int_0^t F(s)ds + \int_0^t G(s)dw(s)$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$, and for which

$\hat{x}(\omega,t) = X(\omega,t)$ for a.e. $(\omega,t) \in \Omega \times [0,T]$, in $\mathcal{L}^2_V(\Omega \times [0,T])$;

$\Big(X(t)\Big)_{t\in[0,T]}$ considered as a H-valued process is continuous (see [11], Th. 2, p. 73). But from Lemma 4.2 we have

$$\hat{v}(s) = e^{-as}\hat{x}(s) \quad \text{a.e. } (\omega,t) \in \Omega \times [0,T].$$

Therefore

$$\hat{v}(s) = e^{-as}\hat{X}(s) \quad \text{a.e. } (\omega,t) \in \Omega \times [0,T]. \tag{24}$$

(ii) Applying the Ito formula to (23) we get

$$(25)\quad E\|e^{-aT}X(T)\|^2 = E\|X_0\|^2 - 2E\int_0^T \langle \mathcal{B}(s), e^{-as}X(s)\rangle ds -$$

$$-2a\int_0^t (\hat{v}(s), e^{-as}X(s))ds + 2E\int_0^T (F(s), e^{-as}X(s))ds + E\int_0^T \|G(s)\|^2 dw(s).$$

Next we set $t = T$ in (22). Thus

$$\hat{u}_N(T) = X_0 - \int_0^T \mathcal{B}_N(s)ds - a\int_0^T \hat{v}_N(s)ds + \int_0^T F_N(s)ds + \int_0^T G_N(s)dw(s).$$

Using Lemma 4.2, Lemma 4.3 and Cor. 5.4 and passing to the limit when $N \to \infty$ we obtain

$$\hat{u} = X_0 - \int_0^T \mathcal{B}(s)ds - a\int_0^T \hat{v}(s)ds + \int_0^T F(s)ds + \int_0^T G(s)dw(s)$$

for a.e. $\omega \in \Omega$. Taking into consideration (23) we have

$$(26)\qquad \hat{u}(\omega) = e^{-aT}X(\omega, T) \qquad \text{for a.e.} \quad \omega \in \Omega.$$

In Lemma 4.2 we proved $\hat{v}_N(T) \rightharpoonup \hat{u}$ in $\mathcal{L}^2_H(\Omega)$. It follows from Prop. 5.2 and (26) that

$$(27)\qquad E\|e^{-aT}X(T)\|^2 \leq \liminf_{N\to\infty} E\|\hat{v}_N(T)\|^2.$$

Put

$$\delta = \limsup_{N\to\infty} E\|\hat{v}_N(T)\|^2 - E\|e^{-aT}X(T)\|^2.$$

From (25), (26) and (27) we obtain
(28)

$$\delta + E\|X_0\|^2 - 2E\int_0^T \langle \mathcal{B}(s), e^{-as}X(s)\rangle ds - 2aE\int_0^T (\hat{v}(s), e^{-as}X(s))ds +$$

$$+2E\int_0^T (F(s), e^{-as}X(s))ds + E\int_0^T \|G(s)\|^2 ds \leq \limsup_{N\to\infty} E\|\hat{v}_N(T)\|^2$$

and from (5), (12) and (13) we get

$$(1+2ah_N)E\|v_n\|^2+2E\langle e^{-at_n}\mathcal{A}x_n,v_n\rangle h_N\le E\|v_{n-1}\|^2+$$
$$+2E(e^{-at_{n-1}}f(t_{n-1},x_{n-1}),v_n)h_N+E\|e^{-at_{n-1}}g(t_{n-2},x_{n-1})\|^2h_N$$

for each $n\in\{2,3,\dots,N\}$, with

$$(1+2ah_N)E\|v_1\|^2+2E\langle e^{-at_1}\mathcal{A}x_1,v_1\rangle h_N\le$$
$$\le E\|X_0\|^2+2E(e^{-at_0}f(t_0,x_0),v_1)h_N$$

in the case $n=1$. Summing these relations from $n=1$ to $n=N$, we obtain

$$E\|\hat{v}_N(T)\|^2\le E\|X_0\|^2-2E\int_0^T\langle\mathcal{B}_N(s),\hat{v}_N(s)\rangle ds- \tag{29}$$
$$-2aE\int_0^T\|\hat{v}_N(s)\|^2ds+2E\int_0^T(F_N(s),\hat{v}_N(s))ds+E\int_0^T\|G_N(s)\|^2ds.$$

In the proofs of Lemma 4.2 and Lemma 4.3 we have seen that

$$E\int_0^T\|e^{-at}\hat{x}_N(t)-\hat{v}_N(t)\|^2dt\to 0,$$
$$E\int_0^T\|\mathcal{B}_N(t)-e^{-at}\mathcal{A}\hat{x}_N(t)\|_*^2dt\to 0,$$
$$E\int_0^T\|F_N(t)-e^{-at}f(t,\hat{x}_N(t))\|^2dt\to 0,$$
$$E\int_0^T\|G_N(t)-e^{-at}g(t,\hat{x}_N(t))\|^2dt\to 0$$

for $N\to\infty$. Consequently,

$$\lim_{N\to\infty}E\int_0^T\|\hat{v}_N(t)\|^2dt=\lim_{N\to\infty}E\int_0^T(e^{-at}\hat{x}_N(t),\hat{v}_N(t))dt,$$

$$\lim_{N\to\infty} E\int_0^T \langle \mathcal{B}_N(t), \hat{v}_N(t)\rangle dt = \lim_{N\to\infty} E\int_0^T \langle e^{-at}\mathcal{A}\hat{x}_N(t), \hat{v}_N(t)\rangle dt,$$

$$\lim_{N\to\infty} E\int_0^T (F_N(t), \hat{v}_N(t))dt = \lim_{N\to\infty} E\int_0^T \langle e^{-at} f(t, \hat{x}_N(t)), \hat{v}_N(t)\rangle dt,$$

$$\lim_{N\to\infty} E\int_0^T \|G_N(t)\|^2 dt = \lim_{N\to\infty} E\int_0^T \|e^{-at} g(t, \hat{x}_N(t))\|^2 dt$$

In these equalities we have omitted writing $\limsup_{N\to\infty}$, because all the above sequences are bounded and thus a subsequence can always be found that is convergent and for which we can use the same notation.

From (28) and (29) it follows that

$$\delta - 2E\int_0^T \langle \mathcal{B}(t), e^{-at}X(t)\rangle dt - 2aE\int_0^T (\hat{v}(t), e^{-at}X(t))dt+ \tag{30}$$

$$+2E\int_0^T (F(t), e^{-at}X(t))dt + E\int_0^T \|G(t)\|^2 dt \leq$$

$$\leq -2\lim_{N\to\infty} E\int_0^T \langle e^{-at}\mathcal{A}\hat{x}_N(t), \hat{v}_N(t)\rangle dt-$$

$$-2a\lim_{N\to\infty} E\int_0^T (e^{-at}\hat{x}_N(t), \hat{v}_N(t))dt+$$

$$+2\lim_{N\to\infty} E\int_0^T \langle e^{-at} f(t, \hat{x}_N(t)), \hat{v}_N(t)\rangle dt+$$

$$+\lim_{N\to\infty} E\int_0^T \|e^{-at} g(t, \hat{x}_N(t))\|^2 dt.$$

Let L be the right part of this inequality. Further, for each natural number N we put

$$L_N = -2E\int_0^T e^{-2at}\langle \mathcal{A}\hat{x}_N(t) - \mathcal{A}X(t), \hat{x}_N(t) - X(t)\rangle dt -$$

$$-2aE\int_0^T e^{-2at}\|\hat{x}_N(t) - X(t)\|^2 dt +$$

$$+2E\int_0^T e^{-2at}(f(t,\hat{x}_N(t)) - f(t,X(t)), \hat{x}_N(t) - X(t))dt +$$

$$+E\int_0^T e^{-2at}\|g(t,\hat{x}_N(t)) - g(t,X(t))\|^2 dt +$$

$$+a_1 E\int_0^T e^{-2at}\|\hat{x}_N(t) - X(t)\|_V^2 dt.$$

We claim that

(31) $\qquad L_N \leq 0 \quad$ for all natural numbers $\quad N.$

Indeed, from hyothesis $(\mathbf{H}_3)$ we have

$$-2E\int_0^T e^{-2at}\langle \mathcal{A}\hat{x}_N(t) - \mathcal{A}X(t), \hat{x}_N(t) - X(t)\rangle dt \leq$$

$$\leq -2a_1 E\int_0^T e^{-2at}\|\hat{x}_N(t) - X(t)\|_V^2 dt + 2a_2 E\int_0^T e^{-2at}\|\hat{x}_N(t) - X(t)\|^2 dt.$$

The Lipschitz continuity of f and g implies that

$$2E\int_0^T e^{-2at}(f(t,\hat{x}_N(t)) - f(t,X(t)), \hat{x}_N(t) - X(t))dt \leq$$

$$\leq (1+\beta)E\int_0^T e^{-2at}\|\hat{x}_N(t) - X(t)\|^2 dt,$$

$$E\int_0^T e^{-2at}\|g(t,\hat{x}_N(t))-g(t,X(t))\|^2dt \le \beta E\int_0^T e^{-2at}\|\hat{x}_N(t)-X(t)\|^2dt,$$

so

$$L_N \le (2\beta+1+2a_2-a_1c-2a)E\int_0^T e^{-2at}\|\hat{x}_N(t)-X(t)\|^2dt,$$

(where c is the positive constant that appears in (20)). For a convenient choice of a (i.e., such that $2\beta+1+2a_2-a_1c-2a<0$ and $a>a_2$), we obtain $L_N\le 0$. Next note that

$$\lim_{N\to\infty} L_N = L+2E\int_0^T\langle\mathcal{B}(t),e^{-at}X(t)\rangle dt - 2E\int_0^T(F(t),e^{-at}X(t))dt+$$

$$(32)\qquad +E\int_0^T(e^{-at}g(t,X(t))-2G(t),e^{-at}g(t,X(t)))dt+$$

$$+2aE\int_0^T(\hat{v}(t),e^{-at}X(t))dt+a_1\lim_{N\to\infty}E\int_0^T e^{-2at}\|\hat{x}_N(t)-X(t)\|_V^2dt.$$

From (30), (31) and (32) it follows

$$\delta+E\int_0^T\|G(t)-e^{-at}g(t,X(t)\|^2+$$

$$+a_1\lim_{N\to\infty}E\int_0^T e^{-2at}\|\hat{x}_N(t)-X(t)\|_V^2dt\le 0.$$

This inequality implies $\delta=0$,

$$(33)\qquad e^{-at}g(t,X(t))=G(t)\quad \text{for a.e.}\quad (\omega,t)\in\Omega\times[0,T],$$

and

$$(34)\qquad \lim_{N\to\infty}E\int_0^T e^{-2at}\|\hat{x}_N(t)-X(t)\|_V^2dt=0.$$

From (34) it follows that $\hat{x}_N\to X$ in $\mathcal{L}_V^2(\Omega\times[0,T])$. We mention that

$(\hat{x}_N)$ here denotes a subsequence of the original sequence $(\hat{x}_N)$.

(iii) Since $\delta = 0$, we have

$$E\|\hat{v}_N(T)\|^2 \to E\|e^{-aT}X(T)\|^2.$$

By using Lemma 4.2 together with relation (26) and Prop. 5.2, it follows that $\hat{v}_N(T) \to e^{-aT}X(T)$ in l. Therefore $\hat{x}_N(T) \to X(T)$; $(\hat{x}_N(T))$ denotes here a subsequence of the original sequence $(\hat{x}_N(T))$. (iv) Taking into consideration the conclusion (ii) of this theorem and the Lipschitz continuity of f, we have

$$e^{-at}f(t,\hat{x}_N(t)) \to e^{-at}f(t,X(t)) \qquad \text{in} \quad \mathcal{L}^2_H(\Omega\times[0,T]).$$

From Lemma 4.3 it follows that

$$e^{-at}f(t,X(t)) = F(t) \qquad \text{for} \quad \text{a.e.} \quad (\omega,t)\in\Omega\times[0,T]. \tag{35}$$

Next we show that

$$e^{-at}\mathcal{A}X(t) = \mathcal{B}(t) \qquad \text{for} \quad \text{a.e.} \quad (\omega,t)\in\Omega\times[0,T]. \tag{36}$$

From the monotonicity of $\mathcal{A}$ we have

$$E\int_0^T \langle e^{-at}(\mathcal{A}\hat{x}_N(t) - \mathcal{A}y(t)), \hat{x}_N(t) - y(t)\rangle dt \ge 0$$

for all $y \in \mathcal{L}^2_V(\Omega\times[0,T])$. Passing to the limit when $N\to\infty$, using Lemma 4.3 and conclusion (ii) of Th. 4.6, we get

$$E\int_0^T \langle \mathcal{B}(t) - e^{-at}\mathcal{A}y(t), X(t) - y(t)\rangle dt \ge 0 \tag{37}$$

for all $y \in \mathcal{L}^2_V(\Omega\times[0,T])$. Let $\mathcal{J}: V \to V^*$ be the duality map between V and V^*. From the properties of this map (see [12], Prop. 32, p. 861) we can write

$$E\int_0^T e^{-at}\langle \mathcal{J}X(t) - \mathcal{J}y(t), X(t) - y(t)\rangle dt \ge 0 \quad \text{for all} \quad y\in\mathcal{L}^2_V(\Omega\times[0,T]). \tag{38}$$

The map $(\mathcal{A}+\mathcal{J})^{-1}: V^* \to V$ is well defined and is a demicontinuous, maximal monotone mapping (see [12], Cor. 32.30, p. 882). Using the monotonicity of $\mathcal{A}$ and the properties of $\mathcal{J}$ it can be shown that

$$\|(\mathcal{A}+\mathcal{J})^{-1}(z_1) - (\mathcal{A}+\mathcal{J})^{-1}(z_2)\|_V \le \|z_1 - z_2\|_{V^*} \quad \text{for all} \quad z_1, z_2 \in V^*. \tag{39}$$

Hence $(\mathcal{A}+\mathcal{J})^{-1}$ is a continuous function and in view of (39), we can write

$$(40)\quad E\int_0^T \|(\mathcal{A}+\mathcal{J})^{-1}(b^*)\|_V^2 dt \le E\int_0^T \|b^*\|_*^2 dt + T\|(\mathcal{A}+\mathcal{J})^{-1}(0^*)\|_V^2,$$

for all $b^* \in \mathcal{L}^2_{V^*}(\Omega\times[0,T])$ (0^* is the zero element in $\mathcal{L}^2_{V^*}(\Omega\times[0,T])$). Hence $(\mathcal{A}+\mathcal{J})^{-1}(b^*) \in \mathcal{L}^2_V(\Omega\times[0,T])$, if $b^* \in \mathcal{L}^2_{V^*}(\Omega\times[0,T])$.

Let $Z(t) = X(t) - (\mathcal{A}+\mathcal{J})^{-1}(\mathcal{J}X(t) + e^{at}B(t))$; from (40) we conclude that $Z \in \mathcal{L}^2_V(\Omega\times[0,T])$. Relation (36) holds if we can prove that $Z = 0$ in the space $\mathcal{L}^2_V(\Omega\times[0,T])$. For this end we consider

$$(41)\qquad Y_1(t) = (\mathcal{A}+\mathcal{J})^{-1}(\mathcal{J}X(t) + e^{at}B(t) + \mathcal{J}Z(t)),$$

which is obviously an element from $\mathcal{L}^2_V(\Omega\times[0,T])$ (we use (39). In (37) and (38) we put $y = Y_1$ and add up these two relations and obtain

$$(42)\quad E\int_0^T e^{-at}\langle e^{at}B(t) + \mathcal{J}X(t) - \mathcal{A}Y_1(t) - \mathcal{J}Y_1(t), X(t) - Y_1(t)\rangle dt \ge 0.$$

This implies

$$-E\int_0^T e^{-at}\langle \mathcal{J}Z(t), X(t) - Y_1(t)\rangle dt \ge 0$$

and then

$$E\int_0^T e^{-at}\langle \mathcal{J}Z(t), Z(t)\rangle dt + E\int_0^T e^{-at}\langle \mathcal{J}Z(t), Y_0(t) - Y_1(t)\rangle dt \le 0.$$

On the other hand the monotonicity of $(\mathcal{A}+\mathcal{J})^{-1}$ and (41) imply

$$E\int_0^T e^{-at}\langle \mathcal{J}Z(t), Y_0(t) - Y_1(t)\rangle dt \le 0.$$

Therefore

$$E\int_0^T e^{-at}\langle \mathcal{J}Z(t), Z(t)\rangle dt \le 0,$$

which imply $Z(t) = 0$ for a.e. $(\omega, t) \in \Omega\times[0,T]$. Consequently, relation (36) holds.

Finally, from (33), (35), (36) and (21) it results that

$$e^{-at}X(t) = X_0 - \int_0^t e^{-as}(\mathcal{A}X(s) + aX(s))ds + \int_0^t e^{-as} f(s, X(s))ds$$

$$+ \int_0^t e^{-as} g(s, X(s))dw(s)$$

for all $t \in [0, T]$ and a.e. $\omega \in \Omega$. By the equivalence between the problems $(\mathbf{P_1})$ and $(\mathbf{P_2})$ it follows that X is solution of $(\mathbf{P_1})$.

In order to show the uniqueness of the solution of $(\mathbf{P_1})$, we assume that $\Big(X(t)\Big)_{t\in[0,T]}$ and $\Big(Y(t)\Big)_{t\in[0,T]}$ are two solutions of $(\mathbf{P_1})$. Let $t \in$ $\in [0, T]$ be fixed for the moment. Then by the Ito formula it follows that

$$E\|X(t) - Y(t)\|^2 = -2E\int_0^t \langle \mathcal{A}X(s) - \mathcal{A}Y(s), X(s) - Y(s)\rangle ds +$$

$$+2E\int_0^t (f(s, X(s)) - f(s, Y(s)), X(s) - Y(s))ds +$$

$$+E\int_0^t \|g(s, X(s)) - g(s, Y(s))\|^2 ds.$$

Since $\mathcal{A}$ is monotone and f, g are Lipschitz continuous, we have

$$E\|X(t) - Y(t)\|^2 \leq (2\beta + 1)E\int_0^t \|X(t) - Y(t)\|^2 dt,$$

so by applying Gronwall's Lemma it follows that

$$E\|X(t) - Y(t)\|^2 = 0.$$

Hence $X(t) = Y(t)$ for a.e. $\omega \in \Omega$. But X and Y have continuous trajectories in H and so

$$P\Big(\sup_{t\in[0,T]} \|X(t) - Y(t)\|^2 = 0\Big) = 1,$$

which means that $(\mathbf{P_1})$ has with probability 1 a unique solution.

(v) Let $(\hat{x}_{N'})$ be an arbitrary subsequence of the original sequence $(\hat{x}_N)$, such that $(\hat{x}_{N'})$ converges weakly to a function $x \in \mathcal{L}^2_V(\Omega\times[0, T])$.

Repeating the argument of all results contained in Lemma 4.2 up to Th. 4.4-(iv) for this subsequence, we conclude that x is a solution of $(\mathbf{P_1})$. From the uniqueness of the solution of problem $(\mathbf{P_1})$ we obtain $x = X$. Thus all weakly convergent subsequences of $(\hat{x}_N)$ have the same limit X. By Prop. 5.1 it follows that $(\hat{x}_N)$ converges weakly to X.

Let $(\hat{x}_{N'})$ be a subsequence of the original sequence $(\hat{x}_N)$. All results contained in Lemma 4.2 up to Th. 4.4-(ii) (applied for $(\hat{x}_{N'})$ in the space $\mathcal{L}^2_V(\Omega \times [0,T])$) imply the existence of a subsequence of $(\hat{x}_{N'})$ that converges strongly to X. Thus each subsequence of $(\hat{x}_N)$ has a subsequence that converges strongly to X. Using Prop. 5.1 it follows that the original sequence $(\hat{x}_N)$ converges strongly to X.

Reasoning as in the proof of the strong convergence of $(\hat{x}_N)$ to X in the space $\mathcal{L}^2_V(\Omega \times [0,T])$, it also can be shown that the whole sequence $(\hat{x}_N(T))$ converges strongly to $X(T)$ in l. $\Diamond$

5. Appendix - Some convergence principles

Proposition 5.1 ([12], Prop. 10.13, p. 480). *Let (x_n) be a sequence in a Banach space S. Then the following assertions hold:*

(i) *If S is reflexive and (x_n) is bounded, then (x_n) has a weakly convergent subsequence. If, in addition, each weakly convergent subsequence of (x_n) has the same limit $x \in S$, then (x_n) converges weakly to x.*

(ii) *If every subsequence of (x_n) has a subsequence that converges strongly to the same limit x (where $x \in S$), then $x_n \to x$.*

Proposition 5.2 ([12], Prop. 21.23, p. 258). *Let (x_n) be a sequence in a Banach space S.*

(i) *If (x_n) converges weakly to x (where $x \in S$), then (x_n) is bounded and*

$$\|x\|_S \le \liminf_{n\to\infty} \|x_n\|_S.$$

(ii) *If S is locally uniformly convex and the sequence (x_n) satisfies $x_n \rightharpoonup x$ and $\|x_n\|_S \to \|x\|_S$ (where $x \in S$), then $x_n \to x$.*

Proposition 5.3 ([12], Prop. 21.27, p. 261). *Let S_1 and S_2 be Banach spaces and let $L : S_1 \to S_2$ be a continuous linear operator. If (x_n) is a sequence in S_1 such that $x_n \rightharpoonup x$ (where $x \in S_1$), then $L(x_n) \rightharpoonup L(x)$.*

By applying Prop. 5.3 we obtain the following corollary.

Corollary 5.4. *If S is a Banach space and if (x_n) is a sequence from*

$\mathcal{L}^2_S(\Omega \times [0,T])$ *that converges weakly to* $x \in \mathcal{L}^2_S(\Omega \times [0,T])$, *then*

(i) $\int\limits_0^t x_n(s)ds \rightharpoonup \int\limits_0^t x(s)ds$, $\int\limits_0^t x_n(s)dw(s) \rightharpoonup \int\limits_0^t x(s)dw(s)$ *in* $\mathcal{L}^2_S(\Omega \times$ $\times [0,T])$;

(ii) $\int\limits_0^T x_n(s)ds \rightharpoonup \int\limits_0^T x(s)ds$, $\int\limits_0^T x_n(s)dw(s) \rightharpoonup \int\limits_0^T x(s)dw(s)$ *in* $\mathcal{L}^2_S(\Omega)$.

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