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OSCILLATION OF TWO DELAYS DIFFERENTIAL EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

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Abstract: Sufficient conditions for the oscillation of all the solutions of the first order delay differential equation with positive and negative coefficients

$$\dot{x}(t) + p(t)x(t-\tau) - q(t)x(t-\sigma) = 0$$

are given.

1. Introduction

The oscillation theory of delay differential equations has been mostly developed during the past few years. This is motivated by the many applications of delay differential equations in physics, biology, ecology and physiology. We refer, for example, to [5], [9], [11], [13], [15], [27], [30], [32] and to the references cited therein.

The purpose of this paper is to consider the delay differential equation

$$\dot{x}(t) + p(t)x(t-\tau) - q(t)x(t-\sigma) = 0, \tag{1}$$

where p and $q \in \mathcal{C}([t_0, +\infty), \mathbb{R}^+)$ and $\tau, \sigma \in \mathbb{R}^+$.

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By a solution of equation (1) on $[t_0, +\infty)$, where $t_0 \geq 0$, we mean a continuous function defined on $[t_0 - \max\{\tau, \sigma\}, +\infty)$, which is a differentiable function x on $[t_0, +\infty)$ and satisfies equation (1) for all $t \geq t_0$. Such a solution is called oscillatory if it has arbitrarily large zeros. Otherwise it is called nonoscillatory.

The first systematic study for the oscillation of all solutions of the delay differential equation with positive coefficient

$$\dot{x}(t) + p(t)x(\tau(t)) = 0 \tag{2}$$

was undertaken by Mishkis. In 1950 [31], he proved that every solution of the equation (2) oscillates if

$$\limsup_{t\to+\infty}[t - \tau(t)] < +\infty, \qquad \liminf_{t\to+\infty}[t - \tau(t)] \cdot \liminf_{t\to+\infty} p(t) > \frac{1}{e}.$$

In 1972, Ladas, Lakshmikantham and Papadakis [25] proved that the same conclusion holds if

$$\limsup_{t\to+\infty} \int_{\tau(t)}^{t} p(s)ds > 1. \tag{3}$$

In 1979, Ladas [24] and in 1982, Koplatadze and Chanturiya [18] improved (3) to

$$\liminf_{t\to+\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{1}{e}. \tag{4}$$

Concerning the constant $1/e$ in (4), it is to be noted that if the inequality

$$\int_{\tau(t)}^{t} p(s)ds \leq \frac{1}{e}$$

holds eventually, then, according to a result in [18], the equation (2) has a non-oscillatory solution.

How to fill the gap between the conditions (3) and (4) when the limit

$$\lim_{t\to+\infty} \int_{\tau(t)}^{t} p(s)ds$$

does not exists, is an interesting problem which has been recently investigated by several authors.

In 1988, Erbe and Zhang [10] developed new oscillation criteria by employing the upper bound of the ratio $x(\tau(t))/x(t)$ for possible non-oscillatory solutions x of the equation (2). Their result says that all the solutions of the equation (2) are oscillatory if $0 < m \leq 1/e$ and

$$M > 1 - \frac{m^2}{4} \tag{5}$$

where

$$m = \liminf_{t \to +\infty} \int_{t-\tau}^{t} p(s)ds \qquad \text{and} \qquad M = \limsup_{t \to +\infty} \int_{t-\tau}^{t} p(s)ds.$$

Since then, several authors obtained better results by improving the upper bound for $x(\tau(t))/x(t)$. Among them, we can cite Chao [4], Yu and Wang [33] and Yu, Wang, Zhang and Qian [34].

In 1990, Elbert and Stravroulakis [8] and in 1991, Kwong [23], using different techniques, improved condition (5) in the case where $0 < m \leq 1/e$ to the conditions

$$M > 1 - \left(1 - \frac{1}{\sqrt{\lambda_1}}\right)^2 \quad \text{and} \quad M > \frac{\ln \lambda_1 + 1}{\lambda_1},$$

respectively, where λ_1 is the smaller root of the equation $\lambda = e^{m\lambda}$.

We also mention that in the case where

$$\int_{\tau(t)}^{t} p(s)ds \geq \frac{1}{e} \quad \text{and} \quad \lim_{t \to +\infty} \int_{\tau(t)}^{t} p(s)ds = \frac{1}{e},$$

this problem has been studied in 1995 by Elbert and Stravroulakis [8], Kozakiewics [19] and Li [28], and in 1996 by Li [29] and by Domshlak and Stavroulakis [6].

In 1998, Domshlak and Stavroulakis [7] and in 1999, Jaros and Stavroulakis [17] established sufficient conditions for the oscillation of all solutions of the equation (1) in the critical state that the corresponding limiting equation admits a non-oscillatory solution.

Among several other works devoted to the study of oscillatory properties of delay differential equations, we can cite the papers by Agwo [1], Arino and Győri [2], Arino, Ladas and Sficas [3], Gopalsamy [12], Gopalsamy and Ladas [14], Győri and Ladas [15], Győri, Ladas and Pakula [16], Kulenovic and Ladas [20], [21], Kulenovic, Ladas and Meimaridou [22] and Ladas and Stavroulakis [26].

In this paper, we provide new sufficient conditions for the oscillation of all the solutions of the equation (1) by means of the generalized characteristic equation.

2. The main result

We first give some results needed in the proof of our main theorem (Th. 2.1).

Lemma 2.1. *Let $x \in \mathbb{R}$ and $r > 0$. Then*

$$re^{rx} \geq rx + \ln er. \tag{6}$$

Proof. Write the right hand side of (6) as

$$rx + \ln er = \ln e^{rx} + \ln er = \ln e + \ln re^{rx} = 1 + \ln re^{rx}.$$

So, inequality (6) becomes

$$re^{rx} \geq 1 + \ln re^{rx} \quad \text{or} \quad re^{rx} - \ln re^{rx} \geq 1. \tag{7}$$

Let $z = re^{rx}$ and consider the function $f(z) = z - \ln z$. In terms of z and $f(z)$, inequality (7) reads $f(z) \geq 1$. Note that $f(1) = 1$ and $\frac{d}{dz} f(z) = \frac{z-1}{z}$, which implies that f admits 1 as a minimum and then $f(z) \geq 1$ for all $z > 0$. $\Diamond$

Lemma 2.2 [15]. *Suppose that $x \in C([t_0, +\infty), \mathbb{R})$ satisfies the inequality*

$$x(t) \leq c + \max_{t-\tau \leq s \leq t} x(s) \quad \textit{for} \quad t \geq t_0,$$

where $c \leq 0$, $\tau \in \mathbb{R}^+$ and $t_0 \in \mathbb{R}$. Then x cannot be a nonnegative function.

Lemma 2.3 [15]. *Let $p_i \in C([t_0, +\infty), \mathbb{R}^+)$ and $\tau_i \in \mathbb{R}^+$, $i = 1, \ldots, n$. The differential inequality*

$$\dot{x}(t) + \sum_{i=1}^{n} p_i(t) x(t - \tau_i) \leq 0, \quad t \geq t_0$$

has an eventually positive solution if and only if the differential equation

$$\dot{y}(t) + \sum_{i=1}^{n} p_i(t) y(t - \tau_i) = 0, \quad t \geq t_0$$

has an eventually positive solution.

Consider the delay differential equation

$$\dot{y}(t) + a(t) y(t - \tau) = 0, \quad t \geq t_0, \tag{8}$$

and let

$$m = \liminf_{t \to +\infty} \int_{t-\tau}^{t} a(s) ds \qquad \text{and} \qquad M = \limsup_{t \to +\infty} \int_{t-\tau}^{t} a(s) ds.$$

Lemma 2.4 [17]. *Suppose that $m > 0$ and the equation (8) has an eventually positive solution y. Then $m \leq 1/e$ and*

$$M \leq c_1 = \frac{1 + \ln \lambda_1}{\lambda_1} - L,$$

where λ_1 *is the smaller root of the equation* $\lambda = e^{m\lambda}$ *and* $L = \liminf_{t \to +\infty} \frac{y(t)}{y(t-\tau)}$.

Lemma 2.5 [33]. *Let* $0 < m \leq 1/e$ *and let* y *be an eventually positive solution of the equation* (8). *Then*

$$\limsup_{t \to +\infty} \frac{y(t-\tau)}{y(t)} \leq c_2 = \frac{2}{1 - m - \sqrt{1 - 2m - m^2}}.$$

Lemma 2.6. *Consider the delay differential equation* (1) *and assume that the following conditions hold:*

(H1): $p, q \in C([t_0, +\infty), \mathbb{R}^+)$, $\tau, \sigma \in \mathbb{R}^+$ *and* $\tau \geq \sigma$,
(H2): $p(t) \geq q(t + \sigma - \tau)$ *for* $t \geq t_0 + \tau - \sigma$,
(H3): $\int_{t-\tau}^{t-\sigma} q(s + \sigma) ds \leq 1$ *for* $t \geq t_0 + \tau$.

Let x *be an eventually positive solution of the equation* (1) *and* X *the function defined by*

$$X(t) = x(t) - \int_{t-\tau}^{t-\sigma} q(s + \sigma) x(s) ds, \qquad t \geq t_0 + \tau - \sigma. \tag{9}$$

Then X *is decreasing and positive.*

Proof. By differentiation, (9) gives

$$\dot{X}(t) = \dot{x}(t) - q(t) x(t - \sigma) + q(t + \sigma - \tau) x(t - \tau).$$

By (1) we see that

$$\dot{X}(t) = -[p(t) - q(t + \sigma - \tau)] x(t - \tau). \tag{10}$$

Because that x is positive for $t \geq t_1 - \tau$, where $t_1 \geq t_0 + \tau$, and from hypothesis (H2), we conclude that

$$\dot{X}(t) < 0 \qquad \text{for} \qquad t \geq t_1 + \tau, \tag{11}$$

which implies that X is decreasing on $[t_1 + \tau, +\infty)$.

Now, by contradiction we prove that X is positive. Suppose that there exists a $t_2 \geq t_1$ such that $X(t_2) \leq 0$. Inequality (11) implies that there exists a $t_3 \geq t_2$ such that $X(t) \leq X(t_3) \leq 0$ for $t \geq t_3$. From (9) it follows that for $t \geq t_3$ we have

$$x(t) = X(t) + \int_{t-\tau}^{t-\sigma} q(s+\sigma)x(s)ds \le X(t_3) + \int_{t-\tau}^{t-\sigma} q(s+\sigma)x(s)ds \le$$
$$\le X(t_3) + \left(\max_{t-\tau\le s\le t-\sigma} x(s)\right)\int_{t-\tau}^{t-\sigma} q(s+\sigma)ds.$$

Hypothesis (H3) yields

$$x(t) \le X(t_3) + \max_{t-\tau\le s\le t-\sigma} x(s) \qquad \text{for} \quad t \ge t_3.$$

By Lemma 2.2, x cannot be nonnegative on $[t_3, +\infty)$, which is a contradiction to the assumptions of our theorem. This completes the proof. ◊

Our main result about the oscillation of all solutions of the equation (1) is embodied in the following:

Theorem 2.1. *Let the hypotheses* (H1) *to* (H3) *of* Lemma 2.6 *are true. Let* $a(t) = p(t) - q(t+\sigma-\tau)$ *and assume that*

(H4): $\liminf_{t\to+\infty} \int_{t-\tau}^{t} a(s)ds > 0$,

(H5): $\limsup_{t\to+\infty} \int_{\bar{t}_0}^{t} a(s)\ln\left(e\int_s^{s+\tau} a(u)du\right) ds = +\infty$, *for some* $\bar{t}_0$.

Then every solution of the equation (1) *is oscillatory.*

Proof. For the sake of contradiction, we assume that (1) has a positive solution x. (For the case that (1)x has a negative solution $\overline{x}$, we simply let $x = -\overline{x}$). By Lemma 2.6, the function X defined by (9) is positive. Also by (10) we have

$$\dot{X}(t) + [p(t) - q(t+\sigma-\tau)]x(t-\tau) = 0.$$

Since $0 < X(t) \le x(t)$ and $p(t) \ge q(t+\sigma-\tau)$ (see (H2)), X satisfies

$$\dot{X}(t) + [p(t) - q(t+\sigma-\tau)]X(t-\tau) \le 0.$$

By Lemma 2.3, the delay differential equation

$$\dot{y}(t) + [p(t) - q(t+\sigma-\tau)]y(t-\tau) = 0 \tag{12}$$

has a positive solution. Let y be such a solution. Note that y is decreasing for sufficiently large t. Dividing both sides of equation (12) by $y(t)$, we have

$$\frac{\dot{y}(t)}{y(t)} + [p(t) - q(t+\sigma-\tau)]\frac{y(t-\tau)}{y(t)} = 0.$$

Integrating both sides of this equation from $t-\tau$ to t, for sufficiently large t, we have

$$\ln \frac{y(t)}{y(t-\tau)} + \int_{t-\tau}^{t} [p(s) - q(s+\sigma-\tau)] \frac{y(s-\tau)}{y(s)} ds = 0.$$

Let $W(t) = \frac{y(t-\tau)}{y(t)}$. By the last equation, we have

$$\int_{t-\tau}^{t} [p(s) - q(s+\sigma-\tau)] W(s) ds = \ln W(t),$$

or

$$W(t) = \exp\left(\int_{t-\tau}^{t} a(s) W(s) ds \right),$$

where $a(t) = p(t) - q(t+\sigma-\tau)$. Multiplying the factor $a(t)$ in both sides of this equation, the function $\alpha(t) = a(t)W(t)$ satisfies the generalized characteristic equation

$$\alpha(t) = a(t) \exp\left(\int_{t-\tau}^{t} \alpha(s) ds \right).$$

Let $r(t) = \int_{t-\tau}^{t} a(s)ds$ and $x(t) = \frac{1}{r(t)} \int_{t-\tau}^{t} \alpha(s)ds$. The equation above is equivalent to

$$r(t)\alpha(t) = a(t) r(t) e^{r(t)x(t)}.$$

By Lemma 2.1 we get

$$r(t)\alpha(t) \geq a(t) \left[r(t)x(t) + \ln er(t) \right],$$

which is equivalent to

$$r(t)\alpha(t) \geq a(t) \left[\int_{t-\tau}^{t} \alpha(s)ds + \ln er(t) \right]$$

or

$$\left(\int_{t-\tau}^{t} a(s)ds \right) \alpha(t) - a(t) \int_{t-\tau}^{t} \alpha(s)ds \geq a(t) \ln \left(e \int_{t-\tau}^{t} a(s)ds \right).$$

Integrating both sides of this inequality from $\bar{t}_0$ to T, for $T > \bar{t}_0$, $\bar{t}_0$ sufficiently large, we have

$$\begin{aligned} (13) \qquad & \int_{\bar{t}_0}^{T} \left(\int_{t-\tau}^{t} a(s)ds \right) \alpha(t)dt - \int_{\bar{t}_0}^{T} a(t) \int_{t-\tau}^{t} \alpha(s)dsdt \geq \\ & \geq \int_{\bar{t}_0}^{T} a(t) \ln \left(e \int_{t-\tau}^{t} a(s)ds \right) dt. \end{aligned}$$

By interchanging the order of integration, we find that

$$\int_{\bar{t}_0}^{T} a(t)\left(\int_{t-\tau}^{t}\alpha(s)ds\right)dt \geq \int_{\bar{t}_0}^{T-\tau}\left(\int_{s}^{s+\tau} a(t)\alpha(s)dt\right)ds =$$

$$= \int_{\bar{t}_0}^{T-\tau}\alpha(s)\left(\int_{s}^{s+\tau} a(t)dt\right)ds = \int_{\bar{t}_0}^{T-\tau}\alpha(t)\left(\int_{t}^{t+\tau} a(s)ds\right)dt.$$

Hence

$$\text{(14)}\quad \begin{aligned} &\int_{\bar{t}_0}^{T}\alpha(t)\left(\int_{t-\tau}^{t} a(s)ds\right)dt - \int_{\bar{t}_0}^{T-\tau}\alpha(t)\left(\int_{t}^{t+\tau} a(s)ds\right)dt \geq \\ &\geq \int_{\bar{t}_0}^{T}\left(\int_{t-\tau}^{t} a(s)ds\right)\alpha(t)dt - \int_{\bar{t}_0}^{T} a(t)\int_{t-\tau}^{t}\alpha(s)dsdt \end{aligned}$$

and therefore, from (13) and (14), it follows that

$$\int_{T-\tau}^{T}\alpha(t)\left(\int_{t-\tau}^{t} a(s)ds\right)dt \geq \int_{\bar{t}_0}^{T} a(t)\ln\left(e\int_{t-\tau}^{t} a(s)ds\right)dt.$$

Taking "limsup" on both sides of this inequality, we have

$$\text{(15)}\quad \begin{aligned} M\cdot\limsup_{t\to+\infty}\int_{t-\tau}^{t}\alpha(s)ds &\geq \limsup_{t\to+\infty}\int_{t-\tau}^{t}\alpha(s)\left(\int_{s-\tau}^{s} a(u)du\right)ds \geq \\ &\geq \limsup_{t\to+\infty}\int_{\bar{t}_0}^{t} a(s)\ln\left(e\int_{s-\tau}^{s} a(u)du\right)ds, \end{aligned}$$

where $M = \limsup\limits_{t\to+\infty}\int_{t-\tau}^{t} a(s)ds$. Since $\alpha(t) = a(t)W(t) = -\frac{\dot{y}(t)}{y(t)}$, inequality (15) gives

$$M\cdot\limsup_{t\to+\infty}\ln\frac{y(t-\tau)}{y(t)} \geq \limsup_{t\to+\infty}\int_{\bar{t}_0}^{t} a(s)\ln\left(e\int_{s-\tau}^{s} a(u)du\right)ds.$$

Using Lemma 2.4 (M is finite) and hypothesis (H5), we deduce that

$$\limsup_{t\to+\infty}\frac{y(t-\tau)}{y(t)} = +\infty.$$

In view of Lemma 2.5, we have a contradiction. Thus the result of Th. 2.1 holds. ◊

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