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MALMQUIST - TAKENAKA SYSTEMS AND EQUILIBRIUM CONDITIONS

Margit **Pap**

University of Pécs, Ifjúság útja 6, H-7624 Pécs, Hungary

Ferenc **Schipp**

Department of Numerical Analysis, Eötvös L. University, Pázmány P. sétány I/D, H-1117 Budapest, Hungary; Computer and automation Research Institute of HAS, Kende u. 13–17, H-1502 Budapest

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Abstract: The Malmquist-Takenaka systems ($\Phi_n^{\mathfrak{a}}, n \in \mathbb{N}^*$) form an orthonormal system on the unite circle $\mathbb{T}$. The restriction of the finite collection ($\Phi_n^{\mathfrak{a}}, n = 1, \cdots, N$) to a subset $\mathbb{T}_N^{\mathfrak{a}}$ of $\mathbb{T}$ is a discrete orthonormal system with respect to the scalar product $[\cdot,\cdot]_N$. It is showed that the set $\mathbb{T}_N^{\mathfrak{a}}$ can be interpreted as a solution of an electrostatic equilibrium problem. The zeros of Jacobi, Laguerre and Hermite polynomials admit a similar interpretation.

E-mail addresses: papm@math.ttk.pte.hu, schipp@ludens.elte.hu
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1. Introduction

In control theory the Malmquist-Takenaka systems $(\Phi_n^{\mathfrak{a}}, n \in \mathbb{N}^*)$ [5], [10] are often used to identify the transfer function of the system [1], [2], [3]. This orthonormal system is generated by a sequence $\mathfrak{a} = = (a_1, a_2, \cdots)$ of complex numbers $a_n \in \mathbb{D}$ $(n \in \mathbb{N}^* := \{1, 2, \cdots\})$ of the unite disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and can be expressed by the Blaschke-functions

$$B_b(z) := \frac{z - b}{1 - \bar{b}z} \quad (b \in \mathbb{D}, z \in \mathbb{C}). \tag{1.1}$$

Namely (see [4]) the systems $\Phi_n = \Phi_n^{\mathfrak{a}}$ $(n \in \mathbb{N}^*)$ in question are defined by

$$\begin{aligned} \Phi_1(z) &:= \frac{\sqrt{1 - |a_1|^2}}{1 - \bar{a}_1 z}, \\ \Phi_n(z) &:= \frac{\sqrt{1 - |a_n|^2}}{1 - \bar{a}_n z} \prod_{k=1}^{n-1} B_{a_k}(z) \quad (z \in \mathbb{C}, n = 2, 3, \cdots). \end{aligned} \tag{1.2}$$

The Malmquist-Takenaka functions form an orthonormal system on the unite circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, i.e.

$$\langle \Phi_n, \Phi_m \rangle := \frac{1}{2\pi} \int_0^{2\pi} \Phi_n(e^{it}) \overline{\Phi_m(e^{it})} \, dt = \delta_{mn} \quad (m, n \in \mathbb{N}^*),$$

where δ_{mn} is the Kronecker symbol (see [3], [4]).

In the special case if $a_n = b$ $(n \in \mathbb{N}^*)$, then $\Phi_n^{\mathfrak{a}} = L_n^b$ $(n \in \mathbb{N}^*)$ is the discrete Laguerre system, and if $a_{2k-1} = a, a_{2k} = b$ $(k \in \mathbb{N}^*)$, then $(\Phi_n^{\mathfrak{a}}, n \in \mathbb{N}^*)$ is the Kautz-system, investigated in [2].

If b belongs to $\mathbb{D}$ then B_b is a $1-1$ map on $\mathbb{D}$ and on $\mathbb{T}$, respectively. Moreover (see [2]) B_b, can be written in the form

$$B_b(e^{it}) = e^{i\beta_b(t)} \quad (t \in \mathbb{R},\ b = re^{i\tau} \in \mathbb{D}), \tag{1.3}$$

where

$$\beta_b(t) := \tau + \gamma_s(t - \tau), \qquad \gamma_s(t) := 2 \arctan\left(s \tan \frac{t}{2}\right)$$
$$\left(t \in [-\pi, \pi)\right),\ s := \frac{1 + r}{1 - r}\right)$$

and γ_s is extended to $\mathbb{R}$ by $\gamma_s(t + 2\pi) = 2\pi + \gamma_s(t)$ $(t \in \mathbb{R})$.

Thus the product $\prod_{j=1}^{N} B_{a_j}$ is of the form

$$\prod_{j=1}^{N} B_{a_j}(e^{it}) = e^{i(\beta_{a_1}(t)+\cdots+\beta_{a_N}(t))} \quad (t \in \mathbb{R}, N = 1, 2, \cdots). \tag{1.4}$$

This implies that the solution of the equation

$$\frac{z-a_1}{1-\bar{a}_1 z}\frac{z-a_2}{1-\bar{a}_2 z}\cdots\frac{z-a_N}{1-\bar{a}_N z} = 1$$

can be written as

$$w_k := e^{it_k}, \quad t_k := \theta_N^{-1}(2\pi(k-1)/N) \quad (k = 1, 2, \cdots, N), \tag{1.5}$$

where θ_N^{-1} is the inverse of the function

$$\theta_N(t) := \frac{1}{N}(\beta_{a_1}(t) + \cdots + \beta_{a_N}(t)) \quad (t \in \mathbb{R}). \tag{1.6}$$

We introduce the weight function ρ_N by

$$\frac{1}{\rho_N(z)} := \sum_{k=1}^{N} \frac{1-|a_k|^2}{|1-\bar{a}_k z|^2} \quad (z \in \mathbb{T},\ N = 1, 2, \cdots) \tag{1.7}$$

and set

$$\mathbb{T}_N := \mathbb{T}_N^{\mathfrak{a}} := \left\{\theta_N^{-1}(2\pi(k-1)/N) : k = 1, 2, \cdots, N\right\} \quad (N = 1, 2, \cdots). \tag{1.8}$$

Theorem 1. *The finite collection of the functions Φ_n $(1 \le n \le N)$ form a discrete orthonormal system with respect to the scalar product*

$$[F, G]_N := \sum_{z \in \mathbb{T}_N} F(z)\overline{G(z)}\rho_N(z), \tag{1.9}$$

namely

$$[\Phi_n, \Phi_m]_N = \delta_{mn} \quad (1 \le m, n \le N).$$

In this paper we show that the points of $\mathbb{T}_N^{\mathfrak{a}}$ are closely connected with an electrostatic equilibrium problem.

2. The equilibrium condition

For any complex number $z \in \mathbb{C}$ set $z^* := 1/\bar{z}$ and introduce the polynomials

$$\omega_1(z) := \prod_{k=1}^{N}(z-a_k), \quad \omega_2(z) := \prod_{k=1}^{N}(1-\bar{a}_k z), \tag{2.1}$$
$$\omega(z) := \omega_1'(z)\omega_2(z) - \omega_2'(z)\omega_1(z) \quad (z \in \mathbb{C}).$$

It is clear that ω is a polynomial of degree $2N-2$. We show (see Lemma 1.) that if $c \in \mathbb{C}$ is a root of ω then c^* is also a root of ω with the same multiplicity. Denote by $c_1, c_1^*, \cdots, c_s, c_s^*$ the pairwise distinct roots of ω with the multiplicity $\nu_1, \nu_1, \cdots, \nu_s, \nu_s$.

Theorem 2. *The numbers* $z_n := w_n \in \mathbb{T}_N^{\mathfrak{a}}$ $(n = 1, 2, \cdots, N)$ *are the solutions of the equilibrium equations*

$$\sum_{k=1, k\neq n}^{N} \frac{1}{z_n - z_k} = \sum_{j=1}^{s}\left(\frac{\nu_j}{2}\frac{1}{z_n - c_j} + \frac{\nu_j}{2}\frac{1}{z_n - c_j^*}\right) \tag{2.2}$$
$$(n = 1, \cdots, N).$$

The points of $\mathbb{T}_N^{\mathfrak{a}}$ can be interpreted as a solution of the following electrostatic equilibrium problem. If N unite "masses" at the variable points $z_1, z_2, \cdots, z_N \in \mathbb{T}$ and $-\nu_1/2, -\nu_1/2, \cdots, -\nu_s/2, -\nu_s/2$ fixed mases at the fixed points $c_1, c_1^*, \cdots, c_s, c_s^*$ are given then $z_1 = w_1, \cdots, z_N = w_N$ is the equilibrium position of the electrostatic forces in question.

We remark that the zeros of Jacobi, Laguerre and Hermite polynomials admit a similar interpretation (see [9], pp. 140, 153).

The roots of ω are described in the following

Lemma 1. *Denote* $a_1, a_2, \cdots, a_r$ *the pairwise distinct roots of* ω_1 *with the multiplicity* $m_1, m_2, \cdots, m_r$. *Then* ω *is of the form*

$$\omega(z) = \Omega(z)\prod_{j=1}^{r}(z-a_j)^{m_j-1}(1-\bar{a}_j z)^{m_j-1} \quad (z \in \mathbb{C}),$$

where

$$\Omega(z) := \sum_{k=1}^{r} m_k(1-|a_k|^2) \prod_{j=1, j\neq k}^{r} (z-a_j)(1-\bar{a}_j z) \quad (z \in \mathbb{C})$$

is a polynomial of degree $2r-2$. *Moreover, if* c *is a root of* Ω *with multiplicity* m *then* c^* *is also a root of* Ω *with the same multiplicity.*

In the case of discrete Laguerre functions $a_1 = a_2 = \cdots = a_N = b$ and consequently

$$\omega(z) = N[(z-b)^{N-1}(1-\bar{b}z)^N + \bar{b}(1-\bar{b}z)^{N-1}(z-b)^N] = \\ = N(1-|b|^2)(z-b)^{N-1}(1-\bar{b}z)^{N-1}.$$

Thus the roots of ω are b and b^* with multiplicity $N-1$ and we get the next claim proved in [6].

Corollary 1. *The numbers* $w_k = e^{i\tau_k}$ $(\tau_k := \beta_b^{-1}(2\pi(k-1)/N)$, $k = 1, \cdots, N)$ *are the solutions of the equilibrium equations*

$$\sum_{k=1,k\neq n}^{N} \frac{1}{w_n - w_k} = \frac{N-1}{2}\left(\frac{1}{w_n - b} + \frac{1}{w_n - b^*}\right)$$
$$(n = 1, \cdots, N).$$

In the case of Kautz system $a_1 = a_3 = \cdots = a_{2N-1} = a \in \mathbb{D}$, $a_2 = \cdots = a_{2N} = b \in \mathbb{D}$ and consequently

$$\omega(z) = \Omega(z)[(z-a)(z-b)(z-a^*)(z-b^*)]^{N-1},$$
$$\Omega(z) := N[(1-|a|^2)(z-b)(1-\bar{b}z) + (1-|b|^2)(z-a)(1-\bar{a}z)].$$

Denote c and c^* the roots of the polynomial Ω.

Corollary 2. *Let* $\theta_2(t) := (\beta_a(t) + \beta_b(t))/2$ *be the argumentum transformation corresponding to the Kautz system. The numbers* $w_k = e^{i\tau_k}$ $(\tau_k := \theta_{2N}^{-1}(\pi(k-1)/N)$, $k = 1, 2, \cdots, 2N)$ *are the solutions of the equilibrium equations*

$$\frac{1/2}{w_n - c} + \frac{1/2}{w_n - c^*} +$$
$$+\frac{N-1}{2}\left(\frac{1}{w_n - a} + \frac{1}{w_n - a^*} + \frac{1}{w_n - b} + \frac{1}{w_n - b^*}\right) =$$
$$= \sum_{k=1,k\neq n}^{2N} \frac{1}{w_n - w_k} \qquad (n = 1, 2, \cdots, 2N).$$

3. Proofs

To prove Th. 1 we use the following closed form of the Dirichlet kernel of the system Φ_n $(n \in \mathbb{N}^*)$.

Lemma 2. *The Dirichlet kernels of the system* Φ_n $(n \in \mathbb{N}^*)$ *can be written in the closed form* (see [4])

$$D_N(z,w) := \sum_{k=1}^{N} \Phi_k(z)\overline{\Phi_k(w)} = \frac{1-\prod_{j=1}^{N} B_{a_j}(z)\overline{B_{a_j}(w)}}{1-z\overline{w}}. \tag{3.1}$$

Proof of Lemma 2. We show (3.1) by induction with respect to N. For $N=1$ we have

$$\frac{1-B_{a_1}(z)\overline{B_{a_1}(w)}}{1-z\overline{w}} = \frac{(1-\bar{a}_1 z)(1-a_1\bar{w})-(z-a_1)(\bar{w}-\bar{a}_1)}{(1-\bar{a}_1 z)(1-a_1\bar{w})(1-z\bar{w})} =$$

$$= \frac{(1-|a_1|^2)(1-z\bar{w})}{(1-\bar{a}_1 z)(1-a_1\bar{w})(1-z\bar{w})} = \Phi_1(z)\overline{\Phi_1(w)}$$

and (3.1) holds for $N=1$.

Suppose that (3.1) is true for N. Then by (1.2) and (3.1)

$$D_{N+1}(z,w) = \frac{1-\prod_{j=1}^{N} B_{a_j}(z)\overline{B_{a_j}(w)}}{1-z\overline{w}} +$$

$$+\frac{(1-|a_{N+1}|^2)\prod_{j=1}^{N} B_{a_j}(z)\overline{B_{a_j}(w)}}{(1-\bar{a}_{N+1}z)(1-a_{N+1}\bar{w})} = \frac{1}{1-z\bar{w}} -$$

$$-\prod_{j=1}^{N} B_{a_j}(z)\overline{B_{a_j}(w)}\frac{(1-\bar{a}_{N+1}z)(1-a_{N+1}\bar{w})-(1-z\bar{w})(1-|a_{N+1}|^2)}{(1-z\bar{w})(1-\bar{a}_{N+1}z)(1-a_{N+1}\bar{w})} =$$

$$= \frac{1-\prod_{j=1}^{N+1} B_{a_j}(z)\overline{B_{a_j}(w)}}{1-z\bar{w}}$$

and (3.1) holds for $N+1$. $\Diamond$

Proof of Theorem 1. By (1.2) and (1.7)

$$\sum_{k=1}^{N} |\Phi_k(z)|^2 = \sum_{k=1}^{N} \frac{1-|a_k|^2}{|1-\bar{a}_k z|^2} = \frac{1}{\rho_N(z)}.$$

Set

$$u_{k\ell} := \Phi_k(w_\ell)\sqrt{\rho_N(w_\ell)} \quad (1 \le k, \ell \le N).$$

In the case $j \neq \ell$ by (1.4), (1.5), (1.6) and (3.1)

$$\sum_{k=1}^{N} u_{kj}\overline{u_{k\ell}} = \sqrt{\rho_N(w_j)\rho_N(w_\ell)}\sum_{k=1}^{N}\Phi_k(w_j)\overline{\Phi_k(w_\ell)} =$$
$$= \sqrt{\rho_N(w_j)\rho_N(w_\ell)}\frac{1 - e^{2\pi i N(\theta_N(t_j)-\theta_N(t_\ell))}}{1 - w_j\overline{w_\ell}} =$$
$$= \sqrt{\rho_N(w_j)\rho_N(w_\ell)}\frac{1 - e^{2\pi i(j-\ell)}}{1 - w_j\overline{w_\ell}} = 0.$$

Obviously for $j = \ell$ we have

$$\sum_{k=1}^{N} u_{kj}\bar{u}_{k\ell} = 1.$$

Thus

$$\sum_{k=1}^{N} u_{kj}\bar{u}_{k\ell} = \delta_{j\ell} \quad (1 \le j, \ell \le N)$$

and consequently the matrix $U = [u_{k\ell}]_{k,\ell=1}^{N}$ is unitary. This fact, (1.7) and (1.9) imply

$$\sum_{k=1}^{N} u_{jk}\overline{u_{\ell k}} = [\Phi_j, \Phi_\ell]_N = \delta_{j\ell} \quad (1 \le j, \ell \le N)$$

and Th. 1 is proved. $\Diamond$

Proof of Lemma 1. To prove Lemma 1 we introduce the following notion. We shall say that the polinomial P of degree n is an *inversion polynomial* if for every $z \in \mathbb{C}$, $z \neq 0$

$$P(z^*) = \bar{z}^{-n}\overline{P(z)}$$

is satisfied.

Obviously

$$Q_a(z) := (z - a)(1 - \bar{a}z) \quad (z \in \mathbb{C})$$

is an inversion polynomial. Indeed

$$Q_a(z^*) = \left(\frac{1}{\bar{z}} - a)(1 - \frac{\bar{a}}{\bar{z}}\right) = \bar{z}^{-2}\overline{Q_a(z)} \quad (z \in \mathbb{C}).$$

It is clear that if c is a root of an inversion polynomial P and $0 \neq c \in \mathbb{C}$ then $P(c^*) = 0$. Moreover, the multiplicity of c and c^* is the same.

Observe that if c is a root of the inversion polynomial P then for the polynomial P/Q_c we have

$$\frac{P(z^*)}{Q_c(z^*)} = \bar{z}^{-(n-2)}\frac{\overline{P(z)}}{\overline{Q_c(z)}}$$

and consequently P/Q_c is also an inversion polynomial. This implies that the roots c and c^* have the same multiplicity.

We show that Ω is an inversion polynomial. Indeed

$$\Omega(z^*) = \sum_{k=1}^{r} m_k(1-|a_k|^2)\prod_{j=1,j\neq k}^{r} Q_{a_j}(z^*) =$$
$$= \bar{z}^{-2(r-1)}\sum_{k=1}^{r} m_k(1-|a_k|^2)\prod_{j=1,j\neq k}^{r} \overline{Q_{a_j}(z)} = \bar{z}^{-2(r-1)}\overline{\Omega(z)}.$$

Thus Lemma 1 is proved. ◊

Proof of Theorem 2. Denote

$$\varphi(z) := \prod_{j=0}^{N-1}\frac{z-a_j}{1-\bar{a}_j z} - 1 \quad (z\in\mathbb{C}).$$

By (1.1), (1.3), (1.4) and (1.5) it is clear that $\varphi(z)=0$ if and only if $z = w_k := e^{it_k}$, $t_k := \theta_N^{-1}(2\pi(k-1)/N)$ $(k=1,\cdots,N)$. Set

(3.2)
$$f(z) := \prod_{k=1}^{N}(z-w_k),$$
$$g(z) := \prod_{j=1}^{N}(z-a_j) - \prod_{j=1}^{N}(1-\bar{a}_j z) =: \omega_1(z)-\omega_2(z) \qquad (z\in\mathbb{C}).$$

The polynomials f and g have the same degree and roots, therefore $f=\lambda g$ with a constant $\lambda\in\mathbb{C}$.

It is easy to see that

$$\text{(3.3)}\quad \frac{1}{2}\frac{g''(w_n)}{g'(w_n)} = \frac{1}{2}\frac{f''(w_n)}{f'(w_n)} = \sum_{k=1,k\neq n}^{N}\frac{1}{w_n-w_k} \quad (n=1,2,\cdots,N).$$

By the definition of w_n

$$\prod_{j=1}^{N} \frac{w_n - a_j}{1 - \bar{a}_j w_n} = \frac{\omega_1(w_n)}{\omega_2(w_n)} = 1 \quad (n = 1, \cdots, N). \tag{3.4}$$

On the other hand by (2.1), (3.2) and (3.4) we get

$$\begin{aligned} \frac{g''(w_n)}{g'(w_n)} &= \frac{\omega_1''(w_n) - \omega_2''(w_n)}{\omega_1'(w_n) - \omega_2'(w_n)} = \\ &= \frac{\omega_2(w_n)\omega_1''(w_n) - \omega_1(w_n)\omega_2''(w_n)}{\omega_2(w_n)\omega_1'(w_n) - \omega_1(w_n)\omega_2'(w_n)} = \frac{\omega'(w_n)}{\omega(w_n)}. \end{aligned} \tag{3.5}$$

By Lemma 1 the roots of ω are of the form $c_1, c_1^*, \cdots, c_s, c_s^*$ with the multiplicity $\nu_1, \nu_1, \cdots, \nu_s, \nu_s$. Consequently in ω'/ω every root appeares with multiplicity 1 and thus we have the partial decomposition

$$\frac{\omega'(z)}{\omega(z)} = \sum_{j=1}^{s} \left(\frac{A_j}{z - c_j} + \frac{\tilde{A}_j}{z - c_j^*} \right).$$

Write $\omega(z) = (z - c_j)^{\nu_j} P_j(z)$, where $P(c_j) \neq 0$. Then

$$\frac{\omega'(z)}{\omega(z)} = \frac{\nu_j P_j(z) + (z - c_j)P_j'(z)}{(z - c_j)P_j(z)}.$$

Consequently

$$A_j = \lim_{z \to c_j} (z - c_j) \frac{\omega'(z)}{\omega(z)} = \nu_j.$$

In a similar way $\tilde{A}_j = \nu_j$ and by (3.3) and (3.5) we get

$$\frac{1}{2} \frac{g''(w_n)}{g'(w_n)} = \frac{1}{2} \frac{\omega'(w_n)}{\omega(w_n)} = \sum_{j=1}^{s} \left(\frac{\nu_j/2}{w_n - c_j} + \frac{\nu_j/2}{w_n - c_j^*} \right)$$

and Th. 2 is proved. ◊

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