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MULTIPLICATIVE SEQUENCES FOR UNIVERSAL SEQUENCES OF MAPPINGS

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Abstract: In this paper we study the universality of (c_nT_n), where (c_n) is a scalar sequence and (T_n) is a universal sequence such that each T_n is valued in a topological vector space. We also show that if T is a hypercyclic operator on a Baire metrizable complex locally convex space then there exists a residual subset in the unit circle such that λT is also hypercyclic for all λ in such a subset.

1. Introduction and notation

In this paper, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{T}$ will denote the set of positive integers, the set of rational numbers, the real line, the field of complex numbers and the unit circle $\{|z| = 1\}$, respectively. Suppose that X is a topological space and that Y is a topological vector space over

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$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. In this note we prove that if X and Y satisfy certain assumptions and $T_n : X \to Y$ $(n \in \mathbb{N})$ is a densely universal sequence of continuous mappings then, for "most" scalar sequences (c_n), the sequence of mappings $(c_n T_n)$ is also densely universal, see Section 2. In addition, we prove that if T is a hypercyclic operator on a complex topological vector space satisfying suitable conditions then there exists a residual subset $B \subset \mathbb{T}$ such that λT is also hypercyclic for all $\lambda \in B$, see Section 3. The reader is referred to [8] for an excellent survey about the relevant facts on universality and hypercyclicity.

A little terminology is needed. Assume that X, Y are topological spaces and that $T_n : X \to Y$ $(n \in \mathbb{N})$ is a sequence in the class $C(X,Y) :$ $:= \{$continuous mappings from X into $Y\}$. A point $x \in X$ is said to be universal for (T_n) whenever the orbit $\{T_n x : n \in \mathbb{N}\}$ is dense in Y. The symbol $U((T_n))$ will stand for the set of universal points of X for (T_n). The sequence (T_n) is called universal if and only if $U((T_n))$ is not empty. If (T_n) is universal then, trivially, Y is separable. (T_n) is said to be densely universal when $U((T_n))$ is dense in Y. Denote by $C(X)$ the set of continuous selfmappings $T : X \to X$, that is, $C(X) = C(X,X)$. If $T \in C(X)$, then T is said to be universal whenever there is a universal point x for T, i.e., a point such that the orbit $\{T^n x : n \in \mathbb{N}\}$ is dense in X, where $T^1 = T$, $T^2 = T \circ T$, ... In other words, T is universal if and only if (T^n) is universal. We will denote $U(T) = U((T^n))$. T is called densely universal when (T^n) is densely universal. From the fact that each power T^m $(m \in \mathbb{N})$ has dense range whenever T is universal, it is easy to see that every universal selfmapping is densely universal. If X is a topological vector space then every universal $T \in L(X) := \{$operators on $X\} = \{T \in C(X) : T$ is linear$\}$ is called hypercyclic.

The motivation for the results furnished in this paper is the following. If T is a hypercyclic operator on a complex Banach space X, then a multiple λT $(\lambda \in \mathbb{C})$ need not be hypercyclic. Indeed, take λ with $|\lambda| \le ||T||^{-1}$. But even if $|\lambda| > ||T||^{-1}$ the operator λT can be non-hypercyclic. As a matter of fact, the only hope lies on the scalars of $\mathbb{T}$, as is shown next. In [4] the author shows that on any separable infinite-dimensional Banach space there exists a hypercyclic operator T which is a compact perturbation of the identity I, that is, $T = I + K$ for some compact operator K. In turn, Chan and Shapiro [6, p. 1446] had proved that if X is a complex Banach space and $T = I + K$ is hypercyclic, with K compact, then K must be quasi-nilpotent, i.e., its spectrum $\sigma(K)$ collapses to $\{0\}$. Hence $\sigma(T) = \{1\}$ and $\sigma(\lambda T) = \{\lambda\}$ ($\lambda \in \mathbb{C}$). On the other hand, it is known that each component of the spectrum of a hypercyclic operator intersects $\mathbb{T}$, see [9, Th. 2.8]. We conclude that for every $\lambda \in \mathbb{C} \setminus \mathbb{T}$ the operator λT is not hypercyclic. Thus it is natural to ask whether λT is hypercyclic for all $\lambda \in \mathbb{T}$ whenever T is hypercyclic. We point out that the answer is trivially affirmative if T is hereditarily hypercyclic, that is, if (T^{n_k}) is hypercyclic for every strictly increasing sequence $(n_k) \subset \mathbb{N}$ (see the end paragraph of this section for an extension). Since λT is hypercyclic if and only if the sequence of operators $(c_n T_n)$ is universal (where $c_n = \lambda^n$ and $T_n = T^n$), then a second question arises in a more general setting: If (T_n) is universal and (c_n) is a scalar sequence, is $(c_n T^n)$ universal? A special case of this situation was considered in [3].

We finish this section with some considerations on hereditary hypercyclicity. In this case everything "works rather well" about multiplicative sequences and multiples.

Proposition. *Let X be a topological space, Y a topological vector space and $(T_n) \subset C(X, Y)$ a sequence with the following weak heredity condition: There is a subsequence (T_{n_k}) so that every subsequence (T_{m_k}) of (T_{n_k}) is universal. Then $(c_n T_n)$ is*

$$0 < \liminf_{n\to\infty} |c_n| \le \sup_{n\in\mathbb{N}} |c_n| < \infty.$$

Proof. Let us fix the subsequence (n_k) of the hypothesis and consider (c_{n_k}). By our assumption on the sequence (c_n), the sequence (c_{n_k}) has a convergent subsequence

$$c_{m_k} \to \gamma \neq 0.$$

Moreover, since (T_{m_k}) is universal, there is an element $x \in X$ that is universal with respect to this sequence of mappings. We can now show that x is also universal for $(c_{m_k} T_{m_k})$, hence also for $(c_n T_n)$. To see this, let $y \in Y$. Then there is there is a subsequence (p_k) of (m_k) with

$$T_{p_k} \to \frac{y}{\gamma},$$

hence

$$c_{p_k} T_{p_k} x \to y.$$

This shows that $(c_n T_n)$ is universal. $\Diamond$

Corollary. *Let X be a complex topological vector space and $T \in L(X)$. Assume that T satisfies the following weak hereditary hypercyclicity con-*

dition: There is a subsequence (T^{n_k}) so that every subsequence (T^{m_k}) of (T^{n_k}) is universal. Then λT is hypercyclic for every $\lambda \in \mathbb{T}$.

Now, by a recent result of J. P. Bès and A. Peris [5], every hypercyclic operator T on a completely metrizable topological vector space X satisfies the well-known Hypercyclicity Criterion if and only if T has the mentioned weak hereditary hypercyclicity condition. If all hypercyclic operators satisfy that criterion (this has not been proved or disproved up to date) then we would have $B = \mathbb{T}$ in the statement of Th. 9 (see Section 3), at least when X is completely metrizable.

2. Multiplicative sequences

If Λ is a topological space then the set $\Lambda^{\mathbb{N}}$ of Λ-valued sequences will be considered endowed with the product topology. For instance, if Λ is a completely metrizable space then $\Lambda^{\mathbb{N}}$ is also completely metrizable because the distance on $\Lambda^{\mathbb{N}}$ given by

$$\rho(c, d) = \sum_{1}^{\infty} \frac{1}{2^n} \cdot \frac{\chi(c_n, d_n)}{1 + \chi(c_n, d_n)}$$

is complete. Here $c = (c_n)$, $d = (d_n)$ and χ is a complete distance on Λ generating its topology. In particular, $\Lambda^{\mathbb{N}}$ is a Baire space in this case. A special instance is the Fréchet space ω of $\mathbb{K}$-valued sequences, see [10, p. 33]. In a Baire space, a subset is residual if and only if it contains a dense G_δ-subset. Residual subsets are topologically "very large". Assume that (T_n) is a universal sequence. Whether $(c_n T_n)$ is universal or not will depend on the nature of the sequence (c_n), see for instance [3]. Nevertheless, it can be asserted that universality is the most general property, at least when (T_n) is densely universal, see Th. 5 below. Before establishing our result, we need the following universality criterion, which is special case of a result due to Grosse-Erdmann [7, Satz 1.2.2].

Theorem 1. *Suppose that X is a Baire topological space, Y is a second-countable topological space and $(T_n) \subset C(X, Y)$. Then (T_n) is densely universal if and only if the set $\{(x, T_n x) : x \in X, n \in \mathbb{N}\}$ is dense in the topological product $X \times Y$ if and only if the set $U((T_n))$ is residual.*

As a consequence, we get an assertion on the structure of the subset of "good" multiplicative sequences. In fact, we can obtain the following more general result.

Theorem 2. *Assume that* $(T_n) \subset C(X,Y)$, *where* X, Y *are second-countable topological spaces and* X *is, in addition, a Baire space. Let* Λ *be a topological space and consider the corresponding space* $\Lambda^{\mathbb{N}}$ *of* Λ*-valued sequences. Suppose that* $\Phi \in C(\Lambda \times Y, Y)$. *Then we have that*

$$(1) \qquad S := \{(c_n) \in \Lambda^{\mathbb{N}} : (\Phi(c_n, T_n(\cdot))) \quad \text{is densely universal } \}$$

is a G_δ*-subset of* $\Lambda^{\mathbb{N}}$.

Proof. Since X and Y are second-countable, $X \times Y$ is also second-countable. There is therefore a countable open basis $\{V_j : j \in \mathbb{N}\}$ for $X \times Y$. We get from Th. 1 that

$$\begin{aligned} S &= \{(c_n) \in \Lambda^{\mathbb{N}} : \{(x, \Phi(c_n, T_n x)) : x \in X,\, n \in \mathbb{N}\} \text{ is dense in } X \times Y\} \\ &= \{(c_n) \in \Lambda^{\mathbb{N}} : \text{for every } j \in \mathbb{N} \text{ there are } x \in X \text{and } n \in \mathbb{N} \\ &\qquad\qquad\qquad\qquad \text{with } (x, \Phi(c_n, T_n x)) \in V_j\}. \end{aligned}$$

Hence

$$(2) \qquad S = \bigcap_{j \in \mathbb{N}} \bigcup_{\substack{x \in X \\ n \in \mathbb{N}}} \varphi_{x,n}^{-1}(V_j),$$

where each $\varphi_{x,n}$ is the mapping

$$c = (c_n) \in \Lambda^{\mathbb{N}} \mapsto \varphi_{x,n}(c) = (x, \Phi(\pi_n(c), T_n x)) \in X \times Y,$$

π_n being the nth projection $c \in \Lambda^{\mathbb{N}} \mapsto c_n \in \Lambda$, which is continuous. Since both of two components of $\varphi_{x,n}$, namely, $c \in \Lambda^{\mathbb{N}} \mapsto x \in X$ (a constant mapping) and $c \in \Lambda^{\mathbb{N}} \mapsto \Phi(\pi_n(c), T_n x) \in Y$, are continuous, one obtains that every subset $\varphi_{x,n}^{-1}(V_j)$ is open and, by (2), S is a G_δ-subset of $\Lambda^{\mathbb{N}}$. $\Diamond$

At this point it is convenient to recall the following crucial result about product spaces due to Kuratowski and Ulam, see for instance [11, p. 56].

Theorem 3. *Let* Z_1 *and* Z_2 *be Baire spaces with* Z_2 *being second-countable. Let further* A *be a residual subset of* $Z_1 \times Z_2$. *Then the set* $\{z \in Z_1 : A_z$ is residual in $Z_2\}$ *is residual in* Z_1, *where* $A_z = \{z \in Z_2 : : (z, z') \in A\}$.

Theorem 4. *Assume that* $(T_n) \subset C(X,Y)$ *is a densely universal sequence, where* X, Y *are second-countable topological spaces and, in addition,* X *is a Baire space and* Y *is a* T_1*–space with no isolated points. Let* Λ *be a topological space such that* $\Lambda^{\mathbb{N}}$ *is a Baire space. Then the* *following properties are equivalent:*

(a) *There is a residual subset of sequences* $(c_n) \in \Lambda^{\mathbb{N}}$ *such that* $(\Phi(c_n, T_n(\cdot)))$ *is densely universal.*

(b) *There is a sequence* $(c_n) \in \Lambda^{\mathbb{N}}$ *such that* $(\Phi(c_n, T_n(\cdot)))$ *is universal.*

(c) *The mapping* Φ *has dense range.*

Proof. It is obvious that (a) implies (b). Now, if for some sequence $(c_n) \in \Lambda^{\mathbb{N}}$ and some $x \in X$ the set $\{\Phi(c_n, T_n x) : n \in \mathbb{N}\}$ is dense in Y, then the set $\Phi(\Lambda \times Y)$ must also be dense in Y, hence (b) implies (c).

Let us suppose that (c) holds. In order to prove (a), we consider the set S defined by (1). Our goal is to prove that S is residual. We will try to apply Th. 3 with $Z_1 = \Lambda$, $Z_2 = X$ and

$$A = \{((c_n), x) \in \Lambda^{\mathbb{N}} \times X : \{\Phi(c_n, T_n x) : n \in \mathbb{N}\} \text{ is dense in } Y\}.$$

Observe that $S = \{z \in Z_1 : A_z \text{ is residual in } Z_2\}$, because by Th. 1 the sequence $(\Phi(c_n, T_n(\cdot)))$ is densely universal if and only if the set $U((\Phi(c_n, T_n(\cdot))))$ is residual. It is easy to see that A is G_δ since Y is second-countable and the T_n ($n \in \mathbb{N}$) are continuous (the technique would be similar to that of the proof of Th. 2). Therefore we would be done as soon as we prove that A is dense in $\Lambda^{\mathbb{N}} \times X$. To see this, let $O \subset \Lambda^{\mathbb{N}}$ and $U \subset X$ be non-empty open sets. Since (T_n) is densely universal, U contains a universal element x. Further, by the product topology, there is an $N \in \mathbb{N}$ and $O_1, ..., O_{N-1} \subset \Lambda$ non-empty and open so that $O_1 \times \cdots \times O_{N-1} \times \Lambda \times \Lambda \times \cdots \subset O$. Now choose a basis $\{V_k : k \in \in \mathbb{N}\}$ of open subsets in Y. Since Φ has dense range there are $d_k \in \Lambda$ and $y_k \in Y$ with $\Phi(d_k, y_k) \in V_k$. Since $(T_n x)$ is dense, Φ is continuous, and Y has no isolated points and is T_1, there is some $n_k \geq N$ with $\Phi(d_k, T_{n_k} x) \in V_k$. We can assume that (n_k) is increasing. Defining $(c_n) \in \Lambda^{\mathbb{N}}$ with $c_{n_k} = d_k$, $c_i \in O_i$ for $i < N$, and all other c_i arbitrary we see that $(c_n) \in O$, $x \in U$ and $\{\Phi(c_n, T_n x) : n \in \mathbb{N}\}$ is dense in Y. Hence A is dense in $\Lambda^{\mathbb{N}}$. The theorem is now completely proved. $\Diamond$

Now, we obtain as a consequence of Th. 4 the following one. This time the "external law" Φ is the multiplication by scalars.

Theorem 5. *Suppose that* $(T_n) \subset C(X, Y)$ *is densely universal, where* X *is a second-countable Baire topological space and* Y *is a metrizable separable topological vector space. Then there is a residual subset of sequences* $(c_n) \in \omega$ *such that* $(c_n T_n)$ *is densely universal, so universal.*

Proof. It suffices to consider the set $\Lambda = \mathbb{K}$ together with the mapping $\Phi : (\lambda, y) \in \mathbb{K} \times Y \mapsto \lambda y \in Y$ and to apply Th. 4. Observe that Φ is onto, hence it has dense range. ◊

Corollary 6. *Given a universal selfmapping $T \in C(X)$, where X is a Baire metrizable topological vector space, there is a residual subset of sequences $(c_n) \in \omega$ such that $(c_n T^n)$ is densely universal, so universal.*

Proof. Just take into account that if T is universal, then it is densely universal and X is second-countable. ◊

3. Multiples of hypercyclic operators

In order to deal with the problem of unimodular multiples of hypercyclic operators, two preliminary statements are needed. The first one is similar to Th. 2.

Lemma 7. *Suppose that Λ, X and Y are topological spaces, in such a way that X, Y are second-countable and X is a Baire space. Assume that $\Phi_n \in C(\Lambda \times X, Y)$ $(n \in \mathbb{N})$. Then the subset*

$$B = \{\lambda \in \Lambda : \ (\Phi_n(\lambda, \cdot)) \ \text{ is densely universal}\}$$

is a G_δ-subset of Λ.

Proof. Pick anew a countable open basis $\{V_j : j \in \mathbb{N}\}$ for $X \times Y$. From Th. 1, $B = \{\lambda \in \Lambda : \{(x, \Phi_n(\lambda, x)) : x \in X, n \in \mathbb{N}\}$ is dense in $X \times Y\} = \{\lambda \in \Lambda :$ for every $j \in \mathbb{N}$ there are $x \in X$ and $n \in \mathbb{N}$ with $(x, \Phi_n(\lambda, x)) \in V_j\}$. Hence

$$B = \bigcap_{j \in \mathbb{N}} \bigcup_{\substack{x \in X \\ n \in \mathbb{N}}} \psi_{x,n}^{-1}(V_j),$$

where each $\psi_{x,n}$ is the mapping

$$\lambda \in \Lambda \mapsto \psi_{x,n}(\lambda) = (x, \Phi_n(\lambda, x)) \in X \times Y.$$

Its first component $\lambda \mapsto x$ is constant, so continuous. Its second component is continuous because Φ is. Hence $\psi_{x,n}$ is continuous and every set $\psi_{x,n}^{-1}(V_j)$ is open in Λ. Thus, B is a G_δ-subset. ◊

The second preliminary statement is the following important property due to S. I. Ansari, see [1, Th. 1] and [2, Note 3].

Theorem 8. *Let T be a hypercyclic operator on a locally convex space X. Then T^m is hypercyclic for every $m \in \mathbb{N}$.*

In fact, $U(T^m) = U(T)$ under the hypotheses of the latter theorem. We conclude with our result on multiples of hypercyclic operators. For the sake of simplicity, we call a complex number λ of $\mathbb{T}$ *rational* whenever there is $r \in \mathbb{Q}$ such that $\lambda = \exp(2\pi i r)$. Note that $\mathbb{T}$ is a Baire space when it is endowed with the euclidean topology.

Theorem 9. *Let X be a Baire metrizable complex locally convex space and $T \in L(X)$. Assume that T is hypercyclic. Denote*

$$B = \{\lambda \in \mathbb{T} : \lambda T \text{ is hypercyclic}\}.$$

Then the following properties are satisfied:

(a) *If $\lambda \in \mathbb{T}$ is rational then $\lambda \in B$.*
(b) *B is residual in $\mathbb{T}$.*
(c) *If $\lambda \in \mathbb{T}$ is rational then $\lambda B = B$.*
(d) *If B has nonempty interior then $B = \mathbb{T}$.*

Proof. Again, X must be separable because T is hypercyclic. Denote by D the set of $\lambda \in \mathbb{T}$ such that λ is rational.

(a) If $\lambda \in D$ then there are integers p, q with $q \in \mathbb{N}$ such that $\lambda = \exp(2\pi ip/q)$, hence $(\lambda T)^q = \lambda^q T^q = T^q$. Since by Th. 8 the operator T^q is hypercyclic, one obtains that $(\lambda T)^q$ is hypercyclic, but then λT is also hypercyclic trivially, i.e., $\lambda \in B$.

(b) Observe that $B = \{\lambda \in \mathbb{T} : \lambda T \text{ is densely universal}\} = \{\lambda \in \mathbb{T} : (\lambda^n T^n) \text{ is densely universal}\}$. By Lemma 7 applied on $\Lambda = \mathbb{T}$, $Y = X$, $\Phi_n(\lambda, x) = \lambda^n T^n x$ ($n \in \mathbb{N}$, $x \in X$), we are done whenever we are able to show that B is dense in $\mathbb{T}$. To see this, consider the subset $D = \{\exp(2\pi ir) : r \in \mathbb{Q}\}$, which is trivially dense in $\mathbb{T}$. But $D \subset B$ from (a), so B is dense.

(c) Assume that $\lambda \in D$ and $\mu \in B$. As in (a), there is $q \in \mathbb{N}$ with $\lambda^q = 1$. Therefore $(\lambda\mu T)^q = \lambda^q \mu^q T^q = \mu^q T^q = (\mu T)^q$, which is again hypercyclic by Th. 8. Then $(\lambda\mu T)^q$ is hypercyclic and, trivially, $\lambda\mu T$ is hypercyclic. But this tells us that $\lambda\mu \in B$, so $\lambda B \subset B$. The reverse inclusion $B \subset \lambda B$ is derived from the fact that $\lambda \in D$ if and only if $\lambda^{-1} \in D$.

(d) By hypothesis, there is an interval (a, b) with $0 < a < b < 1$ such that $B \supset M := \{\exp(2\pi ix) : a < x < b\}$. Choose a positive integer q with $q > 1/(b-a)$ and consider the point $\lambda = \exp(2\pi i/q) \in D$. It is clear that

$$\bigcup_{j=0}^{q-1} \lambda^j M = \mathbb{T}.$$

But, for each $j \in \{0, 1, ..., q-1\}$, $\lambda^j M \subset \lambda^j B = B$ by part (c). The conclusion follows from this. ◊

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