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COMMON FIXED POINT THEOREMS IN CONNECTION TO SOME RESULTS OF J. MATKOWSKI

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Abstract: The aim of this note is to prove two common fixed point theorems in complete metric spaces for a set of self-maps in connection to some fixed point theorems due to J. Matkowski.

1. Introduction

In the paper [7], J. Matkowski (see also Meir-Keeler [9]) has given the following generalization of Banach's principle,

Theorem 1.1. *Let T be a self-mapping of a complete metric space (X, d) and let*

$$(\mathrm{M}_1) \qquad d(Tx, Ty) < d(x, y) \quad \textit{for all} \quad x, y \in X,\ x \neq y.$$

If for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$(\mathrm{M}_2) \qquad \epsilon < d(x, y) < \epsilon + \delta \quad \textit{implies} \quad d(Tx, Ty) \leq \epsilon$$

then T has a unique fixed point; moreover its domain of attraction coincides with the whole of X.

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J. Matkowski has also generalized in the paper [7] a theorem due to Boyd and Wong (see [1]) by the following,

Theorem 1.2. *Let T be a self-mapping of a complete metric space (X, d) such that $d(Tx, Ty) \leq \gamma(d(x, y))$, $x, y \in X$, for some increasing function $\gamma : [0, +\infty[\longrightarrow [0, +\infty[$ fulfilling the condition $\lim_{n\to+\infty} \gamma^n(t) = 0$ for all $t > 0$. Then there exists exactly one fixed point of T and its domain of attraction coincides with the whole of X.*

The object of this note is to provide some common fixed point theorems in connection to the Ths. 1.1, and 1.2. This paper is organized as follows. First we establish Th. 2.1 which may be considered as a generalization of Th. 1.1. We emphasize here that while condition (M_1) forces the mapping T to be continuous on the metric space X, the mappings involved in our result (Th. 2.1) need not to be continuous. So, we observe that no continuity arguments are used, making a difference between this theorem and Th. 1.1 of Matkowski. Secondly we provide Th. 2.2 which generalizes to some extent Th. 1.2 of Matkowski.

2. Results

Theorem 2.1. *Let S, T be two self-mappings of a complete metric space (X, d) and let*

(A_1) $\qquad d(Sx, Ty) \leq d(x, y)$ *for all* $x, y \in X$, $x \neq y$.

If for every $\epsilon > 0$ there exists a $\delta > 0$ such that

(A_2) $\qquad \epsilon \leq d(x, y) < \epsilon + \delta$ *implies* $d(Sx, Ty) < \epsilon$

then S and T have a unique common fixed point $z \in X$. Moreover $\mathrm{Fix}(S) = \mathrm{Fix}(T) = \{z\}$.

Proof. (i) We begin with the construction of a sequence of X. Let x_0 be some point in X. We define

$$x_{2n} = Sx_{2n-1}, \quad n = 1, 2, \ldots$$
$$x_{2n+1} = Tx_{2n}, \quad n = 0, 1, 2, \ldots$$

We put $t_n := d(x_n, x_{n+1})$ for all integer n and we prove that $t_{n_0} = 0$ for some integer n_0. Therefore, we may assume that $t_n > 0$ for all integer n. By using property (A_1), we see that for an each even integer n, we have

$$(1) \quad \begin{aligned} t_n = d(Sx_{n-1}, Tx_n) \leq d(x_{n-1}, x_n) = t_{n-1} = d(Tx_{n-2}, Sx_{n-1}) \leq \\ \leq d(x_{n-2}, x_{n-1}) = t_{n-2}. \end{aligned}$$

The inequalities in (1) show that sequence $(t_n)_n$ is decreasing. Let t be the limit of this sequence and suppose that $t > 0$. Then there exists an even integer n_0 such that $t \leq t_{n_0} < t + \delta(t)$. By the property ($A_2$), we get $d(Tx_{n_0}, Sx_{n_0+1}) = t_{n_0+1} < t$. This is a contradiction. Therefore $t = 0$. We deduce that the sequence $(t_n)_n$ converges to zero.

(ii) Now, we shall prove that $\{x_n\}$ is a Cauchy sequence. Without loss of generality, we can suppose that $0 < \delta(\epsilon) < \epsilon$, for each $\epsilon > 0$ and that neither the sequence $\{x_{2n}\}$ nor the sequence $\{x_{2n+1}\}$ is stationary. Let us fixe an $\epsilon > 0$, then we can find an integer $k_0 > 1$ such that $2r_n < < \delta(\epsilon)$ for each integer $n \geq 2k_0$, where $r_n := t_{n-1} + t_n + t_{n+1}$. We set $\mathcal{B}_0 := \{x \in X : d(x, x_{2k_0}) < \epsilon + \delta(\epsilon)\}$. We shall prove that $\mathcal{B}_0$ is invariant under S and T.

(a) First we prove that $S(\mathcal{B}_0) \subset \mathcal{B}_0$. So let $x \in \mathcal{B}_0$.

If $x = x_{2k_0}$ then (since $x_{2k_0} \neq x_{2k_0+2}$) we have the following inequalities,

$$(2) \quad \begin{aligned} d(Sx_{2k_0}, x_{2k_0}) &\leq d(Sx_{2k_0}, Tx_{2k_0+2}) + d(Tx_{2k_0+2}, x_{2k_0}) \leq \\ &\leq d(x_{2k_0}, x_{2k_0+2}) + d(x_{2k_0+3}, x_{2k_0}) < \\ &< 2r_{2k_0+1} < \delta(\epsilon). \end{aligned}$$

If $x \neq x_{2k_0}$, then there are two cases:

If $0 < d(x, x_{2k_0}) < \epsilon$, then

$$\begin{aligned} d(Sx, x_{2k_0}) &\leq d(Sx, Tx_{2k_0}) + d(Tx_{2k_0}, x_{2k_0}) \leq \\ &\leq d(x, x_{2k_0}) + d(x_{2k_0+1}, x_{2k_0}) < \\ &< \epsilon + r_{2k_0} < \epsilon + \delta(\epsilon). \end{aligned}$$

If $\epsilon \leq d(x, x_{2k_0}) < \epsilon + \delta(\epsilon)$ then, by property (A_2), we have $d(Sx, x_{2k_0+1}) < \epsilon$. We deduce then that

$$d(Sx, x_{2k_0}) \leq d(Sx, x_{2k_0+1}) + d(x_{2k_0+1}, x_{2k_0}) < \epsilon + \delta(\epsilon).$$

(b) Secondly we prove that $T(\mathcal{B}_0) \subset \mathcal{B}_0$. Take x in $\mathcal{B}_0$. If $x = x_{2k_0}$ then we have $Tx = x_{2k_0+1} \in \mathcal{B}_0$. If $x = x_{2k_0-1}$ then since the sequence $(x_{2n+1})_n$ is not stationary, we can write

$$\begin{aligned} d(Tx_{2k_0-1}, x_{2k_0}) &\leq d(Tx_{2k_0-1}, Sx_{2k_0+1}) + d(Sx_{2k_0+1}, x_{2k_0}) \leq \\ &\leq d(x_{2k_0-1}, x_{2k_0+1}) + d(x_{2k_0+2}, x_{2k_0}) \leq \\ &\leq 2r_{2k_0} < \delta(\epsilon). \end{aligned}$$

It remains to handle two cases.

If $0 < d(x, x_{2k_0}) < \epsilon$, and $x \neq x_{2k_0-1}$, then we have

$$d(Tx, x_{2k_0}) = d(Tx, Sx_{2k_0-1}) < d(x, x_{2k_0-1}) \leq$$
$$\leq d(x, x_{2k_0}) + d(x_{2k_0}, x_{2k_0-1}) < \epsilon + \delta(\epsilon).$$

If $\epsilon \leq d(x, x_{2k_0}) < \epsilon + \delta(\epsilon)$ then according to property (A_2), we have $d(Tx, Sx_{2k_0}) < \epsilon$, and by using inequalities (2), we get

$$d(Tx, x_{2k_0}) \leq d(Tx, Sx_{2k_0}) + d(Sx_{2k_0}, x_{2k_0}) < \epsilon + \delta(\epsilon).$$

As a consequence, we deduce that $x_n \in \mathcal{B}_0$, for all integer $n \geq 2k_0$. Thus for all $p, q \geq 2k_0$, we obtain

$$d(x_p, x_q) \leq d(x_p, x_{2k_0}) + d(x_{2k_0}, x_q) < 2[\epsilon + \delta(\epsilon)] < 4\epsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d), thus one may find a point $z = z(S, T) \in X$ such that $x_n \longrightarrow z$ as $n \longrightarrow +\infty$.

(iii) We shall prove that z is a common fixed point for S and T. Since the sequence $(x_{2n})_n$ is not stationary, it must contain a subsequence $(x_{2n(k)})_k$ such that $x_{2n(k)} \neq z$ for every integer k. Therefore for each integer k, we may write

$$\begin{aligned} d(z, Sz) &\leq d(z, x_{2n(k)+1}) + d(x_{2n(k)+1}, Sz) \leq \\ &\leq d(z, x_{2n(k)+1}) + d(Tx_{2n(k)}, Sz) \leq \\ &\leq d(z, x_{2n(k)+1}) + d(x_{2n(k)}, z). \end{aligned} \tag{3}$$

We let $k \longrightarrow +\infty$ in (3) to get $Sz = z$. In a similar manner, we obtain $Tz = z$.

(iv) Uniqueness of z. Suppose that there exists another point $\xi \neq$ $\neq z$ fixed, for instance, by S. We use property (A_2) with $\epsilon := d(\xi, z)$, and get

$$d(\xi, z) = d(S\xi, Tz) < d(\xi, z),$$

a contradiction.

It is proved that there exists a unique point $z \in X$ such that $\mathrm{Fix}(S) = \{z\} = \mathrm{Fix}(T) = \mathrm{Fix}(\{S, T\})$. This completes the proof of our theorem. $\Diamond$

Now, we establish a common fixed point theorem connected to Th. 1.2 of Matkowski. We point out that our result belongs to the class of common fixed point theorems obtained by altering the distances between the points with the use of some suitable functions. Many works (see the references) were devoted to this field of investigations in the fixed point theory. Here we want to contribute by the following

Theorem 2.2. *Let S, T be two self-mappings of a complete metric space (X, d) such that*

$$\text{(B)} \qquad \phi(d(Sx, Ty)) \leq \gamma\big(\phi(d(x, y))\big) \quad \forall x, y \in X \quad \textit{with} \quad x \neq y,$$

where $\gamma, \phi : [0, +\infty[\longrightarrow [0, +\infty[$ are two continuous functions fulfilling the following conditions:
(W_1) $\gamma(t) < t$ *for all* $t > 0$.
(W_2) ϕ *is increasing on* $[0, +\infty[$ *and* $\phi(t) = 0 \Longleftrightarrow t = 0$.
Then S and T have a unique common fixed point $z \in X$. Moreover $\mathrm{Fix}(S) = \mathrm{Fix}(T) = \{z\}$.

Proof. (i) We consider the same sequence as in Proof of Th. 2.1 (a): $x_0 \in X$; $x_{2n} = Sx_{2n-1}$, $(n = 1, 2, \ldots)$; $x_{2n+1} = Tx_{2n}$, $(n = 0, 1, 2, \ldots)$.

We put $t_n := d(x_n, x_{n+1})$ for all integer n and we prove that $t_{n_0} = 0$ for some integer n_0. Therefore, we may assume that $t_n > 0$ for all integer n. In this case, by property (W_2), we get $\phi(t_n) > 0$ for all integer n. By using the assumptions (B) and (W_1) we obtain, for every even integer $n \geq 2$, the following inequalities

$$\begin{aligned} & \phi(t_n) = \phi(d(Sx_{n-1}, Tx_n)) \leq \gamma(\phi(t_{n-1})) < \\ (4) \qquad & < \phi(t_{n-1}) = \phi(d(Sx_{n-1}, Tx_{n-2})) \leq \\ & \leq \gamma(\phi(t_{n-2})) < \phi(t_{n-2}). \end{aligned}$$

Since ϕ is increasing, the inequalities in (4) show that the sequence $(t_n)_n$ is decreasing. Let t be the limit of $(t_n)_n$, and let us suppose that $t > 0$. By the continuity of ϕ and γ, we obtain from the first inequality in (4), and property (W_1) that $0 < \phi(t) \leq \gamma(\phi(t)) < \phi(t)$, which is a contradiction. Thus we must have $t = 0$.

(ii) Now, we shall prove that $\{x_n\}$ is a Cauchy sequence. Since $t = 0$ one needs only to see that $\{x_{2n}\}$ is a Cauchy sequence. To get a contradiction, let us suppose that there is a number $\epsilon > 0$ and two sequences $\{2n(k)\}$, $\{2m(k)\}$ with $2k \leq 2m(k) < 2n(k)$, $(k \in \mathbb{N})$ verifying

$$(5) \qquad d(x_{2n(k)}, x_{2m(k)}) > \epsilon.$$

For each integer k, we shall denote $2n(k)$ the least even integer exceeding $2m(k)$ for which (5) holds. Then

$$d(x_{2m(k)}, x_{2n(k)-2}) \leq \epsilon \quad \text{and} \quad d(x_{2m(k)}, x_{2n(k)}) > \epsilon.$$

For each integer k, we shall put $p_k := d(x_{2m(k)}, x_{2n(k)})$, $s_k := d(x_{2m(k)}, x_{2n(k)+1})$, $q_k := d(x_{2m(k)+1}, x_{2n(k)+1})$, and $r_k := d(x_{2m(k)+1}, x_{2n(k)+2})$, then by using triangular inequalities, we obtain

$$(6)\qquad \begin{aligned} &\epsilon < p_k \le \epsilon + t_{2n(k)-2} + t_{2n(k)-1} \\ &| s_k - p_k | \le t_{2n(k)}, | q_k - s_k | \le t_{2m(k)}, | r_k - s_k | \le t_{2n(k)+1}. \end{aligned}$$

Since the sequence $\{t_n\}$ converges to 0, we deduce from (6) that the sequences $\{p_k\}, \{s_k\}, \{q_k\}$ and $\{r_k\}$ are converging to ϵ. From (5) and these facts, one can deduce that there exists an integer k_0 such that $d(x_{2n(k)+1}, x_{2m(k)}) > 0$, and $\frac{\epsilon}{2} \le p_k - t_{2k} \le d(x_{2n(k)+1}, x_{2m(k)})$, for each integer $k \ge k_0$. Therefore, (for all $k \ge k_0$) we have

$$(7)\qquad \begin{aligned} \phi(r_k) = \phi(d(x_{2n(k)+2}, x_{2m(k)+1})) = \phi(d(Sx_{2n(k)+1}, Tx_{2m(k)})) \le \\ \le \gamma(\phi(s_k)). \end{aligned}$$

We let $k \to \infty$, in (7) and use the properties (W_1) and (W_2), together with the continuity of the functions ϕ and γ to get

$$(8)\qquad 0 < \phi(\epsilon) \le \gamma(\phi(\epsilon)) < \phi(\epsilon).$$

In (8) we have a contradiction. Hence $\{x_n\}$ is the Cauchy sequence in a complete metric space (X, d), thus it converges to a unique point $z = z(S,T) \in X$.

(iii) We shall prove that z is a common fixed point for S and T. Since $t_n > 0$ for all integer n, we see that both subsequences $(x_{2n})_n$ and $(x_{2n+1})_n$ are not stationary. Therefore, we may find a subsequence $(x_{n(k)})_k$ such that $x_{2n(k)+1} \neq z$ for every integer k. Let us suppose that $Tz \neq z$. In this case we can apply the inequality (B) and obtain for all $k \in \mathbb{N}$,

$$(9)\quad \phi(d(x_{2n(k)+2}, Tz)) = \phi(d(Sx_{2n(k)+1}, Tz)) \le \gamma(\phi(d(x_{2n(k)+1}, z))).$$

After letting $k \longrightarrow \infty$, (9) gives

$$(10)\qquad 0 < \phi(d(z, Tz)) \le \gamma(\phi(d(z, Tz))) < \phi(d(z, Tz)),$$

which is a contradiction. Hence $z = Tz$, and in a similar way, it can be shown that $z = Sz$.

(iv) Uniqueness of z. Suppose that there exists another point $y \neq z$ fixed, for instance, by S. Then by the property (B), we have

$$0 < \phi(d(y, z)) = \phi(d(Sy, Tz)) \le \gamma(\phi(d(y, z))) < \phi(d(y, z)),$$

a contradiction. Therefore, we deduce that there exists a unique point $z \in X$ such that $\mathrm{Fix}(S) = \{z\} = \mathrm{Fix}(T) = \mathrm{Fix}(\{S, T\})$. This completes the proof of our theorem. ◊

Remark 2.3. We emphasize that the function ϕ used in Th. 2.3 need not to be strictly increasing.

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