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APPROXIMATELY GENERALIZED CONVEX FUNCTIONS

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Abstract: D.H. Hyers and S.M. Ulam [3] (cf. [2], [5]) have proved the following theorem: If $g : D \to \mathbb{R}$ ($D \subset \mathbb{R}^n$, D open and convex) is an ϵ-convex function, i.e.

$$g(tx + (1-t)y) \le tg(x) + (1-t)g(y) + \epsilon, \quad t \in [0,1],\ x, y \in D,$$

then there exists a convex function $f : D \to \mathbb{R}$ such that

$$|f(x) - g(x)| \le M\epsilon, \quad x \in D,$$

where the constant M depends only on n. We consider this problem for generalized convexity (in Beckenbach sense).

1. Generalized convex functions

In this section we repeat, for the convenience of the reader, two definitions and two theorem from [1].

Definition 1. A family F of continuous real-valued functions φ, defined on an open interval (a,b) is said to be a *two-parameter family* on (a,b) if for any distinct points x_1, x_2 in (a,b) and any numbers y_1, y_2 there exists exactly one $\varphi \in F$ satisfying

$$\varphi(x_i) = y_i, \quad i = 1, 2.$$

Throughout the paper we assume F is a two-parameter family on (a,b).

Definition 2. We say that a function $\psi : (a,b) \to \mathbb{R}$ is *convex (concave) function with respect to the family* F if for any points $a < x_1 < x_2 < b$ the unique $\varphi \in F$ determined by

$$\varphi(x_i) = \psi(x_i), \quad i = 1,2$$

satisfies the inequality

$$\psi(x) \lesseqgtr \varphi(x), \quad x \in [x_1, x_2].$$

Theorem 1. *Let φ_1, φ_2 be distinct elements of the family F and let $c \in (a,b)$. If $\varphi_1(c) = \varphi_2(c)$, then either*

$$\varphi_1(x) > \varphi_2(x), x \in (a,c) \quad \text{and} \quad \varphi_1(x) < \varphi_2(x), x \in (c,b)$$

or

$$\varphi_1(x) < \varphi_2(x), x \in (a,c) \quad \text{and} \quad \varphi_1(x) > \varphi_2(x), x \in (c,b).$$

Theorem 2 (cf. [6]). *Let*

$$a < x_1^n < x_2^n < b \quad \text{and} \quad y_1^n, y_2^n \text{ be real numbers,}$$

for $n = 0,1,2,\ldots$, such that

$$x_i^0 = \lim_{n\to\infty} x_i^n, y_i^0 = \lim_{n\to\infty} y_i^n, \quad i = 1,2.$$

Let φ_n, where $n = 0,1,2,\ldots$, be the element of F determined by the relations

$$\varphi_n(x_i^n) = y_i^n, \quad i = 1,2.$$

Then $\varphi_n \to \varphi_0$ uniformly on every compact subinterval of (a,b).

2. Generalized convex sets

First we give two definitions and one theorem from [4]. Let $A, B \in \in (a,b) \times \mathbb{R}, A = (x_1, y_1), B = (x_2, y_2)$. If $x_1 = x_2$, then

$$[A,B] := \{(x_1, y) : y_1 \le y \le y_2\}, y_1 \le y_2,$$
$$[A,B] := \{(x_1, y) : y_2 \le y \le y_1\}, y_1 > y_2.$$

If $x_1 \neq x_2$, then

$$[A,B] := \{(x, \varphi(x)) : x_1 \le x \le x_2\}, x_1 < x_2,$$
$$[A,B] := \{(x, \varphi(x)) : x_2 \le x \le x_1\}, x_1 > x_2,$$

where $\varphi \in F$ is determined by

$$\varphi(x_i) = y_i, \quad i = 1,2.$$

Definition 3. A set $D \subset (a,b) \times \mathbb{R}$ will be called *convex with respect to the family* F (or briefly F-convex) iff for any $A, B \in D$ we have

$$[A,B] \subset D.$$

Definition 4. Let $D \subset (a,b) \times \mathbb{R}$. The set

$$\text{conv}_F D := \bigcap \{U \subset (a,b) \times R : U \text{ is } F\text{-convex}, \quad D \subset U\}$$

is called the *convex hull of* D *with respect to the family* F.

Theorem 3. *Let* $D, D_1, D_2 \subset (a,b) \times \mathbb{R}$. *Then*

1. *if* D *is* F*-convex, then* $\text{int}\, D$ *and* $\text{cl}\, D$ *are* F*-convex,*
2. $D \subset \text{conv}_F D$,
3. $\text{conv}_F D$ *is the smallest* F*-convex set containing* D,
4. D *is* F*-convex set iff* $D = \text{conv}_F D$,
5. *if* $D_1 \subset D_2$, *then* $\text{conv}_F D_1 \subset \text{conv}_F D_2$.

The Carathéodory theorem is well known in the theory of convex sets. Now we give a similar one. Let $D \subset (a,b) \times \mathbb{R}$ and let

$$D_1 := D, D_2 := \cup\{[A,B] : A, B \in D\}, D_3 :=$$
$$:= \cup\{[A,B] : A \in D, B \in D_2\}.$$

Theorem 4. $\text{conv}_F D = D_1 \cup D_2 \cup D_3$.

This theorem asserts that any point of the set $\text{conv}_F D$ is a "combination" of at most three points from D.

To prove this theorem we need the following two lemmas.

Lemma 1. *Let* $A, B \in D_3$. *Then for every* $C \in [A,B]$ *there exist* $\bar{A} \in D_2, \bar{B} \in D_3$ *such that* $C \in [\bar{A}, \bar{B}]$.

Proof. Let $A, B \in D_3, C \in [A,B]$ and let $A = (x_A, y_A), B = (x_B, y_B)$. Without loss of generality we may assume that $A \neq B$. We consider two cases:

1. $x_A \neq x_B$,
2. $x_A = x_B$.

1. Let for example $x_A < x_B$. Since $A \in D_3$, there exist $A_1, A_2, A_3 \in \in D$ and $A_4 \in [A_1, A_2]$ such that $A \in [A_3, A_4]$. Let $\varphi \in F$ be determined by

$$\varphi(x_A) = y_A, \quad \varphi(x_B) = y_B.$$

Then

$$[A,B] = \{(x, \varphi(x)) : x_A \leq x \leq x_B\}.$$

It is easily seen that there exists

$$\bar{A} \in [A_1, A_2] \cup [A_1, A_3] \cup [A_2, A_3], \bar{A} = (\bar{x}, \bar{y})$$

such that $\bar{x} \le x_A, \varphi(\bar{x}) = \bar{y}$. Hence we have

$$[\bar{A}, B] = \{(x, \varphi(x)) : \bar{x} \le x \le x_B\}.$$

and, as simple consequence, $[A, B] \subset [\bar{A}, B]$. Since $A_1, A_2, A_3 \in D$, we have

$$[A_1, A_2] \cup [A_1, A_3] \cup [A_2, A_3] \subset D_2.$$

Therefore $\bar{A} \in D_2$. Consequently $C \in [\bar{A}, B]$ and $\bar{A} \in D_2, B \in D_3$.

2. In this case $y_A \ne y_B$, because $A \ne B$. Let $y_A > y_B$. Then

$$[A, B] = \{(x_A, y) : y_B \le y \le y_A\}.$$

Let A_1, A_2, A_3, A_4 be as in the case 1. Analysis similar to that in the case 1 shows that there exists

$$\bar{A} \in [A_1, A_2] \cup [A_1, A_3] \cup [A_2, A_3], \bar{A} = (\bar{x}, \bar{y})$$

such that $\bar{x} = x_A, \bar{y} \ge y_A$ and $[A, B] \subset [\bar{A}, B]$. Therefore $C \in [\bar{A}, B]$ and $\bar{A} \in D_2$, $B \in D_3$. This proves the lemma. ◊

Lemma 2. *Let $A \in D_2, B \in D_3$. Then for every $C \in [A, B]$ there exist $\bar{A} \in D$, $\bar{B} \in D_3$ such that $C \in [\bar{A}, \bar{B}]$.*

Proof. Let $A \in D_2, B \in D_3, C \in [A, B]$ and let $A = (x_A, y_A), B = (x_B, y_B)$, $C = (x_C, y_C)$. Since $A \in D_2$ and $B \in D_3$ there exist $A_1, A_2, B_1, B_2, B_3 \in D$ and $B_4 \in [B_1, B_2]$ such that

$$A \in [A_1, A_2], B \in [B_3, B_4].$$

Let $A_i = (x_i, y_i), i = 1, 2$ and let $B_i = (x_{B_i}, y_{B_i}), i = 1, 2, 3, 4$. Without restriction of generality we may assume that $A \ne B$,$C \ne A$,$C \ne B$,$A_1 \ne A_2$,$A \ne A_1$ and $A \ne A_2$. We consider the following cases:

1. $x_{A_1} = x_{A_2}$,
2. $x_{A_1} \ne x_{A_2}$.

1. In this case $x_A = x_{A_1}$ and $y_{A_1} \ne y_{A_2}$, because $A_1 \ne A_2$. Let $y_{A_1} < y_{A_2}$. Then

$$y_{A_1} < y_A < y_{A_2} \tag{1}$$

$(A \ne A_1, A \ne A_2)$ and

$$[A_1, A_2] = \{(x_A, y) : y_{A_1} \le y \le y_{A_2}\}.$$

If $x_A = x_B$, then $x_C = x_A$ and $C \in [A_2, B]$ (if $y_C < y_A$) or $C \in [A_1, B]$ (if $y_C > y_A$).

Let $x_A \ne x_B$. Let for example $x_A < x_B$. Then $x_A < x_C < x_B$, because $C \ne A$ and $C \ne B$. Let $\varphi, \varphi_1, \varphi_2 \in F$ be determined by

$$\varphi(x_A) = y_A, \qquad \varphi(x_B) = y_B,$$

(2) $$\varphi_1(x_{A_1}) = y_{A_1}, \qquad \varphi_1(x_C) = y_C,$$

(3) $$\varphi_2(x_{A_2}) = y_{A_2}, \qquad \varphi(x_C) = y_C,$$

respectively. It follows from the definition of φ and from $C \in [A, B]$ that $\varphi(x_C) = y_C$. Hence, from (1) and from the definitions of $\varphi, \varphi_1, \varphi_2$ we have

$$\varphi(x_C) = \varphi_1(x_C) = \varphi_2(x_C), \varphi_1(x_A) < \varphi(x_A) < \varphi_2(x_A).$$

Therefore we have by Th. 1,

$$\varphi_1(x) < \varphi(x) < \varphi_2(x), \qquad x \in (a, x_C),$$
$$\varphi_1(x) > \varphi(x) > \varphi_2(x), \qquad x \in (x_C, b).$$

Set

$$H := \{(x, y) : x > x_C, \quad \varphi_2(x) < y < \varphi_1(x)\}.$$

Obviously, $B \in H$ and $B \in [B_3, B_4] \subset D_3$.

If $[B_3, B_4] \not\subset H$, then there exists $G \in [B_3, B_4]$, $G = (x_G, y_G)$ such that $x_G \geq x_C$ and $\varphi_1(x_G) = y_G$ or $\varphi_2(x_G) = y_G$. This means that $C \in [A_1, G]$ and $A_1 \in D$, $G \in [B_3, B_4] \subset D_3$ or $C \in [A_2, G]$ and $A_2 \in D, G \in [B_3, B_4] \subset D_3$.

If $[B_3, B_4] \subset H$, then $B_3 \in H, B_3 = (x_3, y_3)$. Thus

(4) $$x_3 > x_C, \varphi_2(x_3) < y_3 < \varphi_1(x_3).$$

Let $\varphi_3 \in F$ be determined by

(5) $$\varphi_3(x_C) = y_C, \quad \varphi_3(x_3) = y_3.$$

From (2), (3), (4) and from (5) we get

$$\varphi_1(x_C) = \varphi_2(x_C) = \varphi_3(x_C), \varphi_2(x_3) < \varphi_3(x_3) < \varphi_1(x_3).$$

Hence

$$\varphi_1(x) < \varphi_3(x) < \varphi_2(x), \quad x \in (a, x_C),$$

by Th. 1. In particular

$$\varphi_1(x_A) < \varphi_3(x_A) < \varphi_2(x_A).$$

This means that the point $E := (x_A, \varphi_3(x_A))$ belongs to $[A_1, A_2]$. Therefore $E \in D_3$. It follows from the definition of φ_3 that

$$C \in [B_3, E] \quad \text{and} \quad B_3 \in D, E \in [A_1, A_2] \subset D_2 \subset D_3.$$

2. The proof is similar, so we omit it. $\Diamond$

Proof of Theorem 4. It is obvious that

$$D = D_1 \subset D_2 \subset D_3 = D_1 \cup D_2 \cup D_3 \subset \mathrm{conv}_F D.$$

Therefore, if we prove that $D_3 \supset \mathrm{conv}_F D$, the assertion follows. Since $D \subset D_3$, it suffices to show that the set D_3 is F-convex. To do this, we have to show the following implication

$$A, B \in D_3 \Rightarrow [A, B] \subset D_3.$$

It follows from Lemmas 1 and 2 that we need only consider the case $A \in D$, $B \in D_3$.

Let $(x_A, y_A) = A \in D, (x_B, y_B) = B \in D_3$ and let $(x_C, y_C) = C \in \in [A, B]$. Since $B \in D_3$, there exist $B_1, B_2, B_3 \in D$ and $B_4 \in [B_1, B_2]$ such that $B \in [B_3, B_4]$. Let $B_i = (x_i, y_i)$ for $i = 1, 2, 3, 4$. Without loss of generality we may assume that $B_1 \neq B_2$, $B \neq B_3$, $B \neq B_4$ and $A \neq B$.

Let us consider two cases:

1. $x_3 \neq x_4$,
2. $x_3 = x_4$.

1. Let $\varphi \in F$ be determined by

$$\varphi(x_3) = y_3, \quad \varphi(x_4) = y_4.$$

First, suppose that $\varphi(x_A) = y_A$. Then $\varphi(x_C) = y_C$ and consequently

$$C \in [A, B_4] \cup [B, B_4] \subset [A, B_4] \cup [B_3, B_4] \subset D_3,$$

because $B \in [B_3, B_4], C \in [A, B]$ and $A, B_3 \in D, B_4 \in D_2$.

Now, assume that $\varphi(x_A) \neq y_A$. Let for example $\varphi(x_A) < y_A$. It is easily seen that there exists

$$\bar{B} \in [B_1, B_3] \cup [B_1, B_4] \quad \text{if} \quad y_1 \leq y_2,$$

or

$$\bar{B} \in [B_2, B_3] \cup [B_2, B_4] \text{if} y_1 > y_2,$$

such that

$$B \in [A, \bar{B}].$$

Hence $[A, B] \subset [A, \bar{B}]$. Therefore, $C \in [A, \bar{B}]$ and $A \in D, \bar{B} \in D_2$, because

$$B_1, B_2, B_3 \in D \quad \text{and} \quad [B_1, B_4], [B_2, B_4] \subset [B_1, B_2].$$

This means that $C \in D_3$.

2. The proof is similar, so we omit it. $\diamondsuit$

3. Approximately generalized convex functions

As in the case of the usual convexity (see [2], [3], [5]), we may introduce the definition of the approximately convex function.

Definition 5. A function $g : (a,b) \to \mathbb{R}$ will be called *ϵ-convex with respect to the family F* ($\epsilon > 0$) iff for any points $a < x_1 < x_2 < b$ the unique $\varphi \in F$ determined by

$$\varphi(x_i) = g(x_i), \quad i = 1, 2$$

satisfies the inequality

$$g(x) \leq \varphi(x) + \epsilon, \quad x \in [x_1, x_2].$$

A function $g : (a,b) \to \mathbb{R}$ will be called *approximately generalized convex with respect to the family F* iff it is ϵ-convex with respect to the family F (for some $\epsilon > 0$).

It turns out that these functions have the same properties as in the classical situation. We shall start from the following

Theorem 5. *Let $g : (a,b) \to \mathbb{R}$ be an ϵ-convex function with respect to the family F ($\epsilon > 0$). Then g is locally bounded at every point of (a,b).*

Proof. As an easy consequence of Def. 5 we obtain that g is locally bounded above at every point of (a,b).

For an indirect proof suppose that there exists an $x_0 \in (a,b)$ such that g is not bounded below on any right-hand neighbourhood of x_0 or on any left-hand neighbourhood of x_0. We consider the first case, the second is similar.

Let $x_0 < x_0' < x_0'' < b$ and let $\varphi_0 \in F$ be determined by

$$\varphi_0(x_0') = g(x_0'), \quad \varphi_0(x_0'') = g(x_0'').$$

By hypothesis and by continuity of φ_0, there exists $x_1 \in (x_0, x_0')$ such that

$$g(x_1) < \min\{\varphi_0(x_1), -1\}. \tag{6}$$

Let $\varphi_1 \in F$ be determined by

$$\varphi_1(x_1) = g(x_1), \quad \varphi_1(x_0'') = g(x_0'').$$

From definitions of φ_0, φ_1 and from (6) we get

$$\varphi_1(x) < \varphi_0(x), \quad x \in (a, x_0''),$$

by Th. 1. By a similar argument, there exists $x_2 \in (x_0, x_1)$ such that

$$g(x_2) < \min\{\varphi_1(x_2), -2\}.$$

Let $\varphi_2 \in F$ be determined by

$$\varphi_2(x_2) = g(x_2), \quad \varphi_2(x_0'') = g(x_0'').$$

Obviously

$$\varphi_2(x) < \varphi_1(x), \quad x \in (a, x_0''),$$

This way we get a sequence of points $x_1, x_2, x_3, \dots$ and a sequence of functions $\varphi_1, \varphi_2, \varphi_3, \dots$ such that

$$x_0 < \dots < x_3 < x_2 < x_1 < x_0' < x_0'',$$

$$\text{(7)} \qquad \varphi_n \in F, \varphi_n(x_0'') = g(x_0''), \varphi_n(x_n) = g(x_n), \quad n = 1, 2, 3, \dots,$$

$$\text{(8)} \qquad \dots < \varphi_3(x) < \varphi_2(x) < \varphi_1(x) < \varphi_0(x), \quad x \in (a, x_0''),$$

$$\text{(9)} \qquad g(x_n) < \min\{\varphi_{n-1}(x_n), -n\}, \quad n = 1, 2, 3, \dots .$$

From (8)

$$\dots < \varphi_3(x_0') < \varphi_2(x_0') < \varphi_1(x_0') < \varphi_0(x_0').$$

Consequently, there exists $c \in \mathbb{R} \cup \{-\infty\}$ such that

$$\lim_{n\to\infty} \varphi_n(x_0') = c.$$

If $c \in \mathbb{R}$, then

$$\text{(10)} \qquad \varphi_n(x_0') > M, \quad n = 1, 2, 3, \dots$$

for some negative integer M. Let $\bar{\varphi} \in F$ be determined by

$$\bar{\varphi}(x_0') = M, \quad \bar{\varphi}(x_0'') = g(x_0'').$$

From definitions of $\varphi_n, \bar{\varphi}$ and from (10) we get

$$\bar{\varphi}(x) < \varphi_n(x), \quad x \in (a, x_0''), n = 1, 2, 3, \dots .$$

Consequently

$$\bar{\varphi}(x_n) < \varphi_n(x_n), \quad n = 1, 2, 3, \dots .$$

Hence and from (7) ($\varphi_n(x_n) = g(x_n), n = 1, 2, 3, \dots$) we see that

$$\text{(11)} \qquad \bar{\varphi}(x_n) < g(x_n), \quad n = 1, 2, 3, \dots .$$

Since $\bar{\varphi}$ is a continuous function, $\bar{\varphi}$ is bounded below on $[x_0, x_0']$. Therefore the sequence $g(x_1), g(x_2), g(x_3), \dots$ is bounded below (see (11)). On the other hand, from (9) we have

$$\lim_{n\to\infty} g(x_n) = -\infty,$$

which is impossible. This contradiction shows that

$$\lim_{n\to\infty} \varphi_n(x_0') = -\infty.$$

Hence there exists a positive integer n_0 such that

(12) $$\varphi_{n_0}(x_0') < \varphi_0(x_0') - \epsilon.$$

By (7)

$$\varphi_{n_0}(x_{n_0}) = g(x_{n_0}), \varphi_{n_0}(x_0'') = g(x_0'').$$

This gives

$$g(x) \le \varphi_{n_0}(x) + \epsilon, \quad x \in [x_{n_0}, x_0''],$$

because g is ϵ-convex function with respect to the family F. Thus

$$g(x_0') \le \varphi_{n_0}(x_0') + \epsilon.$$

Moreover $g(x_0') = \varphi_0(x_0')$ (see (7)). Therefore

$$\varphi_0(x_0') \le \varphi_{n_0}(x_0') + \epsilon, \quad \text{and consequently} \quad \varphi_{n_0}(x_0') \ge \varphi_0(x_0') - \epsilon,$$

which contradicts (12) and completes the proof. ◊

A simple consequence of Th. 5 is

Corollary 1. *If $g : (a, b) \to \mathbb{R}$ is an approximately generalized convex with respect to the family F, then g is bounded on every compact $C \subset \subset (a, b)$.*

Now we shall present two stability type theorems. The first is

Theorem 6. *Let $g : (a, b) \to \mathbb{R}$ be an ϵ-convex function with respect to the family F ($\epsilon > 0$). Then there exists a convex function with respect to the family F $f : (a, b) \to \mathbb{R}$ such that*

$$f(x) \le g(x) \le f(x) + \epsilon, \quad x \in (a, b).$$

Proof. The proof is similar to that used in [2; Th. 2]. Let g be an ϵ-convex function with respect to the family F. Put

$$W_0 := \{(x, y) \in (a, b) \times \mathbb{R} : g(x) = y\}, \quad W := \mathrm{conv}_F W_0.$$

We first show

(13) $$(x, y) \in W \Rightarrow g(x) - \epsilon \le y.$$

Let $(x, y) = C \in W$. It follows from Th. 4 and from definition of W that $C \in W_1 \cup W_2 \cup W_3$, where

$$W_1 := W_0, \quad W_2 := \cup\{[A, B] : A, B \in W_0\},$$
$$W_3 := \cup\{[A, B] : A \in W_0, B \in W_2\}.$$

If $C \in W_1 = W_0$, that obviously $g(x) - \epsilon \le y$. Let $C \in W_2 \setminus W_1$. Then there exist $A, B \in W_0$ such that $C \in [A, B]$ and $A \ne B$, $A \ne C, B \ne C$. Let $A = (x_A, y_A), B = (x_B, y_B)$ (since $A, B \in W_0$ and $A \ne B$, $x_A \ne x_B$) and let $\varphi_{AB} \in F$ be determined by

$$\varphi_{AB}(x_A) = y_A, \quad \varphi_{AB}(x_B) = y_B.$$

Then $y = \varphi_{AB}(x)$ and we have

$$g(x) \le \varphi_{AB}(x) + \epsilon = y + \epsilon,$$

because g is ϵ-convex function, hence $g(x) - \epsilon \le y$.

Now, assume that $C \in W_3 \setminus (W_1 \cup W_2)$. Then there exist $A, B_1, B_2 \in \in W_0$ and $B \in [B_1, B_2]$ such that $C \in [A, B]$. Let

$$A = (x_A, y_A), \quad B = (x_B, y_B), \quad B_1 = (x_{B_1}, y_{B_1}), \quad B_2 = (x_{B_2}, y_{B_2}).$$

Since $A, B_1, B_2 \in W_0$ and $C \notin W_1 \cup W_2$, we conclude that $x_A \ne \ne x_{B_1}, x_A \ne x_{B_2}$ and $x_{B_1} \ne x_{B_2}$. Let for example $x_A < x_{B_1} < x_{B_2}$. Then $x_{B_1} < x_B < x_{B_2}$ and $x_A < x < x_B$. We assume that $x_A < x \le \le x_{B_1}$ (in the case $x_{B_1} < x < x_B$ the proof is similar).

Let $\varphi_{AB}, \varphi_{AB_1}, \varphi_{AB_2} \in F$ be determined by

$$\begin{matrix} \varphi_{AB}(x_A) = y_A, & \varphi_{AB_i}(x_A) = y_A, \\ \varphi_{AB}(x_B) = y_B, & \varphi_{AB_i}(x_{B_i}) = y_{B_i}, \end{matrix} \qquad i = 1, 2.$$

Then $y = \varphi_{AB}(x)$ and $\varphi_{AB_1}(x) > y$ or $\varphi_{AB_1}(x) < y$, because $C \notin W_1 \cup \cup W_2$. If $\varphi_{AB_1}(x) > y$, then

$$\varphi_{AB_2}(z) < \varphi_{AB}(z) < \varphi_{AB_1}(z), \quad z \in (x_A, x_{B_1}].$$

Hence $\varphi_{AB_2}(x) < \varphi_{AB}(x) = y$ and moreover $(x, \varphi_{AB_2}(x)) \in W_2$. By the above, $g(x) - \epsilon \le \varphi_{AB_2}(x)$. Since $\varphi_{AB_2}(x) < \varphi_{AB}(x) = y$, it follows that $g(x) - \epsilon \le y$.

Similar arguments apply to the case $\varphi_{AB_1}(x) < y$ we get

$$(x, \varphi_{AB_1}(x)) \in W_2, \quad \varphi_{AB_1}(x)) < y, g(x) - \epsilon \le \varphi_{AB_1}(x) < y,$$

which proves (13).

(13) allows us to define a function $f : (a, b) \to \mathbb{R}$ by the formula

$$f(x) := \inf\{y \in \mathbb{R} : (x, y) \in W\}, \quad x \in (a, b)$$

and implies the inequality

$$g(x) \le f(x) + \epsilon, \quad x \in (a, b).$$

Otherwise, since $(x, g(x)) \in W_0$ for $x \in (a, b)$, we have

$$f(x) \le g(x), \quad x \in (a, b).$$

It remains to show that f is convex with respect to the family F. To do this fix $a < x_1 < x_2 < b$ and let $\varphi_0 \in F$ be determined by

$$\varphi_0(x_i) = f(x_i), \quad i = 1, 2.$$

By the definition of f, there exist sequences $\left(y_n^1\right)_n, \left(y_n^2\right)_n$ such that $(x_1, y_n^1), (x_2, y_n^2) \in W$ for $n = 1, 2, 3, \ldots$ and

$$y_n^1 \to f(x_1), \quad y_n^2 \to f(x_2).$$

Let $\varphi_n \in F$ be determined by

$$\varphi_n(x_1) = y_n^1, \quad \varphi_n(x_2) = y_n^2,$$

for $n = 1, 2, 3, \dots$. Since $(x_1, y_n^1), (x_2, y_n^2) \in W$ and W is F-convex, $(x, \varphi_n(x)) \in W$ for $x \in [x_1, x_2]$. Hence and from the definition of f we have

$$f(x) \leq \varphi_n(x), \quad x \in [x_1, x_2]. \tag{14}$$

By Th. 2 $\varphi_n \to \varphi_0$ on (a, b). Therefore $f(x) \leq \varphi_0(x)$ on $[x_1, x_2]$, by (14). This means that f is convex with respect to the family F, which completes the proof. ◊

Under an additional assumption we have

Theorem 7. *Let g be as in* Th. 6. *If for any $c \in \mathbb{R}$ and any $\varphi \in F$ we have $c + \varphi \in F$, then there exists a function $f : (a, b) \to \mathbb{R}$ convex with respect to the family F such that*

$$|g(x) - f(x)| \leq \frac{\epsilon}{2}, \quad x \in (a, b).$$

Proof. Let

$$g_1(x) := g(x) + \frac{\epsilon}{2}, \quad x \in (a, b).$$

It is obvious that g_1 is ϵ-convex with respect to the family F, too. By Th. 6 there exists a convex function with respect to the family F $f : (a, b) \to \mathbb{R}$ such that

$$f(x) \leq g_1(x) \leq f(x) + \epsilon, \quad x \in (a, b).$$

Hence

$$f(x) - \frac{\epsilon}{2} \leq g(x) \leq f(x) + \frac{\epsilon}{2}, \quad x \in (a, b),$$

and consequently

$$|g(x) - f(x)| \leq \frac{\epsilon}{2}, \quad x \in (a, b).$$

This completes the proof. ◊

References

[1] BECKENBACH E.F.: Generalized convex functions, *Bull. Amer. Math. Soc.* **43** (1937), 363–371.

[2] CHOLEWA P.W.: Remarks on the stability of functional equations, *Aequationes Math.* **27** (1984), 76–86.

[3] HYERS D.H. and ULAM S.M.: Approximately convex functions, *Proc. Amer. Math. Soc.* **3** (1952), 821–828.

[4] KRZYSZKOWSKI J.: Generalized convex sets, Rocznik Naukowo-Dydaktyczny WSP w Krakowie, *Prace Matematyczne XIV* **189** (1997), 59–68.

[5] KUCZMA M.: An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality, Państwowe Wydawnictwo Naukowe, Uniwersytet Śląski, Warszawa-Kraków-Katowice, 1985.

[6] TORNHEIM L., On n-parameter families of functions and associated convex functions, *Trans. Amer. Math. Soc.* **69** (1950), 457–467.