Mathematica Pannonica
12/1 (2001), 83–92

CHAIN OF DENDRITES OPENLY UNBOUNDED FROM BELOW

David **Opěla**

Department of Mathematical Analysis, Charles University, Sokolovská 83, CZ-186 75 Prague 8, Czech Republic

Pavel **Pyrih**

Department of Mathematical Analysis, Charles University, Sokolovská 83, CZ-186 75 Prague 8, Czech Republic

Robert **Šámal**

Department of Mathematical Analysis, Charles University, Sokolovská 83, CZ-186 75 Prague 8, Czech Republic

Received: March 2000

MSC 2000: 54 C 10, 54 F 50

Keywords: Dendrite, open mapping, chain.

Abstract: We consider an ordering with respect to open mappings on the class of all dendrites and construct a chain of dendrites which does not have a lower bound answering negatively a question of its existence.

1. Preliminaries

All spaces considered in this paper are assumed to be metric. A *continuum* means a nonempty compact connected space. A *simple*

E-mail addresses: opela@karlin.mff.cuni.cz, pyrih@karlin.mff.cuni.cz, samal@atrey.karlin.mff.cuni.cz

Joint research of the Open Problem Seminar, Charles University, Prague.
Research supported by CEZ J13/98113200007 and the grant No. GAUK 165/99 of Charles University.

closed curve is any space which is homeomorphic to the unit circle. A *dendrite* means a locally connected continuum containing no simple closed curve.

A *mapping* means a continuous function. A surjective mapping $f : X \to Y$ is said to be:

– *open* provided that for each open subset U of X its image $f(U)$ is an open subset of Y;

– *monotone* provided $f^{-1}(y)$ is connected for each $y \in Y$;

We shall use the notion of *order of a point* in the sense of Menger-Urysohn (see e.g. [5, §51, I, p. 274], and we denote by $\operatorname{ord}(p, X)$ the order of the continuum X at a point $p \in X$ or just $\operatorname{ord}(p)$ if there is no risk of confusion. Points of order 1 in a continuum X are called *end points* of X, points of order 2 are called *ordinary points* of X and points of order at least 3 are called *ramification points* of X. Throughout the paper, the symbol $\mathbb{N}$ stands for the set of all natural numbers, i.e. $\mathbb{N} = \{1, 2, \ldots, n\}$.

In [3, p. 7] J. J. Charatonik, W. J. Charatonik and J. R. Prajs introduced a quasiordering $\leq_O$ on the class of all dendrites. We recall the definition. If X, Y are dendrites, then $Y \leq_O X$ if there is a surjective open mapping $f : X \to Y$. This quasiordering is not an ordering (i.e. $X \leq_O Y$ & $Y \leq_O X$ does not imply X is homeomorphic to Y). To guarantee this they said that X and Y are $\mathbb{O}$-*equivalent* iff $X \leq_O Y$ and $Y \leq_O X$ and then considered the quotient of the class of all dendrites. The quasiordering $\leq_O$ induces the ordering $\leq_{\mathbb{O}}$ on the quotient.

J. J. Charatonik, W. J. Charatonik and J. R. Prajs posed a question if every chain with respect to this ordering has a nondegenerate lower bound (see [3, §7, Q4(a)$\mathbb{O}$, p. 51]). In Th. 2.4 we give a negative answer to this question.

In the sequel, the range space will be always considered to be nondegenerate. The following well known results will be used in further arguments.

Statement 1.1. *The order of a point is never increased under an open mapping.*

See [7, Chapter 8, (7.31), p. 147].

Proposition 1.2. *Let X be a dendrite and $f : X \to Y$ a surjective open mapping. Then the order ω of a point is preserved by an open mapping.*

See [3, Prop. 6.5, p. 23].

Let (S, T) be a connected space, and let $p \in S$. If $S \setminus \{p\}$ is not connected, then p is called a *cut point of* S.

An *arc* is any space which is homeomorphic to $[0;1]$. We denote by pq an arc with end points p and q. An arc pq in a continuum X is called a *free arc* if $pq \setminus \{p,q\}$ is open in X. A free arc in a space X containing an end point of X will be called an *antenna* in X.

We recall some more known facts about dendrites.

Proposition 1.3. *Let X be a dendrite.*

1. *Each point of X is either a cut point or an end point.*
2. *For each point $x \in X$ we have* $\operatorname{ord}(x) = c(x)$*, where $c(x)$ is the number of components of $X \setminus \{x\}$.*
3. *For every subcontinuum $Z \subset X$ every component of $X \setminus Z$ is open in X.*
4. *Let $f : X \to Y$ be a surjective open mapping. Then Y is a dendrite.*
5. *Let $f : X \to Y$ be a surjective open mapping. Then the image of an antenna in X is an antenna in Y.*

See [6, Chapter X, Th. 10.7, p. 168], [6, Chapter X, Th. 10.13, p. 170], [6, 5.22(a), p. 83], [6, Cor. 13.41, p. 297] and [3, Cor. 6.7, p.23].

We start with a simple lemma which will be used in the next section.

Lemma 1.4. *Let f be a surjective open mapping of a dendrite X onto Y, $x \in X$ a cut point of X and let A be a component of $X \setminus \{x\}$. Then there are exactly two possibilities:*

1. *$f(A)$ is a component of $Y \setminus \{f(x)\}$ and $f^{-1}(f(x)) \cap A = \emptyset$,*
2. *$f(A) = Y$ and there is a point $x' \in A$: $f(x') = f(x)$.*

Proof. We see that Y is a dendrite due to Prop. 1.3(4). If $f^{-1}(f(x)) \cap$ $\cap A = \emptyset$ then $f(A)$ is contained in some component C of $Y \setminus \{f(x)\}$, since $f(A)$ is a connected set. Since A is open in X, $f(A)$ is open in Y and so in C. Since $A \cup \{x\}$ is compact, $f(A) \cup \{f(x)\}$ is closed in Y and so $f(A)$ is closed in C and it follows that $f(A) = C$ due to connectedness. Hence (1) holds.

If $f^{-1}(f(x)) \cap A \neq \emptyset$ then there is a point $x' \in A$ such that $f(x') = f(x)$ and the image of a neighborhood U of x' intersects any component of $Y \setminus \{f(x)\}$. By the same argument as above, $f(A)$ is both open and closed in $Y = \{f(x)\} \cup \bigcup \{C; C$ is a component of $Y \setminus \{f(x)\}\}$, so $f(A) = Y$. Hence (2) holds. $\Diamond$

2. Construction of a chain $\{\mathcal{D}_n\}_{n=1}^{\infty}$

Let a class $\mathcal{S}$ of spaces be given. A member U of $\mathcal{S}$ is said to be *universal for* $\mathcal{S}$ if every member of $\mathcal{S}$ can be embedded in U, i.e., if for every $X \in \mathcal{S}$ there exists a homeomorphism $h : X \to h(X) \subset \subset U$. Accordingly, a dendrite is said to be *universal* if it contains a homeomorphic image of any dendrite.

For each $m \in \{3, 4, \dots, \omega\}$ we denote by D_m the *standard universal dendrite of order* m which is characterized by the following two conditions:

(1) each ramification point of D_m is of order m;
(2) for each arc A contained in D_m the set of all ramification points of D_m which belong to A is a dense subset of A.

See [2, Section 2], [1, p. 168], [4, Th. 6.2, p. 229] and [6, 10.37, p. 181-185].

In particular, D_ω is called the *standard universal dendrite* or *Ważewski universal dendrite.* This means that any dendrite is contained in the Ważewski dendrite D_ω. So any particular dendrite can be described as a result of 'cutting out' some parts of the Ważewski dendrite D_ω. Another procedure is described in [6, 10.37, p. 181-185] using the locally connected fan, the inverse limit and monotone bonding maps.

Now we describe dendrites $\mathcal{A}_n$ which will play an important role in the following construction. Fix $n \in \mathbb{N}$. We choose in D_ω some maximal arc ab and select on this arc n distinct ordinary points $c_1, c_2, \dots, c_n$ and at each point c_j we attach an arc I_j so that $I_j \cap D_\omega = \{c_j\}$, for $j \in \{1, \dots, n\}$. We will denote by $\mathcal{A}_n$ the obtained space

$$\mathcal{A}_n = D_\omega \cup I_1 \cup I_2 \cup \dots \cup I_n.$$

These arcs $I_1, I_2, \dots, I_n$ are antennas in $\mathcal{A}_n$ and the corresponding attaching points c_1, c_2, $\dots$, c_n will be called *basic.* Any end point x of $\mathcal{A}_n$ such that there is a maximal arc xy meeting $\bigcup_{j=1}^{n} I_j$ exactly at basic points will be called a *convenient attaching point* of $\mathcal{A}_n$. Now we show that the dendrites $\mathcal{A}_n$ are unique. Note that the following proof shows that the convenient attaching points cannot be topologically distinguished.

Lemma 2.1. *Let X be a dendrite satisfying the following three conditions.*

(1) *There are exactly n distinct ramification points $c_1, c_2, \dots, c_n$ in*

X of order 3, they are contained in some maximal arc ab, and the other ramification points of X are of order ω.

(2) *For each $k = 1, 2, \ldots, n$ the point c_k is an end point of an antenna I_k of X and $I_k \cap ab = \{c_k\}$.*

(3) *For each arc J contained in X which is disjoint from the antennas $I_1, \ldots, I_n$ the set of all ramification points of X which belong to J is a dense subset of J.*

Then X is homeomorphic to $\mathcal{A}_n$.

Proof. Denote two spaces satisfying conditions (1) – (3) by X and $\tilde{X}$, the corresponding points $c_1, c_2, \ldots, c_n$ and $\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_n$ respectively and let ab (respectively $\tilde{a}\tilde{b}$) be a maximal arc containing them. It is easy to construct a homeomorphism $g : ab \to \tilde{a}\tilde{b}$ satisfying

(1) $g(a) = \tilde{a}$, $g(b) = \tilde{b}$,

(2) $g(x)$ is a ramification point of $\tilde{X} \iff x$ is a ramification point of X,

(3) $g(c_j) = \tilde{c}_j$ holds for $j \in \{1, 2, \ldots, n\}$.

Now we extend this homeomorphism on the whole of X in an obvious way. Each component of $X \setminus ab$ will be homeomorphically mapped onto the corresponding (in the sense of the attaching point) component of $\tilde{X} \setminus \tilde{a}\tilde{b}$. This is possible, since if we cut off all n antennas, the resulting space will be D_ω. So any component of $\tilde{X} \setminus \tilde{a}\tilde{b}$ is topologically either D_ω without an end point or an antenna without an end point. We will use such homeomorphisms that can be continuously extended to the attaching points. Moreover, we select the corresponding components in such a way that each component of $\tilde{X} \setminus \tilde{a}\tilde{b}$ has exactly one component of $X \setminus ab$ which is mapped onto it.

The continuity of the constructed bijection h at points of $X \setminus ab$ is clear since each such point is contained in some component of $X \setminus ab$ (which is open in X due to Prop. 1.3(3)) and h restricted to this component is a homeomorphism. Let U be an ε-neighborhood of $\tilde{a}\tilde{b}$. Only finitely many components of $\tilde{X} \setminus \tilde{a}\tilde{b}$ are not contained in U (because these components are open due to Prop. 1.3(3), mutually disjoint and together with U cover a compact $\tilde{X}$). This observation implies the continuity of h at points of ab. So h is continuous, and since it is a bijection on a compact space, also h^{-1} is continuous.

Since the number of antennas contained in a given dendrite is preserved by a homeomorphism we conclude that $\mathcal{A}_n$ and $\mathcal{A}_m$ are not homeomorphic if $m \neq n$. $\Diamond$

Now we construct the dendrites $\mathcal{D}_n$ contained in our chain. Fix $n \in \mathbb{N}$. We choose in D_ω some maximal arc J connecting two end points a, b of D_ω and denote by $f : J \to [0;1]$ a homeomorphism from J onto the unit segment satisfying $f(a) = 0$, $f(b) = 1$. We construct two sequences of points contained in J, $\{a_k\}_{k=1}^{\infty}$, $\{b_k\}_{k=n}^{\infty}$, such that the following conditions holds:

(1) $f(a_{k+1}) < f(b_k) < f(a_k)$, for $k \in \{n, n+1, \dots\}$,
(2) $f(a_{k+1}) < f(a_k)$, for $k \in \mathbb{N}$,
(3) a_k is an ordinary point of D_ω, for $k \in \mathbb{N}$,
(4) b_k is an ordinary point of D_ω, for $k \in \{n, n+1, \dots\}$,
(5) $\lim_{k\to\infty} f(a_k) = \lim_{k\to\infty} f(b_k) = 0$.

The existence of such sequences is clear since ordinary points of D_ω are dense in J. At each a_k (for $k \in \mathbb{N}$) we attach to D_ω an antenna I_k satisfying $D_\omega \cap I_k = \{a_k\}$ and $\operatorname{diam}(I_k) < \frac{1}{k}$. At each b_k (for $k \in \in \{n, n+1, \dots\}$) we attach a homeomorphic copy A_k of the dendrite $\mathcal{A}_k$ such that $\operatorname{diam}(A_k) < \frac{1}{k}$ and $\{b_k\} = A_k \cap (D_\omega \cup \bigcup_{k=1}^{\infty} I_k)$ is a convenient attaching point of A_k. In this way we obtain the space $\mathcal{D}_n$:

$$\mathcal{D}_n = D_\omega \quad \cup \quad \bigcup_{k=1}^{\infty} I_k \quad \cup \quad \bigcup_{k=n}^{\infty} A_k.$$

See Fig. 1 for $\mathcal{D}_1$.

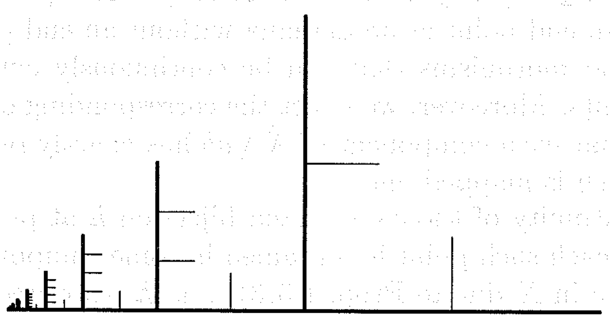

Fig. 1: the dendrite $\mathcal{D}_1$

Thick segments have densely points of order ω, thin segments are antennas.

It is not difficult to see that $\mathcal{D}_n$ is a dendrite. We will denote by B_k^n the component of $\mathcal{D}_n \setminus \{b_k\}$ that contains b. Using Lemma 2.1 one can easily see that B_n^n is homeomorphic to $\mathcal{A}_n \setminus \{e\}$, where e is a convenient attaching point. Now we establish a lemma on a characterization of $\mathcal{D}_n$.

Lemma 2.2. *Let X be a dendrite satisfying the following conditions.*

(1) *There is $n \in \mathbb{N}$ and a ramification point $b_n \in X$ of order 3.*

(2) *Two of the components of* $X \setminus \{b_n\}$ *are homeomorphic (one to another and) to* $\mathcal{A}_n \setminus \{e\}$, *where* e *is a convenient attaching point of* $\mathcal{A}_n$.

(3) *Denote by* D *the closure of the third component of* $X \setminus \{b_n\}$. *Then* D *contains a maximal arc* ab_n *with the following properties:*

For some homeomorphism $g : ab_n \to [0;1]$ *satisfying* $g(a) = 0$ *and* $g(b_n) = 1$ *there are sequences of points* $\{a_k\}_{k=n+1}^{\infty}$ *and* $\{b_k\}_{k=n}^{\infty}$ *such that:*

(i) $g(a_{k+1}) < g(b_k) < g(a_k)$, *for* $k \in \{n+1, n+2, \dots\}$.

(ii) *For each* $k > n$ *the point* a_k *is a ramification point of* X *of order 3, and it is an end point of exactly one antenna* I_k *meeting* ab_n *only at* a_k.

(iii) *For each* $k > n$ *the point* b_k *is a ramification point of order 3, exactly one component* C_k *of* $X \setminus \{b_k\}$ *is disjoint from* ab_n *and homeomorphic to* $\mathcal{A}_k \setminus \{e\}$, *where* e *is a convenient attaching point of* $\mathcal{A}_k$.

(iv) $\lim_{k\to\infty} g(a_k) = \lim_{k\to\infty} g(b_k) = 0$.

(v) *The set*

$$D \setminus \bigcup_{k=n+1}^{\infty} ((I_k \setminus \{a_k\}) \cup C_k)$$

is homeomorphic to the Ważewski universal dendrite.

Then X *is homeomorphic to* $\mathcal{D}_n$.

Proof. There are the sequences $\{\tilde{a}_k\}_{k=n+1}^{\infty}, \{\tilde{b}_k\}_{k=n}^{\infty}$, in $\mathcal{D}_n$ satisfying similar conditions with the mapping f instead of g. The construction of our homeomorphism will start again on the arc ab_n which will be mapped onto $\tilde{a}\tilde{b}_n$ in such a way that the following holds:

(1) $h(a) = \tilde{a}$, $h(b_n) = \tilde{b}_n$, $h(a_k) = \tilde{a}_k$, $h(b_k) = \tilde{b}_k$, for $k \in \{n+1, n+2, \dots\}$,

(2) x is a ramification point of $X \iff h(x)$ is a ramification point of $\mathcal{D}_n$.

Now we can extend the mapping onto the whole of X in a similar way as in the proof of Lemma 2.1 and we finish the proof as previously. ◊

By uniqueness of $\mathcal{D}_n$ (see Lemma 2.2) the points b_k (for $k \geq n$) are the only ones in $\mathcal{D}_n$ that satisfy the two following conditions: (i) they are of order 3 in $\mathcal{D}_n$, and (ii) two of the three components of $\mathcal{D}_n \setminus \{b_k\}$ are not antennas and contain a positive finite number of antennas.

The numbers of antennas in these two components are distinct, except for the point b_n. Since the number of antennas clearly cannot be changed by a homeomorphism, we see that $\mathcal{D}_n$ and $\mathcal{D}_m$ are not homeomorphic for $n \neq m$.

Let a mapping $g_n : \mathcal{D}_n \to \mathcal{D}_{n+1}$ be defined by the conditions:

(1) $g_n \restriction (\mathcal{D}_n \setminus A_n)$ is the identity;

(2) $g_n \restriction A_n : A_n \to B_n^n \cup \{b_n\}$ is a homeomorphism with $g_n(b_n) = b_n$.

Then g_n is clearly open and surjective. See Fig. 2.

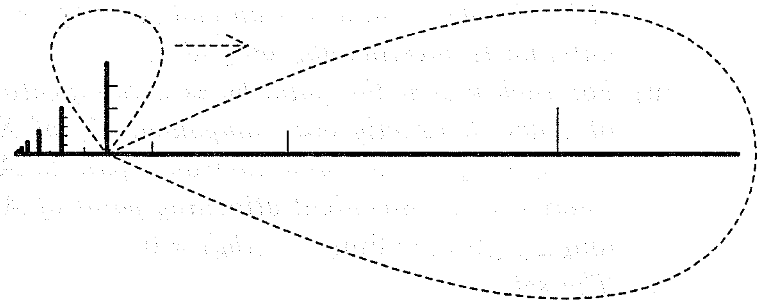

Fig. 2: mapping $\mathcal{D}_3$ onto $\mathcal{D}_4$ sending A_3 to $B_3^3 \cup \{b_3\}$

Our result will easily follow from the following theorem which characterizes all open images of the dendrite $\mathcal{D}_n$.

Theorem 2.3. *Let $n \in \mathbb{N}$ be fixed and let $f : \mathcal{D}_n \to X = f(\mathcal{D}_n)$ be an open mapping. Then there exists $m \geq n$ such that X is homeomorphic to $\mathcal{D}_m$.*

Proof. We will proceed in the following steps:

(α) For each $k \in \mathbb{N}$ there are exactly two possibilities:

(i) $f(A_k) = f(B_k^n) \cup \{f(b_k)\}$,

(ii) $f(A_k) \cap f(B_k^n) = \emptyset$.

Moreover, the first one holds only for finitely many $k \in \mathbb{N}$.

(β) $f(A_k)$ is homeomorphic to $\mathcal{A}_k$, for all $k \in \mathbb{N}$

(γ) There is a natural number $m \geq n$ such that $f(\mathcal{D}_n)$ is homeomorphic to $\mathcal{D}_m$.

Proof of (α) Since $\mathcal{D}_n$ contains an antenna by its definition, $f(\mathcal{D}_n)$ does, according to Prop. 1.3 (5). We will show that $f(\mathcal{D}_n)$ contains infinitely many antennas. Indeed, if not, then there exists an antenna I in $f(\mathcal{D}_n)$ such that for infinitely many k the antenna I_k attached to

D_ω at a_k satisfies $f(I_k) \subset I$. If $f(I_k)$ is a proper subset of I, then $f(a_k)$ is an interior point of I, and therefore points of order ω which are arbitrarily close to a_k also are interior points of I, contrary to Prop. 1.2. Thus $f(I_k) = I$. However, diameters of I_k converge to zero which contradicts the uniform continuity of f. Moreover, there are infinitely many antennas in the image of that component of $\mathcal{D}_n \setminus \{b_k\}$ which contains b_{k+1}, since the rest of $\mathcal{D}_n$ contains only finitely many antennas and so its image.

Now if $f(A_k) = f(B_k^n) \cup \{f(b_k)\}$ for infinitely many $k \in \mathbb{N}$ then for these k there exists $x_k \in A_k$ such that $f(x_k) = f(b)$. Since the distances between x_k and a converges to zero we get by the continuity of f that $f(a) = f(b)$. This implies by openness and continuity that the image of a small neighborhood U of b contains the image of some neighborhood V of a. But V contains all but finitely many A_k, so $f(V)$ contains infinitely many antennas which is the desired contradiction.

We see that $f(A_k) \neq f(\mathcal{D}_n)$ since the first space contains only finitely many antennas while the second one contains infinitely many of them. So due to Lemma 1.4 the set $f(A_k) \setminus \{f(b_k)\}$ is a component of $f(\mathcal{D}_n) \setminus \{f(b_k)\}$. The same argument holds for $f(B_k^n)$. Now it is clear that if $f(A_k) \neq f(B_k^n) \cup \{f(b_k)\}$ then $f(A_k)$ and $f(B_k^n)$ are disjoint. This completes the proof of (α).

Proof of (β) We first show that there are exactly k points of order 3 in $f(A_k)$. Denote by x some point of order 3 in A_k. If ord $f(x) = 2$, then the images of an antenna and the component of $A_k \setminus \{x\}$ which does not contain b_k cover both components of $f(\mathcal{D}_n) \setminus \{f(x)\}$ since they cannot be the same component. This is a contradiction because some of the mentioned images have to contain infinitely many antennas. Similarly for ord $f(x) < 2$. We conclude that ord $f(x) = 3$ due to Statement 1.1.

Let $b_k e$ be a maximal arc in A_k containing all ramification points of order 3 in A_k (i.e. the point e is a convenient attaching point of A_k different from b_k). We show that $f|b_k e$ is injective. If not, there are points x and x' in $b_k e$ such that $f(x) = f(x')$. Suppose that x is contained in the arc $b_k x'$. But now we see that the component of $\mathcal{D}_n \setminus \{x\}$ containing x' will be mapped onto $f(\mathcal{D}_n)$ due to Lemma 1.4 and we obtain a contradiction. This shows that condition (1) of Lemma 2.1 is satisfied for $f(A_k)$.

Statement 1.1 and Prop. 1.2 imply that the other conditions of Lemma 2.1 are satisfied, too. Thus condition (β) follows from Lemma 2.1.

Proof of (γ) Denote by m the maximal $k \in \mathbb{N}$ such that $f(A_{k-1}) = f(B^n_{k-1}) \cup \{f(b_{k-1})\}$ or put $m = 1$ if it does not hold for any $k \in \mathbb{N}$. We notice that open image of D_ω is homeomorphic to D_ω (see [3, Prop. 6.7, p. 23]).

(i) Similarly to the proof of (β) we prove that f is injective on ab.

(ii) Similarly we can also show that $f(A_k)$ meets $f(ab)$ only at $f(b_k)$ for $k > m$ and that $f(I_k) \cap f(ab) = \{f(a_k)\}$, for $k > m$.

Finally due to (α)–(β) we conclude that all conditions of Lemma 2.2 are satisfied with $n = m$ and $X = f(\mathcal{D}_n)$. Hence $f(\mathcal{D}_n)$ is homeomorphic to $\mathcal{D}_m$. $\Diamond$

Now it follows that $\{\mathcal{D}_n\}_{n=1}^{\infty}$ is a chain with respect to $\leq_{\mathbb{O}}$, i.e. $D_{n+1} \leq_O D_n$ for all natural numbers n and D_n, D_m are not $\mathbb{O}$-equivalent for $n \neq m$.

If the chain $\{\mathcal{D}_n\}_{n=1}^{\infty}$ had a lower bound L, then L would be an open image of every $\mathcal{D}_n$, so by Th. 2.3 the continuum L would be homeomorphic to D_m for $m \in \mathbb{N}$ with $m \geq n$ for each $n \in \mathbb{N}$, a contradiction. This is stated in the theorem below.

Theorem 2.4. *The sequence $\{\mathcal{D}_n\}_{n=1}^{\infty}$ is an infinite chain with respect to $\leq_{\mathbb{O}}$ which has no lower bound.*

References

[1] CHARATONIK, J. J.: Monotone mappings of universal dendrites, *Topology Appl.* **38** (1991), 163–187.

[2] CHARATONIK, J. J., CHARATONIK, W. J.: Dendrites, *Aportaciones Mat. Comun.* **22** (1998), 227–253.

[3] CHARATONIK, J. J., CHARATONIK, W. J. and PRAJS, J. R.: Mapping hierarchy for dendrites, *Dissertationes Math.* (Rozprawy Mat.) **333** (1994) 1–52.

[4] CHARATONIK, W. J., DILKS, A.: On self-homeomorphic spaces, *Topology Appl.* **55** (1994), 215–238

[5] KURATOWSKI, K.: Topology, Vol. II, Academic Press, New York, 1968.

[6] NADLER, S. B. Jr.: Continuum theory: an introduction, M. Dekker, Inc., New York-Basel-Hong Kong, 1992.

[7] WHYBURN, G. T.: Analytic topology, Amer. Math. Soc. Colloq. Publ. 28, Providence, 1942, (reprinted with corrections 1971).