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SEMI-CONFLUENT MAPPINGS

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Abstract: Semi-confluent mappings between metric continua are investigated. The obtained results concern compositions, pointed versions of semi-confluence, and inverse limits properties. Some examples related to semi-confluent mappings of hereditarily indecomposable continua are also constructed.

Introduction

Let $f : X \to Y$ be a mapping between topological spaces. Then f is said to be:

- — *confluent* provided that for each subcontinuum Q of Y and for each component C of $f^{-1}(Q)$ we have $f(C) = Q$;
- — *weakly confluent* provided that for each subcontinuum Q of Y there is a component C of $f^{-1}(Q)$ such that $f(C) = Q$;
- — *semi-confluent* provided that for each subcontinuum Q of Y and for every two components C_1 and C_2 of $f^{-1}(Q)$ we have either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$.

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The concept of confluent mappings, introduced by the first named author in 1964, [2], has been extended by A. Lelek in 1971 to weakly confluent mappings, [13], and by T. Maćkowiak in 1973 to semi-confluent ones, [14]. Semi-confluent mappings of continua were investigated in a number of papers, see e.g. [14], [15], [10] and [4]. The present paper contains a further study in this topic.

The paper consists of four sections. In the first of them conditions are discussed which imply that the composition of semi-confluent mappings is semi-confluent. The second section is devoted to localization of the global concept of semi-confluence. The concepts of pointed versions of confluent and weakly confluent mappings, defined in [3], are here extended by introducing pointed versions of semi-confluent mappings, and studying their properties.

Inverse limits are studied in the third section. First, it is shown that if a class of mappings between compact spaces has the inverse limit property, then it has the inverse limit projection property. (This general result is not related to semi-confluent mappings only.) Next, inverse limit property is obtained for pointed versions of semi-confluent and strongly semi-confluent mappings. As a corollary and an application of the general result it is shown that the class of semi-confluent mappings has the inverse limit property and the inverse limit projection property.

Two examples related to hereditarily indecomposable continua are presented in Section 4. The first of them shows that the known characterizations of these continua in terms of confluent and of semi-confluent mappings cannot be extended to wider classes of mappings, viz. to joining mappings. The second example is related to the Eilenberg property. It indicates that the presence of hereditarily indecomposable continua is not necessary (both in domain and in range spaces) for examples showing that the classes of mappings larger than ones of semi-confluent (as locally semi-confluent, weakly confluent, or joining) do not have the Eilenberg property. The paper contains also several open problems related to obtained results.

1. Compositions

It is known that the composition $f_2 \circ f_1$ of two semi-confluent mappings $f_1 : X \to Y$ and $f_2 : Y \to Z$ between continua need not be semi-confluent, see [14, Ex. 3.4, p. 254] and [15, Ex. 5.10, p. 31]. The implication holds under an additional assumption that f_1 is confluent,

see [14, Th. 3.3, p. 254]. Another sufficient condition, now concerning the mapping f_2, is presented below. Recall that a *monotone* mapping is defined as one with connected point-inverses.

1.1. Statement. *If $f_1 : X \to Y$ is semi-confluent and $f_2 : Y \to Z$ is monotone, then the composition $f = f_2 \circ f_1$ is semi-confluent.*

Proof. Let $Q \subset Z$ be a continuum. Since f_2 is monotone, $f_2^{-1}(Q)$ is a continuum, and therefore components of $f^{-1}(Q)$ coincide with components of $f_1^{-1}(f_2^{-1}(Q))$. Taking some two of them, C_1 and C_2, we have either $f_1(C_1) \subset f_1(C_2)$ or vice versa by semi-confluence of f_1, whence it follows that either $f_2(f_1(C_1)) \subset f_2(f_1(C_2))$ or invertedly, as needed. $\Diamond$

The assumption that f_2 is monotone cannot be changed into its openness, even if X, Y and Z are very simple locally connected continua. The following example shows this.

1.2. Example. *If X is a simple triod, and Y and Z are closed segments, then there are a semi-confluent mapping $f_1 : X \to Y$ and an open mapping $f_2 : Y \to Z$ such that the composition $f_2 \circ f_1$ is not semi-confluent.*

Proof. In the plane $\mathbb{R}^2$ let pq stand for the straight line segment with end points p and q. Putting

$$a = (1/2, 1/2),\ b = (0, 1),\ c = (0, 0),\ d = (0, -1),\ e = (1/2, -1/2),$$

define $X = ab \cup bd \cup ce$. Then X is a simple triod with end points a, d and e, and with the center c. Let $f_1 : X \to Y = bd$ be the projection defined by $f_1((x, y)) = (0, y) \in Y$ for each point $(x, y) \in X$, and determine $f_2 : Y \to Z = bc$ by $f_2((0, y)) = (0, |y|) \in Z$ for each point $(x, y) \in Y$. Note that f_1 is semi-confluent, and f_2 is open. The composition $f = f_2 \circ f_1$ is not semi-confluent since for $Q = \{(0, y) \in \in Z : y \in [1/4, 3/4]\}$ the set $f^{-1}(Q)$ has four components, and for some two of them, $C_1 = \{(x, y) \in ab : y \in [1/2, 3/4]\}$ and $C_2 = \{(x, y) \in \in ce : y \in [1/4, 1/2]\}$, we have $f(C_1) = \{(0, y) \in Z : y \in [1/2, 3/4]\}$ and $f(C_2) = \{(0, y) \in Z : y \in [1/4, 1/2]\}$, thus neither of the images is contained in the other. $\Diamond$

The above example shows that Statement 1.1. cannot be extended to MO-mappings (i.e., to compositions of open and monotone mappings, the class being common generalization of the classes of monotone and of open ones, see e.g. [15, p. 15]).

2. Local properties

For the (global) concepts of confluent, semi-confluent and weakly

confluent mappings we have two implications, the first of which follows from the definitions, and the second is shown in [14, Cor. 3.2, p. 254] and in [15, Th. 3.8, p. 13].

2.1. Proposition. (a) *Each confluent mapping is semi-confluent.* (b) *Each semi-confluent mapping between continua is weakly confluent.*

In [3, p. 2] the following pointed version of a confluent mapping have been introduced. A mapping f is said to be:

— *confluent relative to a point* $p \in X$ provided that for each subcontinuum Q of Y with $f(p) \in Q$ the component of $f^{-1}(Q)$ that contains the point p is mapped onto the whole Q under f.

Further, another pointed version has been introduced there, at a point of the range space. However, some assertions related to the other pointed version (at a point of the range space) of confluent mappings are mistaken. To correct these assertions we should change either the formulation of the assertions, or the introduced definition. Here the latter case is applied. Namely we introduce now two concepts of a pointed version of a confluent mapping at a point of the range space. A mapping f is said to be:

— *confluent at a point* $q \in Y$ provided that for each subcontinuum Q of Y with with $q \in Q$ each component of $f^{-1}(Q)$ whose image contains q is mapped onto the whole Q under f;

— *strongly confluent at a point* $q \in Y$ provided that for each subcontinuum Q of Y with with $q \in Q$ each component of $f^{-1}(Q)$ is mapped onto the whole Q under f.

Now the following statement (compare [3, (4), p. 2]) is a consequence of the definitions.

2.2. Statement. *A mapping* $f : X \to Y$ *is confluent at a point* $q \in Y$ *if and only if it is confluent relative to each point of* $f^{-1}(q)$.

Since the concept of mapping that is confluent at a point of its range was used in [3] in the sense of the equivalence of 2.2, all assertions formulated in [3] that concern the mentioned concept (namely [3, (4) and (5), p. 2; Cors. 3 and 6, p. 5; and Cor. 12, p. 8]) remains valid for the (new) definition of confluence at $q \in Y$ as formulated above.

The mapping $f : [0,1] \to [0,1]$ defined by $f(x) = 2x$ for $x \in [0, 1/2]$, and $f(x) = 3/2 - x$ for $x \in [1/2, 1]$ is confluent at $q = 0$, while not strongly confluent at this point. Consequently, by 2.2, it is not confluent relative to the point 0 of the range, which is the only point of the preimage $f^{-1}(q)$. This shows that assertion (4) of [3, p. 2] is

not true if the (old) definition of confluence at q as used in [3] (i.e., of strong confluence as defined above) is applied in this assertion.

The next statement is again a consequence of the definitions.

2.3. Statement. *The following conditions are equivalent for a mapping $f : X \to Y$:*

(a) *f is confluent;*
(b) *f is confluent relative to each point $p \in X$;*
(c) *f is confluent at each point $q \in Y$;*
(d) *f is strongly confluent at each point $q \in Y$.*

Concerning weakly confluent mappings pointed versions can be formulated as follows (for the latter one see [3, p. 2]). A mapping $f : X \to Y$ is said to be:

— *weakly confluent relative to a point $p \in X$ (weakly confluent at a point $q \in Y$)* provided that for each subcontinuum Q of Y with $f(p) \in Q$ (such that $q \in Q$) there exists a component of $f^{-1}(Q)$ that is mapped onto the whole Q under f.

Therefore f is weakly confluent relative to $p \in X$ if and only if it is weakly confluent at $f(p) \in Y$. Hence we have the following statements.

2.4. Statement. *A mapping f is:*

(a) *weakly confluent relative to a point p of its domain if and only if it is weakly confluent at the point $f(p)$;*
(b) *weakly confluent at a point q of its range if and only if it is weakly confluent relative to each (to some) point of $f^{-1}(q)$.*

2.5. Statement. *The following conditions are equivalent for a mapping $f : X \to Y$:*

(a) *f is weakly confluent;*
(b) *f is weakly confluent relative to each point $p \in X$;*
(c) *f is weakly confluent at each point $q \in Y$.*

2.6. Proposition. (a) *Each mapping confluent relative to a point p of its domain is weakly confluent relative to p.*

(b) *Each mapping confluent at a point q of its range is weakly confluent at q.*

Pointed versions of semi-confluence can be defined as follows. A mapping $f : X \to Y$ is said to be:

— *semi-confluent relative to a point $p \in X$* provided that for each subcontinuum Q of Y with $f(p) \in Q$ if C_1 is a component

of $f^{-1}(Q)$ that contains p, then for every component C_2 of $f^{-1}(Q)$ such that $f(p) \in f(C_2)$ we have either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$;

— *strongly semi-confluent relative to a point* $p \in X$ provided that for each subcontinuum Q of Y with $f(p) \in Q$ if C_1 is a component of $f^{-1}(Q)$ that contains p, then for every component C_2 of $f^{-1}(Q)$ we have either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$;

— *semi-confluent at a point* $q \in Y$ provided that for each subcontinuum Q of Y with $q \in Q$ and for every two components C_1 and C_2 of $f^{-1}(Q)$ if $q \in f(C_1) \cap f(C_2)$, then either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$;

— *strongly semi-confluent at a point* $q \in Y$ provided that for each subcontinuum Q of Y with $q \in Q$ and for every two components C_1 and C_2 of $f^{-1}(Q)$ we have either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset \subset f(C_1)$.

Below we present some relations between introduced concepts. Their proofs are consequences of the definitions, so they are left to the reader.

2.7. Statement. *A mapping $f : X \to Y$ is semi-confluent at a point $q \in Y$ if and only if it is semi-confluent relative to each point of $f^{-1}(q)$.*

The next result follows from the previous one.

2.8. Statement. *A mapping $f : X \to Y$ is semi-confluent relative to a point $p \in X$ if and only if it is semi-confluent at $f(p)$.*

For strong confluence of the mapping at a point of the range space we have only one implication.

2.9. Statement. *If a mapping $f : X \to Y$ is strongly semi-confluent at a point $q \in Y$ then it is strongly semi-confluent relative to each point of $f^{-1}(q)$.*

Consequently, if f is strongly semi-confluent at $f(p)$, then it is strongly semi-confluent relative to p. The opposite implication does not hold by the following example.

2.10. Example. *There exists a mapping from an arc onto a simple triod that is strongly semi-confluent relative to the only point p of $f^{-1}(f(p))$ but is not strongly semi-confluent at $f(p)$.*

Proof. Let $X = [0, 14]$. Put in the plane $v = (0, 0)$, $a = (0, 2)$, $b = (-2, 0)$, $c = (2, 0)$, and let Y be the union of three straight line segments va, vb and vc. Additionally denote by a', b' and c' the mid points of the arms of Y. Define $f : X \to Y$ as piecewise linear mapping determined by:

$$f(0) = a',\ f(1) = v,\ f(3) = b,\ f(5) = v,\ f(7) = a,\ f(9) = v,$$
$$f(11) = c,\ f(13) = v,\ f(14) = a'.$$

Then f is strongly semi-confluent relative to $p = 7$ which is the only point of $f^{-1}(p)$, while taking $Q = va \cup vb' \cup vc'$ we see that f is not strongly semi-confluent at $f(p) = a$. $\Diamond$

2.11. Statement. *The following conditions are equivalent for a mapping $f : X \to Y$:*

(a) *f is semi-confluent;*
(b) *f is strongly semi-confluent relative to each point $p \in X$;*
(c) *f is strongly semi-confluent at each point $q \in Y$.*

Strong semi-confluence of the mapping cannot be replaced by its semi-confluence in parts (b) and (c) of 2.11 by the following example.

2.12. Example. *There exists a mapping from the Cantor fan onto the two-cell which is semi-confluent relative to each point of its domain and semi-confluent at each point of its range, while not weakly confluent (and thus not semi-confluent).*

Proof. Let C be the Cantor ternary set lying in the standard way in $[0,1]$, and let $g : C \to [0,1]$ be the well known Cantor-Lebesgue step function (see e.g. [11, §16, II, (8), p. 150]; compare [17, Chapter II, §4, p. 35]). Consider the Cantor fan X as the cone over C with the vertex v, and the two-cell Y as the cone over $[0,1]$ with the vertex v'. For each point $c \in C$ map linearly the segment vc onto the segment $v'g(c)$, and let $f : X \to Y$ be the resulting mapping. Thus $\operatorname{card} f^{-1}(q) \leq 2$ for each $q \in Y$, and since $f|vc$ is a homeomorphism, it follows that f is both semi-confluent relative to each $p \in X$ and (by 2.8) semi-confluent at each $q \in Y$. It is not weakly confluent since components of the preimage of the base segment $[0,1]$ of Y are singletons in C which form the base of X. Applying 2.1 we see that f is not semi-confluent. The argument is complete. $\Diamond$

Relations between pointed versions of confluence and of semi-confluence are formulated in the next statement, which again is a consequence of the definitions.

2.13. Proposition. (a) *Each mapping confluent relative to a point p of its domain is strongly semi-confluent relative to p, hence it is semi-confluent relative to p.*

(b) *Each mapping (strongly) confluent at a point q of its range is (strongly) semi-confluent at q.*

2.14. Proposition. *If a surjective mapping of a compact space is strongly semi-confluent relative to a point p of its domain, then it is weakly confluent relative to p.*

Proof. Let a mapping $f : X \to Y$ of a compact space X be strongly semi-confluent relative to a point $p \in X$, and let a subcontinuum Q of Y contain the point $f(p)$. Consider a family $\mathcal{H}$ of subcontinua H of Q with $f(p) \in H$ and for which there is a continuum $C \subset f^{-1}(Q)$ such that $H = f(C)$. The family $\mathcal{H}$ is nonempty because $\{f(p)\} \in \mathcal{H}$. The family is not only partially ordered, but even ordered by inclusion, whence by compactness of X there is a maximal element M in $\mathcal{H}$. By the definition of $\mathcal{H}$ there is a subcontinuum K of $f^{-1}(Q)$ such that $M = f(K)$. We will show that $M = Q$. Suppose that there is a point $q' \in Q \setminus f(K)$. Take a component C of $f^{-1}(Q)$ such that $q' \in f(C)$. Since f is strongly semi-confluent relative to p, we have either $f(C) \subset f(K)$ or $f(K) \subset f(C)$. Since $q' \in f(C)$ and $q' \notin f(K)$, the first inclusion does not hold. Thus $f(K) \subset f(C) \setminus \{q'\}$, contrary to the maximality of M. Thus $f(K) = Q$, as needed. The proof is complete. $\Diamond$

By Statements 2.9 and 2.4 we have the following corollary to 2.14.

2.15. Corollary. *If a surjective mapping of a compact space is strongly semi-confluent at a point q of its range, then it is weakly confluent at q.*

Note that both assertions 2.14 and 2.15 are pointed versions of, and therefore stronger results than, the above mentioned Prop. 2.1 (b).

3. Inverse limits

The following notation will be used. $\mathbf{S} = \{X_\lambda, f_\lambda^\mu, \Lambda\}$ denotes an inverse system of topological spaces X_λ with (continuous) bonding mappings $f_\lambda^\mu : X_\mu \to X_\lambda$ for any $\lambda \leq \mu$, where $\lambda, \mu \in \Lambda$, and Λ is a set directed by a relation $\leq$. We assume that f_λ^λ is the identity, and we denote by $X = \varprojlim \{X_\lambda, f_\lambda^\mu, \Lambda\}$ the inverse limit space. Further, $f_\lambda : X \to X_\lambda$ denotes the projection from the inverse limit space into the λ-th factor space. Given a point $p \in X = \varprojlim \{X_\lambda, f_\lambda^\mu, \Lambda\}$, we put $p_\lambda = f_\lambda(p) \in X_\lambda$ and we write $p = \langle p^\lambda \rangle$. Obviously we have

$$f_\lambda^\mu(p_\mu) = p_\lambda \quad \text{for any} \quad \lambda, \mu \in \Lambda \quad \text{with} \quad \lambda \leq \mu. \tag{3.1}$$

A point $p \in X$, i.e., a system of points $p_\lambda \in X_\lambda$ for $\lambda \in \Lambda$ satisfying (3.1) is called a *thread*. Besides, we denote by x^λ a point of X_λ, not

necessary being the λ-th coordinate of a thread; similarly, we will use $A_\lambda \subset X_\lambda$ to denote a set of the form $f_\lambda(A)$ for some $A \subset X$, while $A^\lambda \subset X_\lambda$ need not be of this form.

Let two inverse systems $\mathbf{S} = \{X_\lambda, f^\mu_\lambda, \Lambda\}$ and $\mathbf{T} = \{Y_\sigma, g^\tau_\sigma, \Sigma\}$ are given. By a mapping $\mathbf{h}$ of $\mathbf{S}$ to $\mathbf{T}$ we mean a family $\{\varphi, h^\sigma\}$ consisting of a nondecreasing function $\varphi : \Sigma \to \Lambda$ such that the set $\varphi(\Sigma)$ is cofinal in Λ, and of mappings $h^\sigma : X_{\varphi(\sigma)} \to Y_\sigma$ defined for all $\sigma \in \Sigma$ and such that $g^\tau_\sigma \circ h^\tau = h^\sigma \circ f^{\varphi(\tau)}_{\varphi(\sigma)}$, i.e., such that the diagram

$$\begin{array}{ccc} X_{\varphi(\sigma)} & \xleftarrow{f^{\varphi(\tau)}_{\varphi(\sigma)}} & X_{\varphi(\tau)} \\ {\scriptstyle h^\sigma}\downarrow & & \downarrow{\scriptstyle h^\tau} \\ Y_\sigma & \xleftarrow[g^\tau_\sigma]{} & Y_\tau \end{array} \tag{3.2}$$

is commutative for any $\sigma, \tau \in \Sigma$ satisfying $\sigma \leq \tau$. Any mapping $\mathbf{h} : \mathbf{S} \to \to \mathbf{T}$ induces a (continuous) mapping of $X = \varprojlim \mathbf{S}$ to $Y = \varprojlim \mathbf{T}$, called the *limit mapping induced by* $\{\varphi, h^\sigma\}$ and denoted by $h = \varprojlim \{\varphi, h^\sigma\} : : X \to Y$ (see [7, Section 2.5, p. 101].

Let $\mathbf{S} = \{X_\lambda, f^\mu_\lambda, \Lambda\}$ and $\mathbf{T} = \{Y_\sigma, g^\tau_\sigma, \Sigma\}$ be inverse systems, and let $\mathbf{h} = \{\varphi, h^\sigma\}$ be a mapping of $\mathbf{S}$ into $\mathbf{T}$. For some classes $\mathfrak{M}$ of mappings the implication

$$f^\mu_\lambda \in \mathfrak{M} \text{ for each } \lambda, \mu \in \Lambda \text{ with } \lambda \leq \mu \quad \text{implies} \quad f_\lambda \in \mathfrak{M}$$

(called the *inverse limit projection property*), and the implication

$$h^\sigma \in \mathfrak{M} \text{ for } \sigma \in \Sigma \quad \text{implies} \quad h = \varprojlim \{\varphi, h^\sigma\} \in \mathfrak{M}$$

(called the *inverse limit property*) were discussed in several papers (see e.g. [1], [3], [8], [16]). In the present chapter we first prove that the inverse limit property implies the inverse limit projection property, and next we investigate these two properties for semi-confluent mappings between compact spaces.

The first main result of the present section is the following.

3.3. Theorem. *If a class of mappings between compact spaces has the inverse limit property, then it has the inverse limit projection property.*

Proof. Let $\mathfrak{M}$ be a class of mappings and let $\mathbf{S} = \{X_\lambda, f^\mu_\lambda, \Lambda\}$ be an inverse system of compact factor spaces X_λ and bonding mappings f^μ_λ belonging to $\mathfrak{M}$ for all $\lambda \leq \mu$. Fix $\lambda_0 \in \Lambda$, put $\Lambda' = \{\lambda \in \Lambda : \lambda_0 \leq \lambda\}$ and consider two inverse systems. The former, $\mathbf{S}'$, is obtained from $\mathbf{S}$

by restricting the index set Λ to Λ', i.e., $\mathbf{S}' = \{X_\lambda, f_\lambda^\mu, \Lambda'\}$. The latter, $\mathbf{C}$, is a constant one, with all factor spaces equal to X_{λ_0} and with the identities as the bonding mappings: $\mathbf{C} = \{Y_\lambda, g_\lambda^\mu, \Lambda'\}$, where $Y_\lambda = X_{\lambda_0}$, and $g_\lambda^\mu = f_{\lambda_0}^{\lambda_0}$ for all $\lambda, \mu \in \Lambda'$ with $\lambda \leq \mu$. Take a mapping $\mathbf{h} : \mathbf{S}' \to \mathbf{C}$ defined by $h^\lambda = f_{\lambda_0}^\lambda : X_\lambda \to X_{\lambda_0}$ for all $\lambda \in \Lambda'$. By assumption all mappings h^λ are in $\mathfrak{M}$, and since $\mathfrak{M}$ has the inverse limit property, it follows that the limit mapping $h : X \to X_{\lambda_0}$ is also in $\mathfrak{M}$. Consider now the diagram

$$\begin{array}{ccc} X_{\lambda_0} & \xleftarrow{f_{\lambda_0}} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle h} \\ X_{\lambda_0} & \xleftarrow[i]{} & X_{\lambda_0} \end{array}$$

in which $i = f_{\lambda_0}^{\lambda_0}$ stands for the identity mapping on X_{λ_0}. Since the diagram commutes (see [7, (6), p. 101]), we have $i \circ f_{\lambda_0} = i \circ h$. Thus $f_{\lambda_0} = h$, and therefore $f_{\lambda_0} \in \mathfrak{M}$, as needed. The proof is complete. $\Diamond$

In connection with 3.3 it would be interesting to know if the implication is true.

3.4. Remark. The converse implication to that of Th. 3.3 is not true. Namely the class of all open mappings between metric continua has the inverse limit projection property (see [16, Th. 5, p. 61]), while it does not have the inverse limit property (see [5, Ex. 3. 24]).

Now we will show the second main result of this section.

3.5. Theorem. *The class of (strongly) semi-confluent mappings relative to points of the domain spaces has the inverse limit property. More precisely, let* $\mathbf{S} = \{X_\lambda, f_\lambda^\mu, \Lambda\}$ *and* $\mathbf{T} = \{Y_\sigma, g_\sigma^\tau, \Sigma\}$ *be inverse systems, and let* $\mathbf{h} = \{\varphi, h^\sigma\}$ *be a mapping of* $\mathbf{S}$ *into* $\mathbf{T}$. *If, for a point* $p = \langle p_\lambda \rangle \in X$, *all mappings* $h^\sigma : X_{\varphi(\sigma)} \to Y_\sigma$ *are (strongly) semi-confluent relative to* $p_{\varphi(\sigma)} \in X_{\varphi(\sigma)}$, *then the limit mapping* $h : X \to Y$ *is (strongly) semi-confluent relative to* p.

Proof. We will argue for strong semi-confluence; for the other version the proof is almost the same.

Take a subcontinuum $Q \subset Y$ with $h(p) \in Q$. Let $A \subset X$ be the component of $h^{-1}(Q)$ that contains the point p, and let B be any other component of $h^{-1}(Q)$. We have to show that either $h(A) \subset h(B)$ or $h(B) \subset h(A)$.

Let $A^{\varphi(\sigma)}$ stand for the component of $(h^\sigma)^{-1}(g_\sigma(Q))$ that contains

the point $p_{\varphi(\sigma)}$. Since $A_{\varphi(\sigma)} = f_{\varphi(\sigma)}(A) \subset (h^\sigma)^{-1}(g_\sigma(h(A)))$, it follows that

$$A_{\varphi(\sigma)} \subset A^{\varphi(\sigma)} \quad \text{for each } \sigma \in \Sigma. \tag{3.6}$$

Observe that $f^{\varphi(\tau)}_{\varphi(\sigma)}(A^{\varphi(\tau)}) \subset A^{\varphi(\sigma)}$, whence it follows that $\mathbf{A} = \{A^{\varphi(\sigma)}, f^{\varphi(\tau)}_{\varphi(\sigma)} | A^{\varphi(\tau)}, \Sigma\}$ is a well defined inverse system of compact spaces. We claim that

$$\varprojlim \mathbf{A} = A. \tag{3.7}$$

In fact, the left member of 3.7 is a continuum containing p, whose image under h is contained in Q, while A is, by its definition, the component of $h^{-1}(Q)$ that contains p, so one inclusion follows. To see the other one it is enough to note that $A = \varprojlim \{f_{\varphi(\sigma)}(A), f^{\varphi(\tau)}_{\varphi(\sigma)} | f_{\varphi(\tau)}(A), \Sigma\} \subset \varprojlim \mathbf{A}$ by (3.6). So (3.7) is established.

Fix a point $q \in B$ and let $B^{\varphi(\sigma)}$ be the component of $(h^\sigma)^{-1}(g_\sigma(Q))$ that contains the point $q_{\varphi(\sigma)}$. As previously for the component A we can show that $\mathbf{B} = \{B^{\varphi(\sigma)}, f^{\varphi(\tau)}_{\varphi(\sigma)} | B^{\varphi(\tau)}, \Sigma\}$ is an inverse system, and that

$$\varprojlim \mathbf{B} = B, \tag{3.8}$$

analogously to (3.7).

By assumption, for each $\sigma \in \Sigma$ we have at least one of the two inclusions

$$h^\sigma(A^{\varphi(\sigma)}) \subset h^\sigma(B^{\varphi(\sigma)}), \tag{3.9}$$

$$h^\sigma(B^{\varphi(\sigma)}) \subset h^\sigma(A^{\varphi(\sigma)}). \tag{3.10}$$

Define $\Sigma_1 = \{\sigma \in \Sigma : (3.9) \text{ holds}\}$ and $\Sigma_2 = \{\sigma \in \Sigma : (3.10) \text{ holds}\}$. Then $\Sigma = \Sigma_1 \cup \Sigma_2$, so at least one of these sets is cofinal in Σ. Without loss of generality we can assume that Σ_1 is cofinal in Σ. Then by (3.7) and (3.8) we have $A = \varprojlim \{A^{\varphi(\sigma)}, f^{\varphi(\tau)}_{\varphi(\sigma)} | A^{\varphi(\tau)}, \Sigma_1\}$ and $B = \varprojlim \{B^{\varphi(\sigma)}, f^{\varphi(\tau)}_{\varphi(\sigma)} | B^{\varphi(\tau)}, \Sigma_1\}$. Since for each $\sigma \in \Sigma_1$ inclusion (3.9) holds, we infer that $h(A) \subset h(B)$. The proof is finished. ◊

The next statement is a consequence of 3.3 and 3.5.

3.11. Statement. *The class of (strongly) semi-confluent mappings relative to points of the domain spaces has the inverse limit projection property.*

By the equivalence of conditions (a) and (b) in 2.11, we get the following corollary to 3.5 and 3.11.

3.12. Corollary. *The class of semi-confluent mappings has the inverse limit property and the inverse limit projection property.*

4. Hereditarily indecomposable continua

A continuum X is said to be *decomposable* provided that it contains two proper subcontinua whose union is X. Otherwise it is said to be *indecomposable.* A continuum is said to be *hereditarily decomposable* (*hereditarily indecomposable*) if each of its subcontinua is decomposable (indecomposable, respectively). A mapping $f : X \to Y$ between continua is said to be *hereditarily* confluent (semi-confluent) provided that for each subcontinuum X' of X the restriction $f|X' : X' \to f(X') \subset Y$ is confluent (semi-confluent, respectively).

The following statement summarizes known characterizations of hereditarily indecomposable continua.

4.1. Statement. *The following conditions are equivalent for a continuum Y:*

(a) *Y is hereditarily indecomposable;*
(b) *each mapping from a continuum X onto Y is confluent;*
(c) *each mapping from a continuum X onto Y is hereditarily confluent;*
(d) *each mapping from a continuum X onto Y is semi-confluent;*
(e) *each mapping from a continuum X onto Y is hereditarily semi-confluent.*

Proof. Equivalence of conditions (a), (b) and (c) is known (see e.g. [15, (6.11), p. 53], where references to the original proofs are given). Implications from (c) to (e) and from (e) to (d) are obvious. Finally equivalence of (a) and (e) is proved in [9, Th. 5.1, p. 359]. The argument is then complete. ◊

A class of mappings that is wider than the class of semi-confluent ones is the class of joining mappings. Recall that a mapping $f : X \to Y$ between continua is said to be *joining* provided that for each subcontinuum Q of Y and for every two components C_1 and C_2 of $f^{-1}(Q)$ we have $f(C_1) \cap f(C_2) \neq \emptyset$. One can ask if the above characterization of hereditarily indecomposable continua can be extended by adding the

class of (hereditarily) joining mappings to the list presented in 4.1. The answer to this question is negative by the following example.

4.2. Example. *If Y is the one-point union of two hereditarily indecomposable continua, then each mapping from a continuum X onto Y is joining.*

Proof. Let P_1 and P_2 be two hereditarily indecomposable continua such that $P_1 \cap P_2 = \{p\}$, and let $Y = P_1 \cup P_2$. Consider a surjection $f : X \to Y$ from a continuum X. To show that f is joining take a subcontinuum Q of Y and consider three cases.

Case 1. $Q \subset P_1 \setminus P_2$. Let $r : Y \to P_1$ be a retraction that shrinks P_2 to the singleton $\{p\}$. Since P_1 is hereditarily indecomposable, the composition $r \circ f : X \to P_1$ is confluent according to equivalence of conditions (a) and (b) of 4.1. Therefore each component of $f^{-1}(Q) = (r \circ f)^{-1}(Q)$ is mapped onto the whole Q, and thus the condition in the definition of a joining mapping is obviously satisfied.

Case 2. $Q \subset P_2 \setminus P_1$. The argument is the same as for Case 1.

Case 3. $p \in Q$. We will show that for each component C of $f^{-1}(Q)$ we have $p \in f(C)$. Suppose the contrary, i.e., that there is a component C of $f^{-1}(Q)$ such that $p \notin f(C)$. Then either $f(C) \subset P_1 \setminus P_2$ or $f(C) \subset P_2 \setminus P_1$. Assume the former inclusion. Let K be a continuum in X satisfying $f(C) \subsetneq K \subset P_1 \setminus \{p\}$. Then C is a component of $f^{-1}(K)$ with $f(C) \subsetneq K$, contrary to the conclusion of Case 1. Thus $p \in f(C)$ for each component C of $f^{-1}(Q)$, so the needed condition holds. The proof is complete. $\Diamond$

Let $\mathbb{R}$ be the real line, $\mathbb{C}$ be the complex plane, and $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. A mapping $\alpha : X \to \mathbb{S}$ of a separable metric space X is said to be *inessential* (writing $\alpha \sim 1$ on X) provided that it belongs to the same component of the space $\mathbb{S}^X$ as the constant mapping $\alpha_0 : X \to \{1\} \subset \mathbb{S}$ (compare [17, Chapter 11, Part b, §§5–9]). For compact spaces X the condition $\alpha \sim 1$ on X is equivalent to the existence of a mapping $\varphi : X \to \mathbb{R}$ such that $\alpha = p \circ \varphi$, where the *universal covering projection* $p : \mathbb{R} \to \mathbb{S}$ is defined by $p(t) = \exp(2\pi i t)$ for $t \in \mathbb{R}$ ([6, Th. 1, p. 162]; compare [17, Chapter 11, §6, Cor. 6.22, p. 226]).

We say that a mapping $f : X \to Y$ from a space X onto a space Y has the *Eilenberg property* provided that for every mapping $g : Y \to \mathbb{S}$ the implication holds

(4.3) if $g \circ f \sim 1$ on X, then $g \sim 1$ on Y.

Note that the condition $g \sim 1$ on Y obviously implies $g \circ f \sim 1$ on X for every surjection $f : X \to Y$ (namely one can put $\psi = \varphi \circ f : X \to \mathbb{R}$ to have $g \circ f = p \circ \psi$). Thus the implication (4.3) can be replaced by the equivalence of the two conditions $g \circ f \sim 1$ on X and $g \sim 1$ on Y.

A continuum is said to be *hereditarily unicoherent* provided that the intersection of any two its subcontinua is connected. The following result is a consequence of [10, Th. 4.2, p. 350] (where the authors consider mappings g from continua into an arbitrary graph G in place of $\mathbb{S}$).

4.4. Proposition. *Each semi-confluent mapping $f : X \to Y$ of a continuum X onto a hereditarily unicoherent continuum Y has the Eilenberg property.*

According to [10, Problem 1, p. 353] it is not known whether hereditary unicoherence of Y is an essential assumption in the above quoted result. Thus we have the following question which is a particular case of Problem 1 of [10].

4.5. Question. Is it true that each semi-confluent mapping between continua has the Eilenberg property?

A mapping $f : X \to Y$ is said to be *locally semi-confluent* provided that for each point $x \in X$ there is a closed neighborhood V of the point x such that $f(V)$ is a closed neighborhood of $f(x)$ and the partial mapping $f|V$ is semi-confluent. It is known that neither weakly confluent, nor joining, nor locally semi-confluent mappings have the Eilenberg property, even if the continuum Y is hereditarily unicoherent, see [4, Statement 14 and Ex. 15, p. 99]. Namely there exist an arc-like continuum X, a hereditarily unicoherent continuum Y, and a weakly confluent, joining and locally semi-confluent mapping $f : X \to \to Y$ and a surjection $g : Y \to \mathbb{S}$ such that $g \circ f \sim 1$ on X and g non ~ 1 on Y. The range space Y in the example above cannot be hereditarily decomposable, see [4, Remark 16, p. 100]. However, the continuum Y in the constructed example contains the pseudo-arc. Thus one can ask if it is true that each continuum Y having the above discussed properties must contain a hereditarily indecomposable continuum. The answer is negative by the following example.

4.6. Example. *There exist an arc-like continuum X and a hereditarily unicoherent continuum Y, both containing no hereditarily indecomposable subcontinua, and a weakly confluent, joining and locally*

semi-confluent mapping $f : X \to Y$ *and a surjection* $g : Y \to \mathbb{S}$ *such that* $g \circ f \sim 1$ *on* X *and* g *non* ~ 1 *on* Y.

Proof. Let P be an indecomposable arc-like Knaster-type continuum with exactly two end points (see e.g. [12, §48, V, Ex. 3 (Fig. 5), p. 205]). Then each proper subcontinuum of P is an arc. Denote the end points of P by a and b and note that they are in different composants of P. Put

$$X = (P \times \{0,1\})/\{(a,1),(b,0)\} \quad \text{and} \quad Y = P/\{a,b\}.$$

Thus X and Y are hereditarily unicoherent continua no one of which contains any hereditarily indecomposable subcontinuum, and X is arc-like. Define $f : X \to Y$ by $f((s,t)) = s$ for $s \in P$ and $t \in \{0,1\}$. Denoting $p = \{(a,1),(b,0)\} \in X$ and $q = \{a,b\} \in Y$ we have

$$f^{-1}(q) = \{(a,0),p,(b,1)\} \text{ and } f^{-1}(y) = \{(y,0),(y,1)\} \text{ for } y \in Y \setminus \{q\}.$$

The reader can verify that f is weakly confluent and joining; an argument that it is locally semi-confluent is the same as in [4, Ex. 15, p. 99]. $\Diamond$

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