Mathematica Pannonica
12/1 (2001), 3–25

THE INNER PERIODIC STRUCTURE OF A FUNCTION

Béla **Uhrin**

Department of Mathematics, University of Pécs, Ifjúság u. 6., 7624 Pécs, Hungary

Received: May 1999

MSC 2000: 26 D 15; 11 H 99, 42 B 99

Keywords: Integral inequalities, geometry of numbers, Fourier analysis.

Abstract: Let $L \subset \mathbb{R}^n$ be a point-lattice of dimension $r, 1 \le r \le n$, let $f: : \mathbb{R}^n \to \mathbb{R}^1$ be a bounded real valued function vanishing outside of a bounded set and put $\operatorname{supp}(f) := \{x \in \mathbb{R}^n : f(x) \neq 0\}$. In the paper periodic properties of f w.r.t. the lattice L are investigated. For this two new concepts are introduced: a special decomposition of the set $\operatorname{supp}(f)$ defined by L and periodic extendability of f to the whole space, respectively. Connections among these two concepts as well as several characterizations of them are proved. The characterizations are of two types. The first use the set of $u \in L$ contained in the algebraic difference of $\operatorname{supp}(f)$ with itself and a restricted "almost everywhere" form of this set. The second type characterizations are exact conditions of equalitites in inequalities among the L_1-norm of f, sums of some other integrals and some special "Fourier-type" series, respectively, defined by L and f.

1. Introduction

In what follows V means the volume (Lebesgue measure, shortly measure) in $\mathbb{R}^n$, $\int \cdot dx$ stands for the integral and "a.e." stands for almost everywhere, respectively, with respect to the V.

E-mail address: uhrin@sztaki.hu

$A+B:=\{a+b : a\in A, b\in B\}$ is the algebraic (Minkowski) sum of $A, B\subseteq \mathbb{R}^n$, in particular $A-B:=A+(-B)$. If for all $x\in A+B$ there are unique $a\in A, b\in B$ s.t. $x=a+b$, then we write $A\oplus B:=A+B$ (the direct algebraic sum). $\theta\in\mathbb{R}^n$ is the zero vector.

$|\cdot|$ is either the cardinality of a set or the absolute value of a real or complex number (the meaning will be clear form the context).

Given r linearly independent vectors $b_1,\dots,b_r\in\mathbb{R}^n$, $1\le r\le n$, the set

$$L:=\operatorname{int}(b_1,\dots,b_r):=\left\{\sum_{i=1}^{r}u_ib_i : u_i \quad \text{integers}, \quad i=1,\dots,r\right\}$$

is the r-dimensional point-lattice generated by the basis (b_i).

Let $\operatorname{lin}(L)$ be the linear subspace generated by (b_i) and let $T\subset\subset\mathbb{R}^n$ be the orthogonal complement linear subspace to $\operatorname{lin}(L)$ (the dimension of T is $n-r$, for $r=n$ by definition $T=\{\theta\}$). So we have $\mathbb{R}^n=\operatorname{lin}(L)\oplus T$.

Let $Q:=\{\sum_{i=1}^{r}\lambda_ib_i : 0\le\lambda_i<1, i=1,\cdots,r\}$ and put $P:=Q\oplus\oplus T$. P is the basic cell of L in $\mathbb{R}^n$ defined by the basis (b_i) (as one can easily see, P is in a one-to-one correspondence with the quotient space $\mathbb{R}^n/L$).

P and L also give a direct decomposition of $\mathbb{R}^n$, $\mathbb{R}^n=P\oplus L$, i.e., any $x\in\mathbb{R}^n$ can be written uniquely as $x=\varphi(x)+[x]$, $\varphi(x)\in P$, $[x]\in\in L$. The functions $\varphi(x)$ and $[x]$ are so called "canonical projections". For any $A\subseteq\mathbb{R}^n$ write $\varphi(A):=\cup_{x\in A}\{\varphi(x)\}$.

If not specified otherwise, $f:\mathbb{R}^n\to\mathbb{R}^1$ is any real valued function defined and bounded on $\mathbb{R}^n$ and vanishing outside of a bounded subset of $\mathbb{R}^n$. The set supp (f) defined in the Abstract is the *support* of f.

We call the function $g:\mathbb{R}^n\to\mathbb{R}^1$ periodic $(\operatorname{mod} L)$, shortly *L-periodic*, if

$$g(x)=g(x+u), x\in\mathbb{R}^n, \quad u\in L.$$

We have to note, that L is a point-lattice of dimension r, where $r\le n$, while the above concepts are usually defined for $r=n$ (see, e.g., [2], [3], [6], [12]). At first glance this seems to be a formal difference only, but looking at the things more closely one discovers soon that this is not so. For example, any result where the quotient space $\mathbb{R}^n/L$ is assumed to be compact (i.e., P bounded) is meaningless for lower dimensional L, because for $r<n$ the set P is not bounded. (Say, the classical basic theorem of geometry of numbers, the Minkowski-

Blichfeldt theorem, is meaningless, because the volume of P is used in it.) Luckily, refinements of some theorems (e.g., refinements of the Minkowski-Blichfeldt theorem, see [14], [15]) extend without any difficulty to lower dimensional L, see [18], [22], [24] for more details.

The situation is similar with the periodicity of a function. This concept is usually defined for n-dimensional point-lattices, because for multidimensional Fourier analysis the compactness of $\mathbb{R}^n/L$ (the boundedness of P) is crucial (for Fourier analysis on $\mathbb{R}^n$ see, e.g., [9], [13]). A method how to overcome this difficulty for some problems of geometry of numbers with lower dimensional L has been elaborated in [24].

As it is well known (see, e.g., [9]), if we consider $\mathbb{R}^n$ as an Abelian group (with the addition of vectors as the group operation), any discrete subgroup L of $\mathbb{R}^n$ is a point-lattice of some dimension r, $1 \leq r \leq n$, and conversely, any such point-lattice is a discrete subgroup of $\mathbb{R}^n$. This further amphasizes that studying lower dimensional point lattices might be important also from this more general point of view (see some remarks on this in Section 4).

In what follows L will denote a point-lattice of arbitrary dimension r, $1 \leq r \leq n$, and the full dimensional L (i.e., when $r = n$) will be denoted by Λ.

There are well known fundamental results in the geometry of numbers where Λ-periodic functions generated by f are very important (see, e.g., [15], or [6], Section 3). Also, there are fundamental results in the Fourier analysis where Λ-periodic functions generated by f are very important (see, e.g., [9], [13]). A connection of Fourier analysis to the geometry of numbers initiated by Siegel [11] and further developed, e.g., by Hlawka [10] and Bombieri [1] is also well known, see, [6] (see Section 4 for more details). We have to emphasize, that in the results just mentioned rather different Λ-periodic functions generated by f than the periodic structure of f were important. The aim of this paper is to explore the periodic structure of f itself. Our methods proved to work not only in the full dimensional case ($r = n$) but in general for L of any dimension r, $1 \leq r \leq n$.

The paper is divided into three more sections. Section 2 contains results for general f proved by completely elementary methods using a disjoint decomposition of supp(f) generated by f and L. In Section 3 we restrict ourselves to non-negative f and we prove several inequalities whose conditions of equalities exactly characterize the inner L-periodic or L-aperiodic properties of f in question. One of the inequalities contains an interesting new series of Fourier-type and here we also use the extension of Fourier analytic technique to lower dimensional L elaborated in [24]. Finally, Section 4 contains some concluding remarks.

2. Some results involving the support of f

Recall all notation and definitions from the Introduction. Remind that L is any point-lattice of dimension r, $1 \le r \le n$, and $f : \mathbb{R}^n \to \mathbb{R}^1$ is any bounded function with bounded support.

Definition 2.1. We call $x \in \operatorname{supp}(f)$ a *core point* of f w.r.t. L, shortly *cL-point* of f, if

$$x + u \notin \operatorname{supp}(f) \forall \theta \neq u \in L. \tag{2.1}$$

We call $x \in \operatorname{supp}(f)$ a *point of inner L-periodicity* of f, shortly *pL-point* of f, if x is not a cL-point and

$$f(x) = f(x+u) \forall \theta \neq u \in L \quad \text{s.t.} \quad x + u \in \operatorname{supp}(f). \tag{2.2}$$

We call $x \in \operatorname{supp}(f)$ a *point of inner L-aperiodicity* of f, shortly *aL-point* of f, if

$$\exists \theta \neq v \in L \quad \text{s.t.} \quad x + v \in \operatorname{supp}(f) \quad \text{and} \quad f(x) \neq f(x+v). \tag{2.3}$$

Denote by $cL(f)$, $pL(f)$ and $aL(f)$ the sets of cL-, pL-, and aL-points of f, respectively. Any of these sets may be empty, but not all of them, because these sets give a mutually disjoint decomposition of $\operatorname{supp}(f)$.

Definition 2.2. We call f extendable to an L-periodic function (shortly: f is *L-extendable*), if there is an L-periodic function g such that $f(x) = g(x)$, $x \in \operatorname{supp}(f)$. We call the latter function g an *L-extension* of f.

If f is measurable, then we call f almost extendable to an L-periodic function (shortly: f is *almost L-extendable*), if there is an L-periodic function g such that $f(x) = g(x)$ for a.e. $x \in \operatorname{supp}(f)$. We call the latter function g an *almost L-extension* of f. For any set $A \subseteq \mathbb{R}^n$ denote

$$\mathcal{L}(A) := (A - A) \cap L. \tag{2.4'}$$

One can check easily that

$$\mathcal{L}(A) = \{u \in L : A \cap (A - u) \neq \emptyset\}. \tag{2.4}$$

Remark 2.3. It seems that the simple identity (2.4) was recognized first in [16] (for $L := \Lambda$), and lead immediately to improvements of a result of Hadwiger,[7], concerning the number $|\{u \in \Lambda : A \cap (A - u) \neq \emptyset\}|$. It is clear that the dimension of the the lattice L does not matter, as concerns the equality of the two sets in (2.4). The equality of the two sets in (2.4) was (and still is) a source for some improvements of several other results for the cardinality of the set on its right hand side (see, [18], [20], [21], and also some remarks on this phenomenon in Section 4).

The identity (2.4) plays an interesting role also in this paper, e.g., using it we have

Proposition 2.4. *The function f has no pL-points and aL-points, i.e.,* $\operatorname{supp}(f) = cL(f)$, *if and only if*

$$\mathcal{L}(\operatorname{supp}(f)) = \{\theta\}. \tag{2.5}$$

A connection of L-extendability with aL-points is given by

Theorem 2.5. *The function f has no aL-points, i.e.,* $\operatorname{supp}(f) = cL(f) \cup pL(f)$, *if and only if it is L-extendable.*

Proof. The "if" part follows immediately from the definitions.

As to the "only if" part, assume that for any $x \in \operatorname{supp}(f)$ either (2.1) or (2.2) is true. Denote for short $A := \operatorname{supp}(f)$. Define the function $g : \mathbb{R}^n \to \mathbb{R}^1$ as follows:

$$g(x) := \begin{cases} f(x), \ x \in A, \\ f(y), \ x \notin A, \ \exists\, y \in A \quad \text{s.t.} \quad y - x \in L, \\ 0, \ x \notin A, \ (A - x) \cap L = \emptyset, \end{cases} \quad x \in \mathbb{R}^n.$$

This function is well defined, because if $y, z \in A, y \neq z$, such that $y - x, z - x \in L$, then also $z - y \in L$, hence (2.2) implies that $f(z) = f(y + (z - y)) = f(y)$.

To prove that g is L-periodic, first let $x \in A$. For any $u \in L$ we have $(x - (u + x)) \in L$, hence, if $x + u \notin A$ then we have $g(x + u) = f(x) = g(x)$, and if $x + u \in A$ then the property (2.2) implies $g(x) = f(x) = f(x + u) = g(x + u)$.

Secondly, let $x \notin A$. Then, for any $u \in L$ we have either $x + u \in A$ or $x + u \notin A$.

In the first case $g(x) = f(x + u) = g(x + u)$. In the second case $g(x + u) = f(y)$ for some $y \in A$ with $(y - (x + u)) \in L$ and $g(x) = f(z)$ for some $z \in A$ with $z - x \in L$, hence by (2.2) $f(y) = f(z)$, yielding $g(x) = f(z) = f(y) = g(x + u)$. $\Diamond$

Both Prop. 2.4 and Th. 2.5 are too "stiff" in the sense that after changing values of f at a few points, the conditions (2.1), (2.2), (2.3) and (2.5) occuring in them may not be true.

A plausible idea how to get statements that are insensitive to such small changes is to consider everything "almost everywhere". However, it is not quite clear e.g. how to define the a.e. version of the set $(A-A)\cap \cap L$. It is again the identity (2.4) that suggests a useful a.e modification of the latter set.

Namely, for any measurable $A \subseteq \mathbb{R}^n$ introduce the following set

$$\hat{\mathcal{L}}(A) := \{u \in L : \ V(A \cap (A+u)) > 0\}. \tag{2.6}$$

As the identiy (2.4) shows, this set is a natural "restriction" of $\mathcal{L}(A)$.

It is clear that if A is open then

$$\hat{\mathcal{L}}(A) = \mathcal{L}(A) \tag{2.7}$$

and if A' is a measurable subset of A such that $V(A') = V(A)$, then

$$\hat{\mathcal{L}}(A') = \hat{\mathcal{L}}(A). \tag{2.8}$$

These imply that if e.g. A has got an open kernel A^o, i.e., there is an open set A^o such that $A^o \subseteq A$ and $V(A^o) = V(A)$, then

$$\hat{\mathcal{L}}(A) = \mathcal{L}(A^o). \tag{2.9}$$

So for functions f such that supp (f) has got an open kernel, using the latter relation and the Prop. 2.4 one get a result with $\hat{\mathcal{L}}(\text{supp}\,(f))$ instaed of $\mathcal{L}(\text{supp}\,(f))$ at once.

In the general case we have the following flexible versions of the Prop. 2.4 and Th. 2.5, respectively.

Theorem 2.6. *Let f be measurable and assume $V(\text{supp}\,(f)) > 0$. Then*

$$V(\text{supp}\,(f) \setminus cL(f)) = 0 \tag{2.10}$$

holds if and only if

$$\hat{\mathcal{L}}(\text{supp}\,(f)) = \{\theta\}. \tag{2.11}$$

Theorem 2.7. *Let f be measurable and assume $V(\text{supp}\,(f)) > 0$. Then f is almost L-extendable if and only if*

$$V\big(\text{supp}\,(f) \setminus (cL(f) \cup pL(f))\big) = 0. \tag{2.12}$$

The proofs of these theorems depend on exploring the finer structures of the sets

$$B(f,u) := \text{supp}\,(f) \cap (\text{supp}\,(f) - u), \; u \in L,$$

induced by the values of the function f.

By (2.4) we see that

$$B(f,u) = \emptyset, \forall u \in L \setminus \mathcal{L}(\text{supp}\,(f)). \tag{2.13}$$

For our purposes the following disjoint decomposition of each $B(f,u)$, defined by the values of f, are useful:

$$B(f,u) = E(f,u) \cup F(f,u), \; u \in \mathcal{L}(\text{supp}\,(f)), \tag{2.14}$$

where

$$E(f,u) := \{x \in B(f,u) : f(x) = f(x+u)\} \tag{2.15}$$

and

$$F(f,u) := \{x \in B(f,u) : f(x) \neq f(x+u)\}. \tag{2.16}$$

It is clear that $F(f,\theta) = \emptyset$, consequently $E(f,\theta) = \text{supp}\,(f)$, and, of course, any of $E(f,u), F(f,u)$, may be empty (but not both, by (2.4)), depending on the "fine" structure of f.

Now, one can easily check, using the definitions, that

$$aL(f) = \bigcup_{\theta \neq u \in \mathcal{L}(\text{supp}\,(f))} F(f,u) \tag{2.17}$$

and

$$pL(f) = \bigcup_{\theta \neq u \in \mathcal{L}(\text{supp}\,(f))} (E(f,u) \setminus aL(f)). \tag{2.18}$$

If G is any subset of supp (f), then, denoting by $f \mid_G$ the restriction of f to G, the identity (2.17) easily implies that

$$aL(f \mid_G) \subseteq aL(f). \tag{2.19}$$

After above preparations the proofs of both theorems are quite natural and easy.

Proof of Theorem 2.6. One can easily check that

$$cL(f) = \text{supp}\,(f) \setminus \bigcup_{\theta \neq u \in \mathcal{L}(\text{supp}\,(f))} B(f,u). \tag{2.20}$$

For measurable f the sets $B(f,u)$ are clearly measurable. The identity (2.20) shows that $cL(f)$ is also measurable and that (2.10) is true if and only if

$$V(B(f,u)) = 0, \quad \theta \neq u \in \mathcal{L}(\text{supp}\,(f)), \tag{2.21}$$

which condition is, taking into account (2.4) and the definition (2.6), clearly equivalent to (2.11). $\Diamond$

Proof of Theorem 2.7. The sets $B(f,u)$ are measurable, so the restrictions of both functions $f(x)$ and $f(x+u)$ to $B(f,u)$ are measurable functions. This implies that the sets $E(f,u)$ and $F(f,u)$, as "level sets" of measurable functions, are measurable as well. Consequently, by (2.17) and (2.18) we see that both $aL(f)$ and $pL(f)$ are measurable.

Assume first that (2.12) is not true, i.e., that $V(aL(f)) > 0$. The representation (2.17) shows that there is $\theta \neq v \in \mathcal{L}(\text{supp}\,(f))$ such that

$$V(F(f,v)) > 0. \tag{2.22}$$

If now f were L-extendable, then there would be an L-periodic function g such that $g(x) = f(x)$, a.e. $x \in \text{supp}\,(f)$, hence

$$g(x) = f(x), \quad \text{a.e.} \quad x \in B(f,v) \tag{2.23}$$

and

$$g(x+v) = f(x+v), \quad \text{a.e.} \quad x \in B(f,v), \tag{2.24}$$

that is by (2.22) and the definition (2.16) of $F(f,v)$ impossible.

This proves the only if part of the theorem.

To prove the converse direction, assume that (2.12) is true, i.e., that

$$V(aL(f)) = 0. \tag{2.25}$$

Take the set

$$A := \text{supp}\,(f) \setminus aL(f). \tag{2.26}$$

A has the same measure as supp (f), hence by (2.8) we have

$$\hat{\mathcal{L}}(A) = \hat{\mathcal{L}}(\text{supp}\,(f)). \tag{2.27}$$

The conditions (2.26) and (2.19) imply that for the restriction $f \mid_A$ of f to A we have

$$aL(f \mid_A) = \emptyset. \tag{2.28}$$

This implies by Th. 2.5 that $f \mid_A$ is L-extendable, hence (2.27) shows that f is almost L-extendable.

By this the theorem is proved. $\diamondsuit$

The identities (2.14) and (2.17) show that L-extendability of f can be given also the following characterization.

Corollary 2.8 (of Th. 2.5). *The function f is L-extendable if and only if*

$$\begin{gathered} f(x) = f(x+u), \quad \theta \neq u \in \mathcal{L}(\text{supp}\,(f)), \\ x \in \text{supp}\,(f) \cap (\text{supp}\,(f) - u). \end{gathered} \tag{2.29}$$

Proof. The condition (2.29) is equivalent to the condition

$$aL(f) = \emptyset. \tag{2.30}$$

Indeed, using (2.17) and (2.14) we see that (2.30) holds if and only if

$$B(f,u) = E(f,u), \quad \theta \neq u \in \mathcal{L}(\mathrm{supp}\,(f)), \tag{2.31}$$

which condition is, taking into account the definition (2.15) of $E(f,u)$, nothing else than (2.29) ◊

We shall need in the next section the following a.e. version of this characterization

Corollary 2.9 (of Th. 2.7). *Let f be measurable and $V(\mathrm{supp}\,(f)) > 0$. Then f is almost L-extendable if and only if*

$$\begin{aligned} f(x) = f(x+u), \quad \theta \neq u \in \mathcal{L}(\mathrm{supp}\,(f)), \quad a.e. \\ x \in \mathrm{supp}\,(f) \cap (\mathrm{supp}\,(f) - u). \end{aligned} \tag{2.32}$$

Proof. The condition (2.32) is equivalent to the condition

$$V(aL(f)) = 0. \tag{2.33}$$

Indeed, using (2.17) and (2.14) we see that (2.33) holds if and only if

$$V(B(f,u)) = V(E(f,u)), \quad \theta \neq u \in \mathcal{L}(\mathrm{supp}\,(f)), \tag{2.34}$$

which condition is, taking into account the definition (2.15) of $E(f,u)$, nothing else than (2.32). ◊

3. Results involving integrals and sums defined by f

Let us fix again that L is an r-dimensional point-lattice of $\mathbb{R}^n$, $1 \leq \leq r \leq n$, $f : \mathbb{R}^n \to \mathbb{R}^1$ a bounded function vanishing outside of a bounded set and denote

$$\bar{f}(x) := \sum_{u \in L} f(x+u), \quad x \in \mathbb{R}^n.$$

The function $\bar{f}$ is clearly L-periodic.

In what follows, if not specified otherwise, f is assumed to be non-negative and measurable. Denote $g(x) := (f(x))^{1/2}, x \in \mathbb{R}^n$, and recall what the three properties of f, i.e, $V(pL(f)) = 0$, $V(aL(f)) = 0$ and L is almost L-extendable, respectively, mean.

It is clear that the condition

$$\int_{\mathbb{R}^n} g(x)g(x+u)dx = 0 \forall u \in L,\ u \neq \theta \tag{3.1}$$

is equivalent to the condition

$$V(\mathrm{supp}\,(f) \cap (\mathrm{supp}\,(f) - u)) = 0 \quad \forall u \in L,\ u \neq \theta. \tag{3.2}$$

Using this, Ths. 2.6 and 2.7 easily give

Corollary 3.1. *Assume $V(\mathrm{supp}\,(f)) > 0$. Then*

$$\int_{\mathbb{R}^n} f(x)dx \leq \int_{\mathbb{R}^n} g(x)\bar{g}(x)dx \tag{3.3}$$

and equality is in (3.3) if and only if f is almost L-extendable and $pL(f)$ has measure zero.

In fact, the inequality (3.3) is a special case of a whole "continuous" hierarchy of inequalities. For this hierarchy we need the following notation. Let a, b, α, λ be real numbers such that $a, b \geq 0, 0 < \lambda < 1$ and $-\infty < \alpha < \infty, \alpha \neq 0$. Then we define $M_\alpha^{(\lambda)}(a,b)$ to be equal to zero if one of a, b is zero and to the number $(\lambda a^\alpha + (1 - \lambda)b^\alpha)^{1/\alpha}$ otherwise. Taking limits in α one arrives at three more numbers: $M_{-\infty}^{(\lambda)}(a,b) = \min\{a,b\}$; $M_{+\infty}^{(\lambda)}(a,b) = \max\{a,b\}$ if $a, b > 0$ and $= 0$ if $a \cdot b = 0$; $M_0^{(\lambda)}(a,b) = a^\lambda \cdot b^{(1-\lambda)}$. (These numbers might be called "extended means" of a, b. On "usual" means $(\lambda a^\alpha + (1-\lambda)b^\alpha)^{1/\alpha}, -\infty \leq \alpha \leq +\infty$, see, e.g., [8].)

The properties of usual means (see [8]) show that $M_\alpha^{(\lambda)}(a,b)$ is for positive $a, b, a \neq b$, a strictly increasing continuous function of α on $[-\infty, +\infty]$.

Now, we have

Theorem 3.2. *Let $-\infty \leq \alpha \leq \beta \leq +\infty$ and assume $V(\mathrm{supp}\,(f)) > 0$. Then*

$$\begin{aligned} \int_{\mathbb{R}^n} f(x)dx &\leq^1 \sum_{u\in L}\int_{\mathbb{R}^n} M_\alpha^{(\lambda)}(f(x), f(x+u))dx \leq^2 \\ &\leq^2 \sum_{u\in L}\int_{\mathbb{R}^n} M_\beta^{(\lambda)}(f(x), f(x+u))dx. \end{aligned} \tag{3.4}$$

$\leq^1$ is equality if and only if f is almost L-extendable and $pL(f)$ is of zero measure.

For $\alpha < \beta$, $\leq^2$ is equality if and only if f is almost L-extendable.

Proof. The inequalities are clear. $\leq^1$ is clearly equality if and only if

$$(3.5) \qquad \int_{\mathbb{R}^n} M_\alpha^{(\lambda)}(f(x), f(x+u))dx = 0 \forall u \in L, \ u \neq \theta.$$

The non-negativity of f and the definition of $M_\alpha^{(\lambda)}$ ensure that the condition (3.5) is equivalent to (3.3) and by Ths. 2.6, 2.7 we get the assertion on the equality in $\leq^1$.

Let $\alpha < \beta$. In this case $\leq^2$ is equality if and only

$$(3.6) \qquad M_\alpha^{(\lambda)}(f(x), f(x+u)) = M_\beta^{(\lambda)}(f(x), f(x+u))$$

for all $u \in L$ and for almost all $x \in \text{supp}\,(f) \cap (\text{supp}\,(f) - u)$.

By the basic properties of means mentioned above, the latter condition is equivalent to the condition

$$(3.7) \qquad \begin{aligned} f(x) = f(x+u), \theta \neq u \in \mathcal{L}(\text{supp}\,(f)), \quad \text{a.e.} \\ x \in \text{supp}\,(f) \cap (\text{supp}\,(f) - u). \end{aligned}$$

Now, using Cor. 2.9 we get the exact condition of equality in $\leq^2$. $\diamond$

We see that Th. 3.2 is also a simple consequence of the Ths. 2.6, 2.7, but now the Cor. 2.9 also is needed.

Remark 3.3. Notice that exact conditions of equalities both in $\leq^1$ and $\leq^2$ do not depend on α, β. These conditions show that if equality occurs in $\leq^1$, then also $\leq^2$ are equalitites for all α, β, but the converse is not true (f may be almost L-extendable and at the same time also having $pL(f)$ of positive measure.)

Recall from Section 1 that n-dimensional point- lattices in $\mathbb{R}^n$ are denoted by Λ. Let $\{b_1, \dots, b_n\}$ be a basis, P be a basic cell of Λ and $d(\Lambda) := V(P)$ be its determinant. Let Λ^* be the polar lattice of Λ. Λ^* is defined (see, e.g., [6]) as the lattice having the basis $b_1^*, b_2^*, \dots, b_n^*$, where $\{b_1^*, \dots, b_n^*\}$ is the system of vectors orthonormal to the system $\{b_1, \dots, b_n\}$, i.e., Λ^* is the n-dimensional point-lattice such that

$$(3.8) \qquad < u, v > \quad \text{is integer} \quad \forall u \in \Lambda \quad \text{and} \quad \forall v \in \Lambda^*.$$

Then we have the following identity (the generalized Parseval "formula" for real functions, see, e.g., [9], [13]).

Proposition 3.4. *For two f_1, f_2 square-integrable non-negative functions defined on P we have*

$$\int_P f_1(y)f_2(y)dy =$$

(3.9)

$$= \frac{1}{d(\Lambda)} \sum_{v\in\Lambda^*} \int_P\int_P \cos(2\pi < v, y-z >) f_1(y) f_2(z) dydz.$$

For (3.9) it is important that Λ is full-dimensional, because $d(\Lambda)$ occurs in it.

In [24] a method has been proposed how to use the Parseval formula also for lower dimensional point-lattices. Roughly speaking, the method extends L to a full-dimensional point-lattice M so that all points of M contained in supp (f) − supp (f) are points of L. This can be done, e.g., as follows.

Let $b_1, \ldots, b_r \in \mathbb{R}^n$ be the defining basis of L and let $b_{r+1}, \ldots, b_n \in \mathbb{R}^n$ be mutually orthogonal non-zero vectors each of which is orthogonal to each $b_1, \ldots, b_r$. (the vectors $b_{r+1}, \ldots, b_n$, are a basis of the linear subspace T introduced at the beginning of Section 1, T has been used to define the basic cell P of L in $\mathbb{R}^n$). For $r = n$, by definition we take $T = \theta$, in this case by definition $M := L$.

If $r < n$ then let $M := L \oplus \text{int}(b_{r+1}, \ldots, b_n)$, be the direct sum of the L and the point-lattice spanned by $(b_{r+1}, \ldots, b_n)$ and put $D := \{\sum_{i=1}^n \lambda_i b_i : 0 \leq \lambda_i < 1, i = 1 \ldots, n\}$ for the basic cell of M in $\mathbb{R}^n$. Now $d(M) := V(D)$. If $r = n$ then we take by definition $M := L$. Below $\text{lin}(L)$ means the linear subspace spanned by $b_1, \ldots, b_r$.

Recall the meanings of $\mathcal{L}(A)$ and $\hat{\mathcal{L}}(A)$ (see (2.4′),(2.6)) and let $\mathcal{M}(A)$ and $\hat{\mathcal{M}}(A)$ be defined analogously with M instead of L.

Proposition 3.5. *Let $A \subset \mathbb{R}^n$ be any bounded set and assume $1 \leq r < n$. If A satisfies the condition*

$$A \subseteq \text{lin}(L) \oplus \left\{ \sum_{j=r+1}^n \alpha_j b_j : |\alpha_j| < 1/2, j = r+1 \ldots, n \right\} \tag{3.10}$$

then

$$\mathcal{L}(A) = \mathcal{M}(A) \tag{3.11}$$

and if A is also measurable, then (3.10) *implies*

$$\hat{\mathcal{L}}(A) = \hat{\mathcal{M}}(A). \tag{3.12}$$

Proof. Simple checking. ◊

In what follows we shall need point-lattices M depending on f, namely if f is a function occuring in Ths. 3.1, 3.2 and $1 \le r < n$, then choose the above $b_{r+1}, \dots, b_n$ such that

$$(3.13) \quad \operatorname{supp}(f) \subseteq \operatorname{lin}(L) \oplus \Big\{ \sum_{j=r+1}^{n} \alpha_j b_j : |\alpha_j| < 1/2, j = r+1 \dots, n \Big\}.$$

If $r < n$ then denote by M_f the lattice spanned by $b_1, \dots, b_n$ so that (3.13) holds, if $r = n$ then take by definition $M_f := L$.

Prop. 3.5 yields

Proposition 3.6. *Assume* $V(\operatorname{supp}(f)) > 0$. *Then*

$$(3.14) \qquad \mathcal{M}_f(\operatorname{supp}(f)) = \mathcal{L}(\operatorname{supp}(f))$$

and

$$(3.15) \qquad \hat{\mathcal{M}}_f(\operatorname{supp}(f)) = \hat{\mathcal{L}}(\operatorname{supp}(f)).$$

The full-dimensional point-lattice $\mathcal{M}_f$ was used in [24] to prove some interesting results for lower dimensional point-lattice L using Fourier-analysis techniques.

Here we have to go one step further and define one more point-lattice depending on f, namely, the $n+1$-dimensional lattice $W_f \subset \subset \mathbb{R}^{n+1}$ as

$$(3.16) \qquad W_f \;:=\; M_f \;\oplus\; \{\, k(\theta, 1) \;: k \in \mathbb{Z}^1 \,\}\,,$$

where θ is the zero vector of $\mathbb{R}^n$ and $\mathbb{Z}^1$ means the set of integers.

For any set $E \subset \mathbb{R}^{n+1}$ the sets $\mathcal{W}_f(E)$ and $\hat{\mathcal{W}}_f(E)$ are defined taking in (2.4′),(2.6) the lattice W_f instead of L.

The last special notation we shall use is the so called *hypograph of a function*, defined for non-negative functions f as :

$$(3.17) \quad \operatorname{hyp}(f) \;:=\; \{\, (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^1 : 0 \le \xi \le f(x), x \in \operatorname{supp}(f) \,\}.$$

For this set the following quite interesting identity involving $\operatorname{supp}(f)$ holds.

Lemma 3.7. *Assume* $V(\operatorname{supp}(f)) > 0$. *Then for any* $(x, 0) \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}^1$ *we have*

$$\bar{V}(\operatorname{hyp}(f) \cap (\operatorname{hyp}(f) - (x,0))) =$$

$$(3.18) \qquad = \int\limits_{\operatorname{supp}(f) \cap (\operatorname{supp}(f) - x)} \min\{f(y), f(y+x)\} dy \;,$$

where $\bar{V}$ *is the Lebesgue measure in* $\mathbb{R}^{n+1}$.

Proof. First observe that for any set $E \subset \mathbb{R}^{n+1}$ and any $z \in \mathbb{R}^{n+1}$ we have

$$\chi_{E\cap(E-z)}(y) = \chi_E(y) \cdot \chi_E(y+z) \ , \quad y \in \mathbb{R}^{n+1}, \tag{3.19}$$

where χ means the characteristic function.

This implies

$$\bar{V}(E \cap (E-z)) = \int_{\mathbb{R}^{n+1}} \chi_E(y) \cdot \chi_E(y+z)dy \ . \tag{3.20}$$

Now for $x \in \mathbb{R}^n$ let G and F be the sets of elements $(y, \eta) \in \mathbb{R}^{n+1}$ such that $y \in \operatorname{supp}(f) \cap (\operatorname{supp}(f) - x)$, $0 \le \eta \le f(y) \le f(y+x)$ and $0 \le \eta \le f(y+x) < f(y)$, respectively. One of the sets may be empty and it is clear that both are empty if and only if $\operatorname{supp}(f) - x$ does not intersects $\operatorname{supp}(f)$. It is also clear that G, F are measurable and they give a disjoint decomposition of the set $\operatorname{supp}(f) \cap (\operatorname{supp}(f) - x)$

Taking in (3.20) $E := \operatorname{hyp}(f)$ after some calculations we get

$$\bar{V}(\operatorname{hyp}(f) \cap (\operatorname{hyp}(f) - (x,0))) = \int_G f(y)dy + \int_F f(y+x)dy \ . \tag{3.21}$$

This implies that the function $\min\{f(y), f(y+x)\}$ (as a function of y) is measurable and that (3.18) holds. ◊

We shall need rather the following consequence of (3.18) than (3.18) itself.

Proposition 3.8. *If f is such that $V(\operatorname{supp}(f)) > 0$ and*

$$0 < f(x) < 1 \ , \quad x \in \operatorname{supp}(f) \ , \tag{3.22}$$

then

$$\mathcal{W}_f(\operatorname{hyp}(f)) = \mathcal{M}_f(\operatorname{supp}(f)) \tag{3.23}$$

and

$$\hat{\mathcal{W}}_f(\operatorname{hyp}(f)) = \hat{\mathcal{M}}_f(\operatorname{supp}(f)). \tag{3.24}$$

Proof. The definition (3.17) of $\operatorname{hyp}(f)$ and the assumption (3.22) imply that for $(x_1, \xi_1), (x_2, \xi_2) \in \operatorname{hyp}(f)$, the condition $(x_1, \xi_1) - (x_2, \xi_2) \in W_f$ holds if and only if $\xi_1 = \xi_2$ and $x_1 - x_2 \in M_f$, that gives

$$\mathcal{W}_f(\operatorname{hyp}(f)) = \mathcal{M}_f(\operatorname{hyp}(f)) \ . \tag{3.25}$$

But one can check easily that

$$\mathcal{M}_f(\operatorname{hyp}(f)) = \mathcal{M}_f(\operatorname{supp}(f)) \ , \tag{3.26}$$

which gives (3.23).

To prove (3.24) one first observe that

$$\hat{\mathcal{W}}_f(\mathrm{hyp}(f)) \subseteq \mathcal{M}_f(\mathrm{supp}\,(f)) \tag{3.27}$$

as a consequence of (3.23).

Secondly, Lemma 3.7 shows that for any $(x,0) \in \mathbb{R}^{n+1}$, the condition

$$\bar{V}(\mathrm{hyp}(f) \cap (\mathrm{hyp}(f) - (x,0))) > 0 \tag{3.28}$$

holds if and only if

$$V(\mathrm{supp}\,(f) \cap (\mathrm{supp}\,(f) - x)) > 0\,. \tag{3.29}$$

Using (3.27) this implies (3.24). $\Diamond$

We have prepared everything to prove the following

Theorem 3.9. *Assume* $V(\mathrm{supp}\,(f)) > 0$. *Then*

$$\int\limits_{\mathbb{R}^n} f(x)dx \leq^a \tag{3.30}$$

$$\leq^a \frac{1}{d(M_f)} \sum_{v \in M_f^*} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \cos(2\pi < v, x-t >) \min\{f(x), f(t)\} dxdt \leq^b$$

$$\leq^b \frac{1}{d(M_f)} \sum_{v \in M_f^*} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \cos(2\pi < v, x-t >) g(x)g(t)\; dxdt,$$

where g *is the function* $f^{1/2}$.

$\leq^a$ *is equality if and only if* f *is almost* L*-extendable and* $pL(f)$ *is of zero measure.*

$\leq^b$ *is equality if and only if* f *is almost* L*-extendable.*

An interesting "feature" of this theorem is that the conditions of equalities depend on the properties of f with respect to the lattice L only and they do not depend on the lattice M_f.

The Th. 3.9 is a consequence of Th. 3.2 and the

Lemma 3.10. *Assume* $V(\mathrm{supp}\,(f)) > 0$. *Then the following two identities hold.*

$$\sum_{u \in L} \int\limits_{\mathbb{R}^n} f(x)f(x+u)dx = \tag{3.31}$$

$$= \frac{1}{d(M_f)} \sum_{v \in M_f^*} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \cos(2\pi < v, x-t >) f(x)f(t)dxdt$$

and

$$\sum_{u\in L}\int_{\mathbb{R}^n}\min\{f(x),f(x+u)\}dx = \tag{3.32}$$

$$=\frac{1}{d(M_f)}\sum_{v\in M_f^*}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\cos(2\pi<v,x-t>)\min\{f(x),f(t)\}dxdt.$$

Proof. As both (3.31) and (3.32) and also M_f are invariant upon multiplying f by a positive contant, we can assume without the loss of generality that for f the condition (3.22) holds.

As to (3.31), it is a simple consequence of the generalized Parseval formula (3.9) applied to the functions $f, \tilde{f}$ and the lattice M_f, where

$$\tilde{f}(x) := \sum_{u\in M_f} f(x+u),\ x\in\mathbb{R}^n.$$

Namely, let P be the basic cell of M_f (defined by the basis of M_f satisfying (3.13)).

The left hand side of (3.31) is equal to

$$\int_{\mathbb{R}^n} f(x)\bar{f}(x)dx$$

where $\bar{f}$ is defined at the beginning of this section.

On the other hand, the relation (3.14) implies that

$$f(x)\bar{f}(x)=f(x)\tilde{f}(x),\ x\in\mathbb{R}^n\ ,$$

consequently

$$\int_{\mathbb{R}^n} f(x)\bar{f}(x)dx=\int_{\mathbb{R}^n} f(x)\tilde{f}(x)dx. \tag{3.33}$$

Using (1.4) with $\Lambda := M_f$ we can see easily that the latter integral is equal to

$$\int_P (\tilde{f}(y))^2dy\ .$$

Applying (3.9) to $\Lambda := M_f$ with $f_1 = f_2 = \tilde{f}$ we get

$$\int_P (\tilde{f}(y))^2 dy = \tag{3.34}$$
$$= \frac{1}{d(M_f)} \sum_{v \in M_f^*} \int_P \int_P \cos(2\pi < v, y-z >)\tilde{f}(y)\tilde{f}(z)dydz.$$

Using again (1.4) (with $\Lambda := M_f$) and the fact that $< v, u >$ is integer for $v \in M_f^*$ and $u \in M_f$, we easily derive that

$$\int_P \int_P \cos(2\pi < v, y-z >)\tilde{f}(y)\tilde{f}(z)dydz = \tag{3.35}$$
$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \cos(2\pi < v, x-t >)f(x)f(t)dxdt.$$

This proves (3.31) via (3.33).

The proof of (3.32) needs some tricks and a deeper analysis and goes as follows.

Denote by h the characteristic function of the set hyp(f). Let $S := P \oplus \{(\theta, \xi) : 0 \leq \xi < 1\}$ be the basic cell of W_f in $\mathbb{R}^{n+1}$, where P is the basic cell of M_f in $\mathbb{R}^n$ mentioned above.

Applying (3.20) to $E :=$ hyp(f) we see that

$$\sum_{w \in W_f} \bar{V}(\text{hyp}(f) \cap (\text{hyp}(f) - w)) = \tag{3.36}$$
$$= \int_{\mathbb{R}^{n+1}} (h(y) \cdot \sum_{w \in W_f} h(y+w))dy.$$

It is clear that (1.4) is valid also in $\mathbb{R}^{n+1}$ and using it to the right hand side integral of (3.36) for W_f, S instead of Λ, P, we get that

$$\sum_{w \in W_f} \bar{V}(\text{hyp}(f) \cap (\text{hyp}(f) - w)) = \int_S \Big(\sum_{w \in W_f} h(z+w) \Big)^2 dz. \tag{3.37}$$

On the other hand, (3.24), (3.15) and (3.18) together imply that

$$\sum_{w \in W_f} \bar{V}(\text{hyp}(f) \cap (\text{hyp}(f) - w)) = \tag{3.38}$$
$$= \sum_{u \in L} \int_{\mathbb{R}^n} \min\{f(x), f(x+u)\}dx \ ,$$

which yields by (3.37) that

$$(3.39)\qquad \sum_{u\in L}\int_{\mathbb{R}^n}\min\{f(x),f(x+u)\}dx \;=\; \int_S\Big(\sum_{w\in W_f}h(z+w)\Big)^2dz.$$

Applying (3.9) for $\mathbb{R}^{n+1}$ with $f_1(\cdot)=f_2(\cdot)=\sum_{w\in W_f}h(\cdot+w)$, $P=S$, $\Lambda=W_f$, using again (1.4) with W_f, S and using (3.39), similarly to the proof of (3.31) via (3.34) and (3.35), we get

$$(3.40)\qquad \sum_{u\in L}\int_{\mathbb{R}^n}\min\{f(x),f(x+u)\}dx \;=$$

$$=\frac{1}{d(M_f)}\sum_{w\in W_f^*}\int_{\mathbb{R}^{n+1}}\int_{\mathbb{R}^{n+1}}\cos(2\pi < w,z-y >)h(z)h(y)dzdy.$$

The scalar product $<\cdot,\cdot>$ within the cosine is now understood in the space $\mathbb{R}^{n+1}$. Here we also used the fact $d(W_f)=d(M_f)$, which follows from the definition (3.16) of W_f.

We see that the proof of (3.40) is analogous to that of (3.31).

To prove that the right hand sides of (3.32) and (3.40) are the same, one needs a careful analysis, the main steps of which are as follows.

(a): The definition of Λ^* (given after Remark 3.3) shows that $W_f^*=M_f^*\oplus\mathbb{Z}^1$, hence $< w,z-y >$ in (3.40) is equal to $< v,x-t > +$ $+k(\xi-\tau)$ where $v\in M_f^*, x,t\in\mathbb{R}^n, k\in\mathbb{Z}^1, \xi,\tau\in\mathbb{R}^1$.

(b): The relation $\sin\,\alpha=-\sin(-\alpha)$ implies that for all $x,t\in\mathbb{R}^n$ we have

$$\sum_{k\in\mathbb{Z}^1}\int_{\mathbb{R}^1}\int_{\mathbb{R}^1}\sin(2\pi k(\xi-\tau)h(x,\xi)h(t,\tau)d\xi d\tau \;=\; 0\ ,$$

(c): Taking into account the formula $\cos(\alpha\pm\beta)=\cos\,\alpha\cdot\cos\,\beta\mp$ $\mp\,\sin\,\alpha\cdot\sin\beta$, (a) and (b) together imply that the right hand side of (3.40) is equal to

$$(3.41)\qquad \begin{aligned}&\frac{1}{d(M_f)}\sum_{v\in M_f^*}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\cos(2\pi < v,x-t >)\cdot\\ &\cdot\Big(\sum_{k\in\mathbb{Z}^1}\int_{\mathbb{R}^1}\int_{\mathbb{R}^1}\cos(2\pi k(\xi-\tau)\Big)h(x,\xi)h(t,\tau)d\xi d\tau)dxdt.\end{aligned}$$

(d): Taking into account the definition of h, the inner sum in (3.41) boils down to

$$\sum_{k\in\mathbb{Z}^1} \int_0^{f(t)} \int_0^{f(x)} \cos(2\pi k(\xi-\tau))d\xi d\tau. \tag{3.42}$$

(e): Performing the two integrations, (3.42) turns to:

$$\begin{aligned} & f(x)\cdot f(t) + \frac{1}{2\pi^2}\Big(\sum_{k\geq 1}\frac{\cos(2\pi k(f(x)-f(t)))}{k^2} + \sum_{k\geq 1}\frac{1}{k^2} - \\ & -\sum_{k\geq 1}\frac{\cos(2\pi k f(x))}{k^2} - \sum_{k\geq 1}\frac{\cos(2\pi k f(t))}{k^2}\Big). \end{aligned} \tag{3.43}$$

(f): For the calculations of four sums in (3.43) we use the well known identity (see, e.g., [5]): for $-1 \leq \eta \leq +1$ we have

$$\sum_{k\geq 1}\frac{\cos(2\pi k\eta)}{k^2}) = \frac{\pi^2}{6} - \pi^2|\eta| + \pi^2\eta^2 . \tag{3.44}$$

(g): Finally, using (3.44) we see that (3.43) is equal to

$$-\frac{1}{2}|f(x)-f(t)| + \frac{1}{2}(f(x)+f(t)) = \min\{f(x), f(t)\}. \tag{3.45}$$

Putting (a)-(g) together, we see that the right hand side of (3.40) is equal to that of (3.32). ◊

Proof of Theorem 3.9. Write the special case of Th. 3.2 with $\alpha = -\infty, \beta = 0, \lambda = 1/2$ and use Lemma 3.10 with g instead of f in (3.31). ◊

Remark 3.11. The identity (3.31) can be considered as an extension of an identity due to Bombieri, [1]: if the dimension r of L is equal to n,i.e., our lattice is the lattice Λ, then by definition $M_f = \Lambda$ and after some calculations one can see that (3.31) gives the identity of Bombieri (see Section 4).

The identity (3.32) seems to be of a different flavor even for the full dimensional L. The most surprising is that the the same sort of a "two variable" function created by f, namely the function $\min\{f(\cdot), f(\cdot)\}$ occurs on both sides of (3.32).

The series occuring on the right hand side of (3.32) seems to be something new, even in the case of full dimensional point-lattices Λ. While the right hand side of (3.31), i.e., that of (3.9), is for full dimensional point-lattices Λ a sum of "scalar products" in the space $L^2(P)$, the right hand side of (3.32) seems to have no such a simple interpretation (an interesting task would be to find some interpretation for it at all).

Also the following interesting question arise as concerns the material of this section: can the results of this section be extended to any, not necessarily non-negative f? The answer is partly yes as it is seen from the following theorems.

Corollary 3.12. *Let f be measurable with $V(\text{supp}(f)) > 0$. Then the inequalities* (3.3), ($3.4, \leq^1$) *and* ($3.30, \leq^a$) *are true with $|f|$ instaed of f and with $g := |f|^{1/2}$. The conditions of equalities in these new inequalities are the same as those being in* Cor. 3.1 *and* Ths. 3.2 *and* 3.9, *respectively.*

Proof. According to the Ths. 2.6, 2.7, the conditions of equalities in inequalities in question depend only on $\text{supp}(f)$ and clearly $\text{supp}(f) = \text{supp}(|f|)$. $\diamondsuit$

For the remaining cases only the following weaker, a little "asymmetric", theorem holds.

Corollary 3.13. *Let f be measurable with $V(\text{supp}(f)) > 0$. Then the inequalities* ($3.4, \leq^2$) *and* ($3.30, \leq^b$) *are true with $|f|$ instead of f and with $g := |f|^{1/2}$. If f is almost L-extendable, then both new inequalities turn to equalities.*

Proof. Cor. 2.9 shows that if f is almost L-extendable, then the same is true for $|f|$. $\diamondsuit$

Remark 3.14. It is clear that the converse is in general not true: $|f|$ can be almost L-extendable, while f need not. This is also a consequence of Cor. 2.9, because $|f(x)| = |f(x+u)|$ may mean that $f(x) = -f(x+u)$.

4. Remarks

4.1. As the results of Section 2 shows, the two periodic properties of a function studied in this paper are closely related to the classical basic question of the geometry of numbers: how many lattice points are there in the difference set of a given set? (For basic results concerning this question in the case of n-dimensional point-lattices $\Lambda \subset \mathbb{R}^n$, see, e.g., [2], [3], [6], [12].)

The inequalities proved in Section 3 are through the two periodicity properties of f directly connected, via the theorems of Section 2, to the problematics of finding some consequences of the fact that $A - A$ contains no non-zero lattice points. One of the classical results of this sort is due to Siegel, [11], which has been further developed by Bombieri,[1] (the so called Siegel-Bombieri formula), Hlawka,[10], and others, see, e.g., [6] for more details. Their results has been both improved and extended to point lattices of any dimension, see, e.g., [14] for point lattices of full dimension and [22], [24] for any L. For more details on these developments, see, [24], [25], [26].

4.2. The Siegel-Bombieri formula mentioned above is in fact a Fourier-analytic identity for the volume $V(A)$ of A, i.e., for the integral of the characteristic function χ_A of a bounded set $A \subset \mathbb{R}^n$. The formula is a consequence of an identity for functions proved by Bombieri, [1] (on the Bombieri identity see Remark 3.11). So, a natural question arise: what about the periodic properties of the "simplest" function $f := \chi_A$? It is clear that any χ_A is L-extendable and also that $aL(\chi_A) = \emptyset$, so the results around these concepts hold automatically. But it turned out that the remaining parts of the "theory" presented here can be for this special case both deepened and generalized. This deepening and generalization has been done in [25].

4.3. The recent paper [26] contains interesting extensions of the some results of the paper [25] to so called "coloured sets" ($\sim$ positive functions f of finite range). The results of [26] are natural deepenings of those for general functions (proved here in Section 2) to the functions of finite range.

4.4. It is clear that for any set $S \subset \mathbb{R}^n$ and any point lattice L the condition $S = \emptyset$ is equivalent to the condition $\varphi(S) = \emptyset$ (of course) and for measurable S the condition $V(S) = 0$ is equivalent to the condition $V(\varphi(S)) = 0$. So all results in this paper concerning the set $pL(f)$ (e.g., it is empty or have zero measure) can be formulated equivalently using $\varphi(pL(f))$. The papers [23], [25], [26] use the latter formulation, in the latter papers the set $\varphi(pL(f)) \subseteq P$ is called the periodic part of f (or of a set, if f is the characteristic function of a set).

4.5. As we have mentioned in the Introduction, in our approach L is any discrete subgroup of $\mathbb{R}^n$. It turned out that (4.1) holds not only for any L, but also in any Abelian locally compact topological group G for any so called "sufficiently large" discrete subgroup Γ of of G (say, if Γ is torsion free, then it is sufficiently large, see [17] for more details).

A very interesting question is: to what extent can the results proved for $(\mathbb{R}^n, L)$, say those in [18], [22], [24], [25], [26] , or those proved in the present paper, be extended to (G, Γ) ? As our techniques work in many situations also for general G, Γ, one has a feeling that some of the results might be extended to these more general structures. (We note that the results based on (4.4) or (4.5) depend on the concept of affine dimension and on the "geometry" of $\mathbb{R}^n$, so the question is the existence of some analogous "geometry" in G.)

An interesting "extension of the geometry of numbers" has been studied in [19] : one takes instaed of L an arbitary closed subgroup of $\mathbb{R}^n$. It is sure that not only (4.1) but also the "second main theorem" of the geometry of numbers, the successive minima theorem, can, to some extent, be extended to this more general situation, [19]. What about the "periodic w.r.t. the closed subgroup" structure of a function?

References

[1] BOMBIERI, E.: Sulla dimostrazione di C. L. Siegel del teorema fondamentale di Minkowski nella geometria dei numeri, *Boll. U. Mat. Ital.*, Ser. III, **17** (1962), 283–288.

[2] CASSELS, J.W.S.: An Introduction to the Geometry of Numbers, Springer, Berlin-Göttingen-Heidelberg, 1959.

[3] ERDŐS, P., GRUBER, P.M. HAMMER, J.: Lattice Points, Longman Sci., London, 1989.

[4] FREIMAN, G., HEPPES, A., UHRIN, B.: A lower estimation for the cardinality of finite difference sets in $\mathbb{R}^n$, In: K.Győry, G.Halász, Eds., Number Theory, Budapest, 1987 , Coll. Math. Soc. J. Bolyai **51**, North-Holland, Amsterdam-New York, 1989, 125–139.

[5] GRADSTEIN, I.S., RYZHIK, I.M.: Tables of integrals, sums, series and products, Fizmatgiz, Moscow, 1963,(in Russian).

[6] GRUBER, P.M., LEKKERKERKER, C.G.: Geometry of Numbers, 2nd Ed., North–Holland, Amsterdam-New York, 1987.

[7] HADWIGER, H.: Überdeckung des Raumes durch translationsgleiche Punktmengen und Nachbarnzahl, *Monatsh. f. Math.* **73** (1969), 213–217.

[8] HARDY, G.H., LITTLEWOOD, J.E., PÓLYA, G.: Inequalities, Cambridge Univ. Press, Cambridge, 1951.

[9] HEWITT, E. and ROSS, K.A.: Abstract Harmonic Analysis, Vol. I,II, Springer, Berlin-Heidelberg-New York, 1963, 1970.

[10] HLAWKA, E.: Zur Geometrie der Zahlen, *Math. Z.* **49** (1943), 285–312.

[11] SIEGEL, C.L.: Über gitterpunkte in konvexen Körpern und ein damit zusammenhängendes Extremalproblem, *Acta Math.* **68** (1935), 307–323.

[12] SIEGEL, C.L.: Lectures on Geometry of Numbers, Springer, Berlin-Heidelberg-New York, 1967.

[13] STEIN, E.M., WEISS, G.: Introduction to Fourier Analysis on Euclidean Spaces, Princeton Univ. Press, Princeton, N.J., 1971.

[14] UHRIN, B.: Some useful estimations in geometry of numbers, *Period. Math. Hungar.* **11** (1980), 95–103.

[15] UHRIN, B.: On a generalization of Minkowski convex body theorem, *J. of Number Th.* **13** (1981), 192–209.

[16] UHRIN, B.: A remark to the paper of H. Hadwiger Überdeckung des Raumes durch translationgleiche Punktmengen und Nachbarnzahl, *Monatsh. f. Math.* **104** (1987), 149–152.

[17] UHRIN, B. Some remarks about the lattice points in difference sets, In: J. Szabados, K. Tandori, Eds., Proc. of the A. Haar Memorial Conference, Budapest, 1985, Coll. Math. Soc. J. Bolyai **49**, North-Holland, Amsterdam-New York, 1987, 929–937.

[18] UHRIN, B.: A description of the segment $[1, T]$, where T is the meeting number of a set-lattice, *Period. Math. Hungar.* **26** (1993), 139–156.

[19] UHRIN, B.: The measure of covering the Euclidean space by group translates of a set, In: G.M. Rassias, Ed., The Mathematical Heritage of C.F. Gauss, World Scientific, Singapore-New Jersey-London, 1991, 785–805.

[20] UHRIN, B.: Two identities in geometry of numbers with applications, In: K. Böröczky, G. Fejes Tóth, Eds., Intl. Conf. on Intuitive Geometry, Szeged (Hungary), 1991, Coll. Math. Soc. J. Bolyai, **63**, North-Holland, 1994, 481–503.

[21] UHRIN, B.: The set of neighbours of a set in a point-lattice, *Discrete Math.* **133** (1994), 259–266.

[22] UHRIN, B.: The index of a point-lattice in a set, *J. Number Th.* **54** (1995), 232–247.

[23] UHRIN, B.: On the periodic structure of a function, Research Report, IRL-1-1995, Comp. and Automat. Inst., HAS, Informatics Res. Laboratory, pp. 1–23.

[24] UHRIN, B.: New lower bounds for the number of lattice points in a difference set, *Acta Sci. Math. (Szeged)* **62** (1996), 81–99.

[25] UHRIN, B.: Inner aperiodicities and partitions of sets, *Linear Algebra and Appl.* **241-243** (1996), 851–876.

[26] UHRIN, B.: Periodic properties of functions and coloured sets, *Publicationes Math. (Debrecen)* **56** (2000), 657–676.

[27] WEIL, A.: Basic Number Theory, Springer, Berlin-Heidelberg-New York, 1967.