Mathematica Pannonica
11/2 (2000), 279–282

A RÉDEI TYPE FACTORIZATION RESULT FOR A SPECIAL 2-GROUP

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Dedicated to Professor Ludwig Reich on his 60th birthday

Received: November 1999

MSC 2000: 20 K 01; 52 C 22

Keywords: Factorization of finite and infinite abelian groups, Hajós-Rédei theory.

Abstract: If a finite abelian 2-group is a direct product of two cyclic groups and also the group is a direct product of its subsets whose orders are either four or two, then at least one of these subsets must be periodic.

1. Introduction

Let G be a finite abelian group with identity element e. If G is a direct product of cyclic subgroups of order $t_1, \ldots, t_s$, then we express this fact saying that G is of *type* $(t_1, \ldots, t_s)$. Let $A_1, \ldots, A_n$ be subsets of G. If each element g of G can be written uniquely in the form

$$g = a_1, \ldots a_n, \qquad a_1 \in A_n, \ldots, a_n \in A_n,$$

then the product $A_1, \ldots, A_n$ is direct and is equal to G. We express this fact saying that G is *factored* into its subsets $A_1, \ldots, A_n$ or simply

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Research partly supported by National Science Foundation Grant No. T022925.

that $G = A_1 \dots A_n$ is a *factorization* of G. If $e \in A_1, \dots, e \in A_n$, then the factorization is called *normed*. The n tuple $(|A_1|, \dots, |A_n|)$ is called the *type of the factorization*. A subset A of G is defined to be periodic if there is an element g of G such that $Ag = A$ and $g \neq e$.

L. Rédei [3] proved that if a finite abelian group is factored into subsets with prime order, then at least one of the factors is periodic.

Rédei's theorem suggests the following problem. Given a finite abelian group G find the factorization types for which one of the factors must be periodic. In [2] this problem was considered for 2-groups. It was proved (Th. 3) that if G is of type $(2^\lambda, 2^\mu)$ and the factorization is of type $(4, 2, \dots, 2)$, then at least one of the factors must be periodic.

In this note we will prove the next extension of the above result. If G is of type $(2^\lambda, 2^\mu)$ and the factorization is of type $(4, \dots, 4, 2, \dots, 2)$, then at least one of the factors is periodic. The prof heavily depends on a result of [1] about the size of an annihilator in a factorization of a p-group of type (p^λ, p^μ).

2. The result

If A is a subset and χ is a character of a finite abelian group G, then $\chi(A)$ will denote the sum

$$\sum_{a \in A} \chi(a).$$

The set of all characters χ of G satisfying $\chi(A) = 0$ is called the *annihilator* set of A and will be denoted by $\mathrm{Ann}(A)$.

Theorem 1. *If $G = A_1 \dots A_n$ is a normed factorization, where G is a group of type $(2^\lambda, 2^\mu)$ and $|A_i| = 2$ or $|A_i| = 4$ for each i, $1 \leq i \leq n$, then at least one of the factors $A_1, \dots, A_n$ is periodic.*

Proof. Let G be a group of type $(2^\lambda, 2^\mu)$ and consider a normed factorization $G = A_1 \dots A_n$ of G such that $|A_i| = 2$ or $|A_i| = 4$ for each i, $1 \leq i \leq n$. We may assume that

$$|A_1| = \dots = |A_s| = 4, \quad |A_{s+1}| = \dots = |A_n| = 2$$

since this is only a matter of reindexing the factors. We proceed by induction on s.

If $s = 0$, then by Rédei's theorem one of the factors is periodic. For the remaining part we assume that $s \geq 1$. Let $A_s = \{e, a, b, c\}$ and introduce the elements d_a, d_b, d_c defined by the equations

$$a = bcd_a, \quad b = acd_b, \quad c = abd_c.$$

Note that if $d_c = e$, then $A_s = \{e, a, b, ab\} = \{e, a\}\{e, b\}$. Now s decreases and by the inductive assumption one of the factors is periodic. Thus we may assume that $d_a \neq e$, $d_b \neq e$, $d_c \neq e$.

To the factor AIi of G we assign the subgroup K_i of G defined by

$$K_i = \bigcap_{\chi \in \mathrm{Ann}(A_i)} \mathrm{Ker}\,\chi.$$

We claim that it may be assumed that $K_s \neq \{e\}$. In order to prove this claim we distinguish two cases depending on A_s as whether it contains a second order element or not.

In the first case suppose that $a^2 = e$ and denote the subgroup $\langle d_b, d_c \rangle$ by L. Note that $L \neq \{e\}$. We will show that $L \subset \mathrm{Ker}\,\chi$ for each $\chi \in \mathrm{Ann}(A_s)$. Let us consider

(1) $$0 = \chi(A) = 1 + \chi(a) + \chi(b) + \chi(c).$$

If $\chi(a) = 1$, then from (1) it follows that $\chi(b) = \chi(c) = -1$. Hence

$$\chi(d_b) = \chi(b)\chi(a^{-1})\chi(c^{-1}) = \chi(b)[\chi(a)]^{-1}[\chi(c)]^{-1} = 1,$$
$$\chi(d_c) = \chi(c)\chi(a^{-1})\chi(b^{-1}) = \chi(c)[\chi(a)]^{-1}[\chi(b)]^{-1} = 1.$$

If $\chi(a) = -1$, then from (1) it follows that $\chi(b) = -\chi(c)$ and so

$$\chi(d_b) = \chi(b)\chi(a^{-1})\chi(c^{-1}) = \chi(b)[\chi(a)]^{-1}[\chi(c)]^{-1} = 1,$$
$$\chi(d_c) = \chi(c)\chi(a^{-1})\chi(b^{-1}) = \chi(c)[\chi(a)]^{-1}[\chi(b)]^{-1} = 1.$$

Let us turn to the second case when A_s does not contain any element of order two. Since the group G is of type $(2^\lambda, 2^\mu)$ it follows that the product of the subgroups $\langle d_a \rangle$, $\langle d_b \rangle$, $\langle d_c \rangle$ cannot be direct. Thus we may assume that after a suitable reordering of the elements d_a, d_b, d_c the inequality $\langle d_a \rangle \cap \langle d_b, d_c \rangle \neq \{e\}$ holds. Let us denote the subgroup on the left hand side by L. Since $L \neq \{e\}$ it will be enough to show that $L \subset \mathrm{Ker}\,\chi$ for each $\chi \in \mathrm{Ann}(A_s)$. From (1) it follows that

(2) $$(1 + \chi(a))(1 + \chi(b)) = (1 - \chi(d_c))\chi(ab)$$

(3) $$(1 + \chi(a))(1 + \chi(c)) = (1 - \chi(d_b))\chi(ac)$$

(4) $$(1 + \chi(b))(1 + \chi(c)) = (1 - \chi(d_a))\chi(bc).$$

Also from (1) it follows that at least one of $\chi(a), \chi(b), \chi(c)$ is equal to -1. If $\chi(a) = -1$, the by (2) and (3) we get $\chi(d_c) = \chi(d_b) = 1$. If $\chi(a) \neq -1$, then by (4) we get $\chi(d_a) = 1$.

Summing up our argument we may assume that $K_s \neq \{e\}$. Similarly, we may assume that $K_i \neq \{e\}$ for each i, $1 \leq i \leq s$.

Let $A_n = \{e, a\}$. Note that in the $a^2 = e$ case A_n is periodic, so we assume that $a^2 \neq e$. This in turn implies that

$$\{e\} \neq \langle a^2 \rangle \subset K_n.$$

Similarly, we may assume that $K_i \neq \{e\}$ for each i, $s+1 \leq i \leq n$. Therefore, $K_i \neq \{e\}$ may be assumed for each factor. By Th. 1 of [1] this is not possible. The contradiction completes the proof. ◊

References

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