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WEIERSTRASS POINTS AND WEIERSTRASS PAIRS ON SINGULAR PLANE CURVES

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Dedicated to Professor Ludwig Reich on his 60th birthday

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Abstract: Fix integers d, t with $d \geq 4$ and $0 \leq t \leq (d-2)(d-1)/2-2$. Here we prove that the normalization of a general plane curve of degree d with t nodes has only ordinary Weierstrass points. We prove a corresponding result for Weierstrass pairs and for Weierstrass triples.

Let X be a smooth complex projective curve of genus $g \geq 2$ and $P \in X$. The set $N(P,X) := \{t \in \mathbb{N} : h^0(X, \mathbf{O}_X(tP)) > h^0(X, \mathbf{O}_X((t-1)P)\}$ (i.e. the set of all integers $t \geq 0$ such that there exists a rational function on X which is regular on $X \setminus \{P\}$ and which has a pole of order t at P) is an additive semi-group of $\mathbb{N}$; $N(P,X)$ is called the semigroup of *non-gaps* of P. The set $G(P,X) := \mathbb{N} \setminus N(P,X)$ is called the set of *gaps* of P. We have $\operatorname{card}(G(P,X)) = g$ ([2], Ex. I-E). For general $P \in X$ we have $G(P,X) = \{1,\dots,g\}$; if $G(P,X) \neq \{1,\dots,g\}$, P is called a *Weierstrass point* of P. Set $w(P) := \sum_{1 \leq t \leq 2g-2} h^0(X, \mathbf{O}_X(tP)) -$

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$-g(g-1)/2$. The integer $w(P)$ is called the *weight* of P. We have $w(P) \geq 0$ for every P. P is a Weierstrass point of X if and only if $w(P) > 0$. We have $\sum_{P \in X} w(P) = g(g+1)(g-1)/6$ for every X and in particular every X has at least a Weierstrass point. It is unknown what are the semi-groups $S \subset \mathbb{N}$ with $\text{card}(\mathbb{N} \setminus S) = g$ and which are of the form $N(P, X)$ for some P and some smooth curve X of genus g (see the introduction of [6]). Recall (see [2], p. 42) that a Weierstrass point P of X is said to be *normal* if its gap sequence is given by the integers t with $1 \leq t \leq g-1$ and the integer $g+1$ or, equivalently, if $h^0(X, \mathbf{O}_X((g-1)P)) = 1$, $h^0(X, \mathbf{O}_X(gP)) = 2$ and $h^1(X, \mathbf{O}_X((g+ +1)P))) = 0$, or, equivalently, if it has weight 1. A Weierstrass point P of X is said to be *ordinary* if $h^1(X, \mathbf{O}_X((g+1)P)) = 0$ (and hence, since it is a Weierstrass point, $h^0(X, \mathbf{O}_X(gP)) = 2$). Let $\mathbf{P}^2$ be the complex projective plane. Fix integers d, t with $d \geq 4$ and $0 \leq t \leq \leq (d-2)(d-1)/2 - 2$. Let $A(d,t)$ be the set of all integral plane curves with degree d and with exactly t nodes as only singularities. It is known (see e.g. [11] or [7]) that $A(d,t)$ is an equidimensional smooth scheme of dimension $(d^2+3d)/2-t$. J. Harris proved in [7] the so-called Severi conjecture, i.e. that $A(d,t)$ is irreducible. The varieties $A(d,t)$ are called Severi varieties. Here we study the Weierstrass points of the normalization, X, of a general member of $A(d,t)$. Note that X has genus $(d-2)(d-1)/2 - t$. In the unique section of this paper we will prove the following result.

Theorem 0.1. *Fix integers d, t with $d \geq 4$ and $0 \leq t \leq (d-2)(d- -1)/2-2$. Then the normalization of a general member of $A(d,t)$ has only normal Weierstrass points.*

The proof of Th. 0.1 will give an interesting result (Th. 1.1) for Weierstrass pairs and Weierstrass triples on the normalization of a general degree d nodal curve with t nodes (see the beginning of Section 1 for the corresponding definitions). To prove these results we need to control, up to high codimension, the cohomology of zero-dimensional subschemes of $\mathbf{P}^2$, Z, with $\text{card}(Z_{\text{red}})$ small (see Lemma 1.2 for the case $\text{card}(Z_{\text{red}}) = 1$). This seems to be very delicate but of independent interest. We stress the notion of prolongation introduced in the proof of Lemma 1.2. We found that a good class of zero-dimensional schemes for the postulation with respect to degree m forms are given by the ones contained in a smooth curve of degree m.

1. The main results. Let X be a smooth, connected projective curve of genus $g \geq 2$; fix an integer $s > 0$ and integers $a(1), \dots, a(s)$ with $a(i) > 0$ for every i and $\sum_{1 \leq i \leq s} a(i) \geq g$; fix distinct points $P(1), \dots, P(s) \in X$. The ordered s-ple $(P(1), \dots, P(s))$ is called a *Weierstrass s-ple* of type $\geq (a(1), \dots, a(s))$ if

$$h^1(X, \mathbf{O}_X(\sum_{1 \leq i \leq s} a(i)P(i))) \neq 0.$$

This is the notion studied in [3]. For $s = 2$ this is the notion of *Weierstrass pair* given in [10] and related but different from the notion of Weierstrass pair given in [2], p. 365. For any fixed s-ple (a(1), ... ,a(s)) the set of all Weierstrass s-ples of type $\geq (a(1), \dots, a(s))$ is an algebraic locally closed subset of X^s. The aim of this section is the proof of Th. 0.1 and of the following result.

Theorem 1.1. *Let d and t be two integers such that $d \geq 4$ and $0 \leq \leq t \leq (d-1)(d-2)/2 - 2$. Set $g := (d-1)(d-2)/2 - t$. Fix integers $a(1) > 0$, $a(2) > 0$ and $a(3) > 0$. Let X be the normalization of the general member of $A(d,t)$. The following facts holds:*

(i) *if $a(1) + a(2) \geq g + 2$ then X has no Weierstrass pair of type $\geq (a(1), a(2))$;*
(ii) *if $a(1) + a(2) \geq g + 1$ then X has only finitely many Weierstrass pairs of type $\geq (a(1), a(2))$;*
(iii) *if $a(1) + a(2) + a(3) \geq g + 2$ then X has no Weierstrass triple of type $\geq (a(1), a(2), a(3))$;*
(iv) *if $a(1) + a(2) + a(3) \geq g + 1$ then X has only finitely many Weierstrass triples of type $\geq (a(1), a(2), a(3))$;*
(iv) *if $a(1) + a(2) + a(3) \geq g$ then X has no 2-dimensional family of Weierstrass triples of type $\geq (a(1), a(2), a(3))$.*

Fix $P \in \mathbf{P}^2$ and positive integers z, m. Set $M(P,z) := \{Z : : Z$ is a curvilinear length z subscheme of $\mathbf{P}^2$ with $Z_{\text{red}} = \{P\}\}$. By the theory of the local Hilbert scheme (see [8] or [9] or [4]) $M(P,z)$ is smooth and irreducible of dimension $z - 1$. If $z \leq (m+1)(m+ + 2)/2$ and $w \geq 0$, set $M(P,z;m,w)' := \{Z : Z$ is a curvilinear length z subscheme of $\mathbf{P}^2$ with $Z_{\text{red}} = \{P\}$, $h^1(\mathbf{P}^2, \mathbf{I}_Z(m)) = w\}$ and $M(P,z;m,w) := \{Z : Z$ is a curvilinear length z subscheme of $\mathbf{P}^2$ with $Z_{\text{red}} = \{P\}$ and $h^1(\mathbf{P}^2, \mathbf{I}_Z(t)) \geq w\}$; we use the same notation also for $z > (m+1)(m+2)/2$, but of course if $w < (m+1)(m+2)/2 - z$, then $M(P,z;m,w)' = \emptyset$. If $z < (m+2)(m+1)/2$, set $M(P,z)(*) := \{Z \in \in M(P,Z)$: every degree m plane curve containing Z is singular at $P\}$; set $M(P,z;m,w)(*) := M(P,z;m,w) \cup M(P,z)(*)$. Set $M(P,(m+$

$+1)(m+2)/2; m,^*) := \{Z \in M(P,(m+1)(m+2)/2) : Z$ is contained in a plane curve of degree m which is singular at $P\}$. More generally, fix strictly positive integers $y, z(1), \dots, z(y)$, m, w with $\sum_{1\le i\le y} z(i) \le$ $\le (m+1)(m+2)/2$ and y distinct points $P_i, 1 \le i \le y$, of $\mathbf{P}^2$. Set $M(P_1,\dots,P_y; z(1),\dots,z(y)) := \{Z : Z$ is a curvilinear subscheme of $\mathbf{P}^2$ with y connected components $Z(1),\dots,Z(y)$ with length$(Z(i)) =$ $= z(i)$ and $Z(i)_{\mathrm{red}} = P_i\}$. By the theory of the local Hilbert scheme (see [8] or [9] or [4]) $M(P_1,\dots,P_y; z(1),\dots,z(y))$ is smooth and irreducible of dimension $\sum_{1\le i\le y} z(i) - y$. Set $M(P_1,\dots,P_y; z(1),\dots,z(y);$ $; m, w) := \{Z \in M(P_1,\dots,P_y; z(1),\dots,z(y)) : h^1(\mathbf{P}^2, \mathbf{I}_Z(m)) \ge w\}$ and $M(P_1,\dots,P_y; z(1),\dots,z(y); m, w)' := \{Z \in M(P_1,\dots,P_y; z(1),\dots$ $\dots, z(y)) : h^1(\mathbf{P}^2, \mathbf{I}_Z(m)) = w\}$. If $z(1)+\dots+z(y) < (m+1)(m+2)/2$, set $M(P_1,\dots,P_y; z(1),\dots,z(y); m)(^*) := \{Z \in M(P_1,\dots,P_y; z(1),\dots$ $\dots, z(y); m)$ such that every degree m plane curve containing Z is singular at at least one point $P_i\}$ and $M(P_1,\dots,P_y; z(1),\dots,z(y);$ $; m, w)(^*) := M(P_1,\dots,P_y; z(1),\dots,z(y); m, w) \cup M(P_1,\dots,P_y;$ $; z(1),\dots,z(y); m)(^*)$. For all integers $z(i)$, $1 \le i \le y$, with $z(i) > 0$ for every i and $\sum_{1\le i\le y} z(i) = (m+2)(m+1)/2$, set $M(P_1,\dots,P_y; z(1),\dots$ $\dots, z(y); m,^{**}) := \{Z \in M(P_1,\dots,P_y; z(1),\dots,z(y)) : Z$ is contained in a degree m plane curve which is singular at one of the points $P_i\}$.

Lemma 1.2. (a) *For all integers z, m with $z \le (m+1)(m+2)/$ $/2M(P,z;m,1)$ has codimension ≥ 1 in $M(P,z)$;*

(b) *for all integers z with $z < (m+1)(m+2)/2$ $M(P,z;m,2)(^*)$ has codimension ≥ 2 in $M(P,z)$;*

(c) *$M(P,(m+1)(m+2)/2; m,^{**})$ has codimension ≥ 2 in $M(P,(m+1)(m+2)/2)$.*

Proof. The first assertion is trivial because $M(P,z)$ is irreducible. The second assertion is trivial for all pairs (z,m) with $z \le m$. In particular we may assume $m \ge 3$. Fix $m \ge 3$. By induction on m it is sufficient to prove the second assertion when $z \ge m(m+1)/2 - 1$. By induction on z (for the fixed integer m) we may assume the result for all integers $z' < z$. By the theory of the local Hilbert scheme (see [8] or [9] or [4]) for every $W \in M(P, z-1)$ the algebraic set $q(W) :=$ $= \{B \in M(P,z) : W \subset B\}$ is irreducible and one-dimensional; $q(W)$ will be called the *prolongation set* of W and every element of $q(W)$ will be called a length z prolongation (or just a *prolongation*) of W. Every $Z \in M(P,z)$ is the prolongation of some $W \in M(P,z-1)$. Since $M(P,z-1;m,2)(^*)$ has codimension at least two in $M(P,z-1)$ by the inductive assumption, the set of all prolongations of elements of $M(P, z-1; m, 2)(*)$ has codimension at least two in $M(P, z)$. Hence to prove part (b) of the lemma it is sufficient to prove the following two assertions:

(i) for every $W \in M(P, z-1) \setminus M(P, z-1; m, 2)(*)$ a general $Z \in q(W)$ is not an element of $M(P, z; m, 2)(*)$;

(ii) for a general $W \in M(P, z-1) \setminus M(P, z-1; m, 2)(*)$ no $Z \in q(W)$ is an element of $M(P, z; m, 2)(*)$.

Proof of (i). Fix $W \in M(P, z-1) \setminus M(P, z-1; m, 2)(*)$ and let D be a degree m plane curve with $W \subset D$ and $P \in D_{\text{reg}}$. There is a unique prolongation, Z', of W which is contained in D. For every other prolongation, Z, of W we have $h^0(\mathbf{P}^2, \mathbf{I}_Z(m)) < h^0(\mathbf{P}^2, \mathbf{I}_W(m))$ and hence $Z \notin M(P, z; m, 2)$. Since $h^0(\mathbf{P}^2, \mathbf{I}_W(m)) \geq 2$ there exists a degree m plane curve D' with $D' \neq D$, $W \subset D'$ and $P \in D'_{\text{reg}}$. The prolongation, Z'', of W along D' is obviously an element of $M(P, z) \setminus M(P, z)(*)$. Since $Z'' \neq Z'$ we have $Z'' \notin M(P, z; m, 2)$. Hence we conclude by the semicontinuity of cohomology and the openness of the condition "to be contained in a degree m curve smooth at P" among zero-dimensional subschemes with constant cohomology.

Proof of (ii). Let D be an integral plane curve with $\deg(D) = m$ and see D as embedded in $\mathbf{P}(H^0(D, \mathbf{O}_D(m)))$ by the complete linear system $H^0(D, \mathbf{O}_D(m))$. Since we are in characteristic zero, for a general $Q \in D$ the osculating hyperplane to D in $\mathbf{P}(H^0(D, \mathbf{O}_D(m)))$ has contact of order $h^0(D, \mathbf{O}_D(m)) - 1 = m(m+3)/2$ with D at Q. This means that $h^0(D, \mathbf{O}_D(m)(-(m(m+3)/2)Q)) = 0$. Up to an element of $\text{Aut}(\mathbf{P}^2)$ (i.e. changing D) we may assume $P = Q$. Let W be the Cartier divisor of order $z-1$ on D and Z the Cartier divisor of order z on D. Z is a prolongation of W and the vanishing of $h^0(D, \mathbf{O}_D(m)(-(m(m+3)/2)Q))$ implies $h^0(D, \mathbf{O}_D(m)(-Z)) < h^0\ (D, \mathbf{O}_D(-W))$. Hence $h^0(\mathbf{P}^2, \mathbf{I}_Z(m)) < h^0(\mathbf{P}^2, \mathbf{I}_W(m))$. By semicontinuity the same properties (existence of D and good postulation) are true for elements in an open subset, Ω, of $M(P, Z)$. We want to find $A \in \Omega$ such that there exists a degree m plane curve D' with P ordinary node of D' and A contained in one of the two smooth branches of D' at P. We will show by induction on the integer y that for every integer y with $1 \leq y \leq (m+2)(m+1)/2 - 2$ a general $B \in M(P, y)$ is contained in one of the two smooth branches of a plane curve D'' which has P as an ordinary node. This assertion is trivial if $y \leq 2$. Assume that it is true for the integer $y' := y-1 \geq 2$ and take such a pair (B, D'') with $B \in M(P, y-$ $-1)$. By the generality of B we may assume that B is contained in a degree m plane curve which is smooth at P. If $y-1 \geq 3$ call B' the prolongation of B along the smooth branch of D'' containing B; if $y-1=2$ we take as D'' a curve such that the Zariski tangent space of B' is one of the two lines of the tangent cone of D'' at P and call B' the prolongation of B along this branch of D''. For any zero-dimensional scheme J with $J_{\mathrm{red}} = \{P\}$ and $2 \leq \mathrm{length}(J) < (m+2)(m+1)/2$ the set of all equations of the plane curves containing J which are singular at P is either $H^0(\mathbf{P}^2, \mathbf{I}_J(m))$ or a hyperplane of $H^0(\mathbf{P}^2, \mathbf{I}_J(m))$ because any plane curve containing J has the tangent line to J at P in its tangent cone at P. Hence for $J = B$ this set is a linear space of dimension at least 2 whose general member is a curve with an ordinary node at P. Hence we may repeat the proof of (i) and obtain $A \in \Omega$ contained in a degree m curve with an ordinary node at P; here we use that $y-1 \leq$ $\leq z-2 \leq (m+2)(m+1)/2-3$. Now we may conclude the proof of (ii) and hence of part (b) of the lemma. Call Φ the open non-empty subset of Ω corresponding to the schemes, W, contained in a degree m curve D'' with an ordinary node at P and with $h^1(\mathbf{P}^2, \mathbf{I}_W(m)) = 0$. Take $Z \in q(W)$. If Z is not the prolongation of W along the corresponding branch of D'', then $h^0(\mathbf{P}^2, \mathbf{I}_Z(m)) < h^0(\mathbf{P}^2, \mathbf{I}_Z(m)) < h^0(\mathbf{P}^2, \mathbf{I}_W(m))$ and $Z \in M(z, m)$ because the set of degree m curves singular at P is a hyperplane of the hyperplane of $\mathbf{P}(H^0(\mathbf{P}^2, \mathbf{I}_W(m)))$ parametrizing the curves which are singular at P. Now we may prove also the third assertion. Fix $W \in M(P, (z+1)(z+2)/2-1)$. If there is $Z \in q(W)$ such that $Z \in M(P, (z+1)(z+2)/2; m, **)$, then $W \in M(P, (z+1)(z+$ $+2)/2-1; m, 2)(*)$. Hence part (c) follows from part (b) for the integer $z = (m+2)(m+1)/2-1$. ◊

The same inductive proof gives the following results.

Lemma 1.3. *Fix integers* $z(1) > 0, z(2) > 0$ *and* $m \geq 4$ *with* $z(1)+$ $+z(2) \leq (m+1)(m+2)/2$. *Fix distinct points* P_1 *and* P_2 *of* $\mathbf{P}^2$. *Then* $M(P_1, P_2; z(1), z(2); m, 1)$ *has codimension* ≥ 1 *in* $M(P_1, P_2;$ $z(1), z(2); m)$ *and* $M(P_1, P_2; z(1), z(2); m, 2)(*)$ *has codimension* ≥ 2 *in* $M(P_1, P_2; z(1), z(2))$. *If* $z(1)+z(2) = (m+2)(m+1)/2$, *then* $M(P_1, P_2; z(1), z(2); m, **)$ *has codimension* ≥ 2 *in* $M(P_1, P_2;$ $z(1), z(2))$.

Lemma 1.4. *Fix integers* $z(1) > 0$, $z(2) > 0$, $z(3) > 0$ *and* $m \geq 4$ *with* $z(1)+z(2)+z(3) \leq (m+1)(m+2)/2$. *Fix* 3 *non collinear points*

P_1, P_2 and P_3 of $\mathbf{P}^2$. Then $M(P_1, P_2, P_3; z(1), z(2), z(3); m, 1)$ has codimension ≥ 1 in $M(P_1, P_2, P_3; z(1), z(2), z(3))$ and $M(P_1, P_2, P_3; z(1), z(2), z(3); m, 2)(*)$ has codimension ≥ 2 in $M(P_1, P_2, P_3; z(1), z(2), z(3))$. If $z(1) + z(2) + z(3) = (m + 2)(m + 1)/2$, then $M(P_1, P_2, P_3(1), z(2); z(3); m, **)$ has codimension ≥ 2 in $M(P_1, P_2; z(1), z(2))$.

Lemma 1.5. *Fix integers $z(1) > 0$, $z(2) > 0$, $z(3) > 0$ and $m \geq 4$ with $z(1)+z(2)+z(3) \leq (m+1)(m+2)/2$. Fix 3 collinear points P_1, P_2 and P_3 of $\mathbf{P}^2$. Then $M(P_1, P_2, P_3; z(1), z(2), z(3); m, 1)$ has codimension ≥ 1 in $M(P_1, P_2, P_3; z(1), z(2), z(3))$ and $M(P_1, P_2, P_3; z(1), z(2), z(3); m, 2)(*)$ has codimension ≥ 2 in $M(P_1, P_2, P_3; z(1), z(2), z(3))$. If $z(1)+ +z(2)+z(3) = (m+2)(m+1)/2$, then $M(P_1, P_2, P_3; z(1), z(2); z(3); m, **)$ has codimension ≥ 2 in $M(P_1, P_2, P_3; z(1), z(2), z(3))$.*

We do not know how many points of the plane we may control in this way for large m.

Proof of Th. 0.1. We divide the proof into 3 steps. In the third step we pass from the statement "only ordinary Weierstrass points" to the statement "only normal Weierstrass points".

STEP 1. Here we assume $g > (d-2)(d-3)/2$. Set $x := (d - -1)(d-2)/2 - g$. Hence $0 \leq x \leq d-3$. Fix a general $S \subset \mathbf{P}^2$ and set $A(S, d) := \{$ integral nodal degree d plane curves with S as singular locus$\}$. Let $D(S, d-3)$ be the set of all degree $d-3$ curves containing S. If z in an integer > 0, set $C(S, z) := \{(P, Z) : P \in (\mathbf{P}^2 \setminus S)$ and Z is a curvilinear length z subscheme of $\mathbf{P}^2$ with $Z_{\mathrm{red}} = \{P\}\}$. By the theory of the local Hilbert scheme (see [8] or [9] or [4]) $C(S, z)$ is smooth and irreducible of dimension $z+1$. Let $\Gamma(d, S, s) := \{(C, P, Z) : C \in A(S, d)$, $(P, Z) \in C(S, z)$ and $Z \subset C\}$ be the incidence correspondence. Let $\pi_1(z) : \Gamma(d, S, z) \to A(S, d)$ and $\pi_2(z) : \Gamma(d, S, z) \to C(s, z)$ be the projections. We will use $C(S, z)$ for $z = g-1$, g and $g+1$. Since $h^0(\mathbf{P}^2, \mathbf{O}_{\mathbf{P}^2}(d-3)) = g + x$, for every $(P, Z) \in C(S, g-1)$, there is $A \in D(S, d-3)$ with $Z \subset A$. Set $C(S, z, =) := \{(P, Z) \in C(S, z)$: there is $A \in D(S, d-3)$ with $Z \subset A\}$.

Claim. *For every $z \geq g + 1 C(S, z, =)$ has codimension ≥ 2 in $C(S, z)$.*

Proof of the Claim. First we assume $x = 0$, i.e. $S = \emptyset$. By the first assertion of Lemma 1.2 $C(\emptyset, g, =)$ is a proper subset of $C(\emptyset, g)$. We will use the notion of prolongation introduced in the proof of Lemma 1.2. For every $W \in C(\emptyset, g) \setminus C(\emptyset, g, =)$ and every prolongation, Z, of W we have $Z \in C(\emptyset, g+1) \setminus C(\emptyset, g+1, =)$. Hence to check the claim for $C(\emptyset, g+1, =)$ it is sufficient to prove the existence of an algebraic subset Γ of $C(\emptyset, g)$ with codimension at least two in $C(\emptyset, g)$ and such that for every $W \in (C(\emptyset, g) \setminus \Gamma)$ a general prolongation of W does not belong to $C(\emptyset, g+1, =)$. Set $\Gamma := M(P, d-3; g, 2)(*)$. By part (c) of Lemma 1.2 we have $\dim(M(P, d-3; g-1, 2)(*)) \leq g-2$. Hence we obtain $\dim(\Gamma) \leq g-1$, as wanted. Now assume $z > g+1$ and that $\mathrm{codim}(C(\emptyset, z-1) \setminus C(\emptyset, z-1, =)) \geq 2$. For every $W \in C(\emptyset, z-1) \setminus \setminus C(\emptyset, z-1, =)$ and every prolongation, Z, of W we have $Z \in C(\emptyset, z) \setminus \setminus C(\emptyset, z, =)$. Hence every element of $C(\emptyset, z, =)$ is the prolongation of some element of $C(\emptyset, z-1, =)$. Thus every irreducible component of $C(\emptyset, z, =)$ has dimension at most $\dim(C(\emptyset, z-1, =)) + 1$ and hence $C(\emptyset, z, =)$ has codimension at least two by the claim for the integer $z - -1$. Now we will prove by induction on x the case $x > 0$. Assume $x \neq 0$ and that the Claim is true for $x-1$. Take $S' \subset \mathbf{P}^2$ with $\mathrm{card}(S') = = x-1$ and S' general. Fix one general point $Z(i)$, $1 \leq i \leq \alpha$, of every irreducible component of $C(S', z, =)$. Note that for a general $Q \in \mathbf{P}^2$ and every 0-dimensional scheme W we have $h^0(\mathbf{P}^2, \mathbf{I}_{W \cup S' \cup \{Q\}}(d-3)) = = \max\{0, h^0(\mathbf{P}^2, \mathbf{I}_{W \cup S'}(d-3)) - 1\}$. Apply this trivial observation for every $Z(i)$, $1 \leq i \leq \alpha$ and set $S := S' \cup \{Q\}$ with Q general. Since $C(S', z)$ and $C(S, z)$ are open subschemes of $C(\emptyset, z)$ and passing from $x-1$ to x we drop by 1 the geometric genus, we obtain the Claim for the pairs (x, z) with $z \geq g+2$. Now assume $z = g+1$. The proof just given works if we know that the set $C(S', g+1, =)' := \{(P, Z) \in C(S', g+1)$: there is $A \in D(S', d-3)$ with $Z \subset A\}$ is a proper subset of $C(S', g+1)$. This is easily shown by induction, but it is a triviality, just meaning that the general point of the normalization, X', of a general plane curve of degree d with $x-1$ nodes is not a Weierstrass point of X'; indeed in characteristic 0 every smooth curve of genus ≥ 2 has only finitely many Weierstrass points. $\Diamond$

Since $3x < (d+2)(d+1)/2$ and $(x, d) \neq (9, 6)$, $A(S, d)$ is a non empty open subset of a projective space of dimension $d(d+3)/2 - 3x$ ([1], Prop. 4.1). Since $3x < 3d := h^0(\mathbf{P}^2, \mathbf{O}_{\mathbf{P}^2}(d)) - h^0(\mathbf{P}^2, \mathbf{O}_{\mathbf{P}^2}(d-3))$ and a curve has dimension 1, using the Claim and its proof we will check that a general $C \in A(S, d)$ contains no scheme Z with Z_{red} a point of $C \setminus S$, $\mathrm{length}(Z) \geq g+1$ and Z contained in a degree $d-3$ adjoint curve to C. Fix S, an integer $z \geq g+1$ and $P \in (\mathbf{P}^2 \setminus S)$. Set $U(P, S, z) := \{C \in A(S, d) : P \in C$ and the length z subscheme of C supported by P is contained in a degree $d-3$ curve containing $S\}$. Since S is general and x is small, $H^0(\mathbf{P}^2, \mathbf{I}_S(d))$ has no base point outside S and hence $h^0(\mathbf{P}^2, \mathbf{I}_{S\cup\{P\}}(d)) = h^0(\mathbf{P}P^2, \mathbf{I}_S(d)) - 1$. Hence by the Claim we obtain that $U(P, S, z)$ has codimension ≥ 3 in $A(S, d)$ for a general $P \in (\mathbf{P}^2 \setminus S)$. There is at most a one-dimensional subset Ω of $\mathbf{P}^2 \setminus S$ such that for every $P \notin \Omega$ $U(P, S, z)$ has codimension 2 in $A(S, d)$; here we use that in the proof of the Claim we may take as point Q any point outside a suitable one-dimensional subset of $\mathbf{P}^2$. For every $P \in (\mathbf{P}^2 \setminus S) U(P, S, z)$ is a proper subset of $A(S, d)$ because for $d \geq 4$ and $x \leq d - 3$ and for a general $SH^0(\mathbf{P}^2, \mathbf{I}_S(d-3))$ has no base point outside S. Varying P in $\mathbf{P}^2 \setminus S$ and looking at the dimensions of the fibers of the projection $\pi_2(z)$, we conclude the checking.

Let X be the normalization of a general $C \in A(S, d)$. Now we will check that for every $P \in S$ there is $C \in A(S, d)$ such that the two local branches of C at P have a length g scheme $Z \in M(P, g; d-3, 0)'$ as intersection with the $(g-1)^{\text{th}}$ infinitesimal neighborhood of P in $\mathbf{P}^2$. Since this is an open condition, it is sufficient to prove it for one $P \in S$. By Lemma 1.2 we know that $M(P, g; d-3, 1)$ is a proper closed subscheme of the irreducible variety $M(P, g)$. Take a general $Z \in M(P, g)$. Since $S \setminus \{P\}$ is general, we have $h^0(\mathbf{P}^2, \mathbf{I}_{Z\cup(S\setminus\{P\})}(d-3)) = 0$. Since $3(x-1)+1+g \leq (d^2+3d)/2$ there is $U \in A(S, d)$ with $Z \subset U$, concluding the checking. Hence for a general $C \in A(S, d)$ the counterimages of the nodes of C are not Weierstrass points of X. Alternatively, one could use the proof of (ii) made in the proof of Lemma 1.2. Hence X has only normal Weierstrass points.

STEP 2. Now we assume $g \leq (d-2)(d-3)/2$. Let y be the unique integer with $4 \leq y < d$ and such that $(y-2)(y-3)/2 < g \leq$ $\leq (y-1)(y-2)/2$. Set $x := (y-1)(y-2)/2 - g$. Hence $0 \leq x \leq$ $\leq y - 3$. We apply the first part of the proof to the pair of integers (y, x). We obtain an irreducible degree y plane curve T with x nodes as only singularities and whose normalization, Z, is a smooth genus g curve with only normal Weierstrass points; to obtain "normal" instead of "ordinary", see Step 3. We fix $d - y$ general lines D_i, $1 \leq i \leq d - y$, and set $Y := T \cup (\cup_{1\leq i\leq d-y} D_i)$. Hence Y is a nodal curve. For each integer i with $1 \leq i \leq d - y$ we fix one point, say P_i, of $T \cap D_i$. With the language of the theory of nodal plane curves (see [11]) we take the points P_i, $1 \leq i \leq d - y$, as unassigned nodes of Y, while we take the remaining $(d-1)(d-2)/2 - g$ singular points of Y as assigned nodes. By the theory in [11] there is a one-dimensional family of plane curves with Y as a special fiber, with an irreducible nodal curve of geometric genus g as a general fiber and such that the total space of this flat family of plane curves has $(d-1)(d-2)/2-g$ disjoint sections which on the general fiber have as images the singular points and on the special fiber Y have as images the assigned nodes. After a further base change we may take a partial normalization of the total space along these sections. We obtain a one-dimensional smoothing of the union, W, of Z and $d-y$ smooth rational curve, R_i, each of them intersecting Z at one point (corresponding to the point P_i through the normalization map $Z \to T$). Note that W is a curve of compact type and that, with the terminology of [5], the curves R_i are rational tails. Now we apply the theory of limit linear series of Eisenbud-Harris ([5] and [6]).

STEP 3. Let X be the normalization of a general $C \in A(S,d)$. We checked at the end of Step 1 that the points of X going to the nodes of C are not Weierstrass points of X. Take a point Q of X whose image, P, in C is a smooth point of C. It is sufficient to check that $h^0(X, \mathbf{O}_X((g-1)Q)) = 1$ and $h^1(X, \mathbf{I}_X((g+1)Q)) = 0$ because if these equalities are satisfied either $h^0(X, \mathbf{O}_X(gQ)) = 2$ (i.e. Q is a normal Weierstrass point) or $h^0(X, \mathbf{O}_X(gQ)) = 1$ (i.e. Q is not a Weierstrass point). The assertion on $h^0(X, \mathbf{O}_X((g-1)Q))$ (resp. $h^1(X, \mathbf{O}_X((g+1)Q)))$ is true for a general $C \in A(S,d)$ by the Claim concerning the codimension of $C(S, g-1, =)$ (resp. $C(S, g+1, =)$). Then the inductive proof given in Step 2 works for normal Weierstrass points. $\Diamond$

Proof of Th. 1.1. Using Lemma 1.3, 1.4 and 1.5 instead of Lemma 1.2 in the proof of Th. 0.1 we obtain Th. 1.1; to apply Lemma 1.5 for the proof of parts (iii), (iv) and (v) of Th. 1.1 note that the set of collinear triples of points of $\mathbf{P}^2$ has dimension 5. $\Diamond$

We believe that the interested reader may use the same method to prove the existence of nodal plane curves with a certain type of non normal Weierstrass points in the following way. We fix integers d, t as in the statement of Th.0.1 and a general $S \subset \mathbf{P}^2$ with card$(S) = t$. Set $g := (d-1)(d-2)/2-t$. We fix an integer z with $g+1 \leq z \leq (d^2+3d)-t$ and a general $P \in (\mathbf{P}^2 \setminus S)$. We look for a curvilinear subscheme Z with $Z_{\text{red}} = \{P\}$, length$(Z) = z$ and with $h^0(\mathbf{P}^2, \mathbf{I}_Z(d-3)) \neq 0$, say with $h^0(\mathbf{P}^2, \mathbf{I}_Z(d-3)) = 1$. This is easy: just use a smooth degree $d-3$ curve Y with $S \subset Y$ and $P \subset Y$. By Bertini's theorem it is easy (at least for certain d, t and z) to show that a general degree d curve C with $Z \subset C$ and with $S \subseteq \text{Sing}(C)$ is an irreducible nodal curve with $S = \text{Sing}(C)$. Let X be the normalization of C. By construction P is a non-ordinary

Weierstrass point of X such that zP is a special divisor of X. We need to check that for general Z (i.e. for general Y) and the general such C the divisor $(z+1)P$ is a non-special divisor of X. Again, this is easy for certain d, t and z. By construction $z+1$ would be the last gap value of P as Weierstrass point of X. It would be nice to prove the existence of (Z, X) such that, with this constraint, the gap sequence of P has the smallest weight, i.e. the gap sequence is $1, \ldots, g-1, z+1$. To obtain this result it is sufficient to find Z such that $h^0(\mathbf{P}^2, \mathbf{I}_{S\cup Z'}(d-3)) = 1$, where Z' is the unique subscheme of Z with $\text{length}(Z') = g-1$. Then we would like to check as in the proof of Th. 0.1 that P is the only non-normal Weierstrass point of X. Again, this is easy for certain d, t and z, but we do not know if such result is true in full generality. Note that the checking of the two conditions "gap sequence $1, \ldots, g-1, z+1$" and "P is the only non-normal Weierstrass point of X" are independent and that both are open conditions. The interested reader may do the same on the Hirzebruch surfaces F_e, $e \geq 0$.

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