Mathematica Pannonica
11/2 (2000), 249–260

MULTIFUNCTIONS WITH SELECTIONS OF CONVEX AND CONCAVE TYPE

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Dedicated to the 60th birthday of Prodessor Ludwig Reich

Received: December 1998

MSC 2000: 26 B 25, 26 E 25; 54 C 65

Keywords: Multifunctions, selections, convex (quasiconvex, subadditive, sublinear) functions.

Abstract: We characterize multifunctions $H : X \longrightarrow \mathrm{cc}(\mathbb{R})$ which have selections $h_1 \in \mathcal{F}$ and $h_2 \in -\mathcal{F}$, where $\mathcal{F}$ is the family of convex, midconvex, subadditive, sublinear, quasiconvex and nondecreasing functions, respectively. The problem of the existence of selections belonging to $\mathcal{F} \cap -\mathcal{F}$ for such multifunctions is also discussed.

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Introduction

Let $\mathcal{F}$ be a family of real functions defined on a set X, $-\mathcal{F} = \{-f : f \in \mathcal{F}\}$ and $\mathcal{H} = \mathcal{F} \cap -\mathcal{F} \neq \emptyset$. Denote by $\mathrm{n}(\mathbb{R})$ the family of all non–empty subsets of $\mathbb{R}$ and by $\mathrm{cc}\,(\mathbb{R})$ the family of all convex and compact members of $\mathrm{n}(\mathbb{R})$. Recall that a function $h : X \longrightarrow \mathbb{R}$ is said to be a *selection* of a multifunction $H : X \longrightarrow \mathrm{n}(Y)$ if $h(x) \in H(x)$ for all $x \in X$.

The aim of this paper is to characterize multifunctions $H : X \longrightarrow \mathrm{cc}\,(\mathbb{R})$ which have selections $h_1 \in \mathcal{F}$ and $h_2 \in -\mathcal{F}$, in the case, where $\mathcal{F}$ is the family of convex, midconvex, subadditive, sublinear, quasiconvex and nondecreasing functions, respectively. We also consider the following problem: suppose that a multifunction H has selections $h_1 \in \mathcal{F}$ and $h_2 \in -\mathcal{F}$. Does there exist a selection of H belonging to $\mathcal{H}$? We give full answer to this problem for the classes $\mathcal{F}$ mentioned above.

1. Multifunctions with convex and concave selections

We start with a theorem characterizing multifunctions having a convex and a concave selection.

Theorem 1. *Let D be a convex subset of a real vector space X. A multifunction $H : D \longrightarrow \mathrm{cc}\,(\mathbb{R})$ has a convex and a concave selection if and only if*

$$H\Big(\sum_{i=1}^{n} t_i x_i\Big) \cap \sum_{i=1}^{n} t_i H(x_i) \neq \emptyset \tag{1}$$

for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in D$ and $t_1, \ldots, t_n \geq 0$ with $t_1 + \cdots + t_n = 1$.

Proof. Put $f(x) = \inf H(x)$ and $g(x) = \sup H(x)$, $x \in D$. Then $H(x) = [f(x), g(x)]$ and $\sum_{i=1}^{n} t_i H(x_i) = \Big[\sum_{i=1}^{n} t_i f(x_i), \sum_{i=1}^{n} t_i g(x_i)\Big]$. Hence H satisfies (1) iff f and g satisfy the system of inequalities

$$(2) \qquad f\Big(\sum_{i=1}^{n} t_i x_i\Big) \le \sum_{i=1}^{n} t_i g(x_i),$$

$$(3) \qquad g\Big(\sum_{i=1}^{n} t_i x_i\Big) \ge \sum_{i=1}^{n} t_i f(x_i).$$

By the sandwich theorem proved in [1] (cf. also [4]), f and g satisfy (2) iff they can be separated by a convex function, i.e. iff H has a convex selection. Similarly, f and g satisfy (3) iff H has a concave selection. This finishes the proof. ◊

Remark 1. It follows from the theorem proved in [6] that a multifunction $H : I \longrightarrow \mathrm{cc}(\mathbb{R})$, where $I \subset \mathbb{R}$ is an interval, has a convex and a concave selection iff it has an affine selection. From the same theorem we get also that $H : I \longrightarrow \mathrm{cc}(\mathbb{R})$ has an affine selection iff

$$H\big(tx_1 + (1-t)x_2\big) \cap \big[tH(x_1) + (1-t)H(x_2)\big] \ne \emptyset$$

for all $x_1, x_2 \in I$ and $t \in (0,1)$ (cf. [8]).

Analogous results are not true for multifunctions defined on arbitrary convex set $D \subset X$, where $\dim X > 1$. As a counterexample we can take the multifunction $H(x) = \big[f(x), g(x)\big]$ described in [6, Remark 2], which is defined on a square $D \subset \mathbb{R}^2$, has a convex and a concave selection (and so it satisfies (1)), but it has no affine selection.

2. Multifunctions with midconvex and midconcave selections

Let D be a convex subset of a real vector space. A function $f :$ $: D \longrightarrow \mathbb{R}$ is said to be *midconvex* (or *Jensen convex*) if

$$(4) \qquad f\Big(\frac{x+y}{2}\Big) \le \frac{f(x)+f(y)}{2}, \quad x, y \in D;$$

f is *midconcave* (*Jensen concave*) if $-f$ is midconvex. If f satisfies (4) with equality, it is called a *Jensen function.*

Theorem 2. *A multifunction* $H : D \longrightarrow \mathrm{cc}(\mathbb{R})$ *has a midconvex and a midconcave selection if and only if*

$$(5) \qquad H\Big(\frac{1}{2^n}\sum_{i=1}^{2^n} x_i\Big) \cap \frac{1}{2^n}\sum_{i=1}^{2^n} H(x_i) \ne \emptyset$$

for all $n \in \mathbb{N}$ *and* $x_1, \ldots, x_{2^n} \in D$.

Proof. Let $H(x) = \big[f(x), g(x)\big]$, $x \in D$. Then H satisfies (5) iff f and g satisfy

$$f\left(\frac{1}{2^n}\sum_{i=1}^{2^n} x_i\right) \le \frac{1}{2^n}\sum_{i=1}^{2^n} g(x_i)$$

and

$$g\left(\frac{1}{2^n}\sum_{i=1}^{2^n} x_i\right) \ge \frac{1}{2^n}\sum_{i=1}^{2^n} f(x_i).$$

By the result on separation by midconvex functions (cf. [4, Cor. 5]), the above inequalities mean that f and g can be separated by a midconvex and a midconcave function, i.e. H has a midconvex and a midconcave selection. ◊

Remark 2. Condition (5) does not imply that H has a Jensen selection (even if $D \subset \mathbb{R}$). The following example gives a multifunction $H : \mathbb{R} \longrightarrow \operatorname{cc}(\mathbb{R})$ which has a midconvex and a midconcave selection, but it has no Jensen selection.

Example 1. Let $\Gamma = \{h_1, h_2, h_\alpha : \alpha \in A\}$ be a Hamel base of $\mathbb{R}$ over $\mathbb{Q}$. For $x \in \mathbb{R}$, $x = r_1 h_1 + r_2 h_2 + \sum_{\alpha\in A} r_\alpha h_\alpha$, $(r_1, r_2, r_\alpha \in \mathbb{Q})$, we put

$$f(x) = |r_1 - r_2|, \qquad g(x) = 2 - |r_1 + r_2|$$

and

$$H(x) = \Big[\min\{f(x), g(x)\}, \max\{f(x), g(x)\}\Big].$$

It is easy to check that f is midconvex and g is midconcave. Thus H has a midconvex and a midconcave selection. Suppose that H admits a Jensen selection h. It has to be of the form $h = c + a$, where $c \in \mathbb{R}$ and a is an additive function. Then

$$\text{(6)}\quad \min\{f(x), g(x)\} \le \le c + r_1 a(h_1) + r_2 a(h_2) + \sum_{\alpha\in A} r_\alpha a(h_\alpha) \le \max\{f(x), g(x)\}.$$

In particular, taking $x_n = n h_\alpha$, $n \in \mathbb{N}$, we get

$$0 \le c + n a(h_\alpha) \le 2, \quad n \in \mathbb{N},$$

which implies that $a(h_\alpha) = 0$ for every $\alpha \in A$. Thus (6) reduces to

$$\text{(7)}\quad \min\{f(x), g(x)\} \le c + r_1 a(h_1) + r_2 a(h_2) \le \max\{f(x), g(x)\}.$$

Taking in (7) $x_1 = h_1 + h_2$ and next $x_2 = -h_1 - h_2$, we get

$$0 \le c + a(h_1) + a(h_2) \le 0$$

and

$$0 \le c - a(h_1) - a(h_2) \le 0.$$

Hence $c = 0$. On the other hand, taking in (7) $x_3 = h_1 - h_2$ and next $x_4 = -h_1 + h_2$, we get

$$2 \le c + a(h_1) - a(h_2) \le 2$$

and

$$2 \le c - a(h_1) + a(h_2) \le 2,$$

which implies that $c = 2$. The obtained contradiction shows that there is no Jensen selection of H.

3. Multifunctions with subadditive and superadditive selections

Let $(S, +)$ be an abelian semigroup. A function $f : S \longrightarrow \mathbb{R}$ is called *subadditive* if

$$f(x+y) \le f(x) + f(y), \quad x, y \in S;$$

f is *superadditive* if $-f$ is subadditive.

Theorem 3. *A multifunction $H : S \longrightarrow \mathrm{cc}(\mathbb{R})$ has a subadditive and a superadditive selection if and only if*

$$H\Big(\sum_{i=1}^{n} x_i\Big) \cap \sum_{i=1}^{n} H(x_i) \ne \emptyset \tag{8}$$

for all $n \in \mathbb{N}$ and $x_1, \dots, x_n \in S$.

Proof. If $H(x) = \big[f(x), g(x)\big]$, $x \in S$, then (8) is equivalent to

$$f\Big(\sum_{i=1}^{n} x_i\Big) \le \sum_{i=1}^{n} g(x_i) \quad \text{and} \quad g\Big(\sum_{i=1}^{n} x_i\Big) \ge \sum_{i=1}^{n} f(x_i).$$

These inequalities mean that f and g can be separated by a subadditive and a superadditive function (cf. [5, Th. 1] or [4, Cor. 1]). This finishes the proof. ◇

Remark 3. Condition (8) does not imply that H has an additive selection. The example below (cf. [5, Ex. 2]) gives a multifunction $H :$ $: [0, \infty) \longrightarrow \mathrm{cc}(\mathbb{R})$ which has a subadditive and a superadditive selection but it has not any additive selection.

Example 2. Consider the functions $f, g : [0, \infty) \longrightarrow \mathbb{R}$ defined by

$$f(x)=\begin{cases}0, & x\neq 1\\ 1, & x=1\end{cases},\quad g(x)=\begin{cases}\sqrt{x}, & x\in[0,2)\\ x, & x\geq 2\end{cases}.$$

Put $H(x)=\big[f(x),g(x)\big]$, $x\in[0,\infty)$. It is easily seen that the functions

$$h_1(x)=\sqrt{x}\quad\text{and}\quad h_2(x)=\begin{cases}0, & x\in[0,1)\\ 1, & x\in[1,2)\\ x, & x\geq 2\end{cases}$$

are, respectively, a subadditive and a superadditive selections of H. Suppose that $a:[0,\infty)\longrightarrow\mathbb{R}$ is an additive selection of H. Since $H(1)=\{1\}$, we have $a(1)=1$ and, consequently, $a(x)=x$ for all $x\in\mathbb{Q}\cap[0,\infty)$. Hence $a(x)>g(x)$ for $x\in(1,2)\cap\mathbb{Q}$. The obtained contradiction shows that H has no additive selection.

4. Multifunctions with sublinear and superlinear selections

Let X be a real vector space. Recall that a function $f:X\longrightarrow\mathbb{R}$ is called *sublinear* if it is subadditive and $f(tx)=tf(x)$ for all $x\in X$ and $t\geq 0$; f is *superlinear* if $-f$ is sublinear.

Theorem 4. *A multifunction $H:X\longrightarrow \mathrm{cc}(\mathbb{R})$ has a sublinear and a superlinear selection if and only if*

$$H\Big(\sum_{i=1}^{n}t_ix_i\Big)\cap\sum_{i=1}^{n}t_iH(x_i)\neq\emptyset \tag{9}$$

for all $n\in\mathbb{N}$, $x_1,\dots,x_n\in X$ and $t_1,\dots,t_n\geq 0$.

Proof. Let $H(x)=\big[f(x),g(x)\big]$, $x\in X$. Condition (9) means that

$$f\Big(\sum_{i=1}^{n}t_ix_i\Big)\leq\sum_{i=1}^{n}t_ig(x_i)\quad\text{and}\quad g\Big(\sum_{i=1}^{n}t_ix_i\Big)\geq\sum_{i=1}^{n}t_if(x_i).$$

By the theorem on separation by sublinear functions (cf. [5, Th.] or [4, Cor. 9]), these inequalities are equivalent to the fact that f and g can be separated by a sublinear and a superlinear function. $\Diamond$

Remark 4. In the case where $X=\mathbb{R}$ or $X=\mathbb{R}^2$ condition (9) is equivalent to the fact that H has a linear selection (cf. Ths. 4a and 4b below). However for $X=\mathbb{R}^3$ analogous result is not true.

Example 3. Consider the functions $f,g:\mathbb{R}^3\longrightarrow\mathbb{R}$ defined by

$$f(x,y,z)=\begin{cases} 2z-x, & x\geq 0\\ 2z, & x<0\end{cases},\quad g(x,y,z)=\sqrt{x^2+y^2}+z$$

and put $H(w)=\Big[\min\{f(w),g(w)\},\max\{f(w),g(w)\}\Big]$, $w=(x,y,z)\in\in\mathbb{R}^3$. It is easy to check that f is sublinear and g is superlinear. Of course these functions are selections of H. Suppose now that H has a linear selection h. It has to be of the form

$$h(x,y,z)=ax+by+cz$$

with some $a,b,c\in\mathbb{R}$. Since $f(0,1,1)=g(0,1,1)=2$, $f(-1,0,1)==g(-1,0,1)=2$ and $f(1,0,2)=g(1,0,2)=3$, we also have

$$h(0,1,1)=2,\ h(-1,0,1)=2\quad\text{and}\quad h(1,0,2)=3.$$

Hence $b+c=2$, $-a+c=2$ and $a+2c=3$, which implies that

$$h(x,y,z)=\frac{1}{3}(-x+y+5z).$$

But this leads to a contradiction, because

$$f\big(-\sqrt{2},\sqrt{2},2\big)=g\big(-\sqrt{2},\sqrt{2},2\big)=4\quad\text{and}\quad h\big(-\sqrt{2},\sqrt{2},2\big)\neq 4.$$

Thus H has no linear selection.

Theorem 4a. *Let* $H:\mathbb{R}\longrightarrow \mathrm{cc}\,(\mathbb{R})$. *The following conditions are equivalent:*

(i) $H(t_1x_1+t_2x_2)\cap\big[t_1H(x_1)+t_2H(x_2)\big]\neq\emptyset$, $x_1,x_2\in\mathbb{R}$, $t_1,t_2\geq 0$;
(ii) *H has a sublinear and a superlinear selection;*
(iii) *H has a linear selection.*

Proof. (i) $\Rightarrow$ (ii) Let $H(x)=\big[f(x),g(x)\big]$, $x\in\mathbb{R}$. By (i) we have

$$f(t_1x_1+t_2x_2)\leq t_1g(x_1)+t_2g(x_2),$$
$$g(t_1x_1+t_2x_2)\geq t_1f(x_1)+t_2f(x_2).$$

These inequalities imply that f and g can be separated by a sublinear and a superlinear function (cf. [5, Th. 3]). Thus H has a sublinear and a superlinear selection.

(ii) $\Rightarrow$ (iii). (For other proof see [5, Cor. 1].) Let h_1 be a sublinear and h_2 — a superlinear selection of H. Consider the multifunction $H_0:\,:\mathbb{R}\longrightarrow\mathrm{cc}\,(\mathbb{R})$ defined by

$$H_0(x)=\Big[\min\{h_1(x),h_2(x)\},\max\{h_1(x),h_2(x)\}\Big]. \tag{10}$$

It is clear that $H_0(x)\subset H(x)$, $x\in\mathbb{R}$, $H_0(0)=\{0\}$ and (by Th. 4)

$$(11)\qquad H_0\Big(\sum_{i=1}^{n} t_i x_i\Big) \cap \sum_{i=1}^{n} t_i H_0(x_i) \neq \emptyset$$

for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in \mathbb{R}$ and $t_1, \ldots, t_n \geq 0$.

We will show that H_0 has a linear selection. Since the space of all linear functions $l : \mathbb{R} \longrightarrow \mathbb{R}$ is one–dimensional, it is enough to show that for every two points $x_1, x_2 \in \mathbb{R}$ there exists a linear function $l : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$(12)\qquad l(x_i) \in H_0(x_i), \quad i = 1, 2$$

(cf. [2, Cor. 3]). Let us fix $x_1, x_2 \in \mathbb{R}$. If $x_1 = 0$ or $x_2 = 0$, then the existence of a linear function satisfying (12) is clear. If $x_1, x_2 > 0$ or $x_1, x_2 < 0$, then $x_2 = sx_1$ with some $s > 0$. By (11) we have

$$H_0(x_2) \cap sH_0(x_1) \neq \emptyset,$$

that is there exist $y_1 \in H_0(x_1)$ and $y_2 \in H_0(x_2)$ with $y_2 = sy_1$. Define $l : \mathbb{R} \longrightarrow \mathbb{R}$ to be linear and such that $l(x_1) = y_1$. Then $l(x_2) = sl(x_1) = sy_1 = y_2$, i.e. l satisfies (12). If $x_1 < 0 < x_2$ or $x_2 < 0 < x_1$, then $0 = sx_1 + (1-s)x_2$ with some $s \in (0, 1)$. By (11)

$$H_0(0) \cap \big[sH_0(x_1) + (1-s)H_0(x_2)\big] \neq \emptyset.$$

Since $H_0(0) = \{0\}$, we have $0 = sy_1 + (1-s)y_2$ with some $y_1 \in H_0(x_1)$ and $y_2 \in H_0(x_2)$. Define $l : \mathbb{R} \longrightarrow \mathbb{R}$ to be affine and such that $l(x_i) = y_i$, $i = 1, 2$. Then $l(0) = sl(x_1) + (1-s)l(x_2) = sy_1 + (1-s)y_2 = 0$, which means that l is linear.

Since the implication (iii) $\Rightarrow$ (i) is obvious, the proof is finished. ◊

Theorem 4b. *Let $H : \mathbb{R}^2 \longrightarrow \mathrm{cc}(\mathbb{R})$. The following conditions are equivalent:*

(i) $H(t_1x_1 + t_2x_2 + t_3x_3) \cap \big[t_1H(x_1) + t_2H(x_2) + t_3H(x_3)\big] \neq \emptyset$, $x_1, x_2, x_3 \in \mathbb{R}$, $t_1, t_2, t_3 \geq 0$;

(ii) *H has a sublinear and a superlinear selection;*

(iii) *H has a linear selection.*

Proof. The implication (i) $\Rightarrow$ (ii) follows from the theorem on separation by sublinear functions (cf. [5, Th. 3]).

(ii) $\Rightarrow$ (iii) Let h_1 be a sublinear and h_2 be a superlinear selection of H. We will show that the multifunction $H_0 : \mathbb{R}^2 \longrightarrow \mathrm{cc}(\mathbb{R})$ defined by (11) has a linear selection. Since the space of all linear functions $l : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is two–dimensional, it is enough to show that for every three points $x_1, x_2, x_3 \in \mathbb{R}^2$ there exists a linear function $l : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that $l(x_i) \in H_0(x_i)$, $i = 1, 2, 3$ (cf. [2, Cor. 3]). Let us fix $x_1, x_2, x_3 \in \mathbb{R}^2$. Since the restriction of H_0 to every straight line passing through 0 has, in view of Th. 4a, a linear selection, we may assume that these points do not lie on any such straight line. Then two of them are linearly independent and the third one is their linear combination. Assume that $x_3 = sx_1 + tx_2$ with some $s, t \in \mathbb{R}$. The following three cases are possible:

I. $s, t \geq 0$. Then in view of Th. 4 we have

$$H_0(x_3) \cap \left[sH_0(x_1) + tH_0(x_2)\right] \neq \emptyset,$$

i.e. there exist $y_i \in H_0(x_i)$, $i = 1, 2, 3$ with $y_3 = sy_1 + ty_2$. Define $l : \mathbb{R}^2 \longrightarrow \mathbb{R}$ to be linear and such that $l(x_i) = y_i$, $i = 1, 2$. Then also $l(x_3) = sl(x_1) + tl(x_2) = sy_1 + ty_2 = y_3$. Thus $l(x_i) \in H_0(x_i)$, $i = 1, 2, 3$.

II. $s, t \leq 0$. Then $0 = x_3 - sx_1 - tx_2$ and by Th. 4

$$H_0(0) \cap \left[H_0(x_3) - sH_0(x_1) - tH_0(x_2)\right] \neq \emptyset.$$

Since $H_0(0) = \{0\}$, we have $0 = y_3 - sy_1 - ty_2$ with some $y_i \in H_0(x_i)$, $i = 1, 2, 3$. Define $l : \mathbb{R}^2 \longrightarrow \mathbb{R}$ to be affine and such that $l(x_i) = y_i$, $i = 1, 2, 3$ (note that the points x_1, x_2, x_3 are affinely independent because x_1, x_2 are linearly independent and $0 \in \operatorname{conv}\{x_1, x_2, x_3\}$). Then

$$l(0) = l\Big(\frac{1}{1-s-t}x_3 - \frac{s}{1-s-t}x_1 - \frac{t}{1-s-t}x_2\Big) = \\ = \frac{1}{1-s-t}(y_3 - sy_1 - ty_2) = 0.$$

Thus l is linear and $l(x_i) \in H_0(x_i)$, $i = 1, 2, 3$.

III. $s \cdot t < 0$. Assume, for instance, that $s > 0$ and $t < 0$. Then $x_1 = \frac{1}{s}x_3 - \frac{t}{s}x_2$. Since the vectors x_2, x_3 are also linearly independent and the coefficients $\frac{1}{s}$ and $-\frac{t}{s}$ are positive, this case reduces to the first one.

This finishes the proof because the implication (iii) $\Rightarrow$ (ii) is obvious. $\Diamond$

5. Multifunctions with quasiconvex and quasiconcave selections

Let D be a convex subset of a real vector space X. Recall that a function $f : D \longrightarrow \mathbb{R}$ is called *quasiconvex* if

$$f\big(tx + (1-t)y\big) \leq \max\{f(x), f(y)\}, \quad x, y \in D,\ t \in [0,1]; \tag{13}$$

f is *quasiconcave* if

(14) $$f\big(tx+(1-t)y\big) \ge \min\{f(x), f(y)\}, \quad x, y \in D,\ t \in [0,1];$$

If f satisfies both conditions (13) and (14), it is called *quasimonotonic* (or *quasiaffine*). Obviously, if $X = \mathbb{R}$, then the quasimonotonicity coincides with the usual monotonicity.

Theorem 5. *A multifunction $H : D \longrightarrow \mathrm{cc}(\mathbb{R})$ has a quasiconvex and a quasiconcave selection iff*

(15) $$H\Big(\sum_{i=1}^{n} t_i x_i\Big) \cap \operatorname{conv}\big[H(x_1) \cup \ldots \cup H(x_n)\big] \neq \emptyset$$

for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in D$ and $t_1, \ldots, t_n \ge 0$ with $t_1 + \cdots + t_n = 1$.

Proof. Let $H(x) = \big[f(x), g(x)\big]$, $x \in D$. Then

$$\operatorname{conv}\big[H(x_1) \cup \ldots \cup H(x_n)\big] = \\ = \Big[\min\big\{f(x_1), \ldots, f(x_n)\big\}, \max\big\{g(x_1), \ldots, g(x_n)\big\}\Big]$$

and a condition (15) is equivalent to the system of inequalities

$$f\Big(\sum_{i=1}^{n} t_i x_i\Big) \le \max\big\{g(x_1), \ldots, g(x_n)\big\},$$

$$g\Big(\sum_{i=1}^{n} t_i x_i\Big) \ge \min\big\{f(x_1), \ldots, f(x_n)\big\}.$$

The first of them means that f and g can be separated by a quasiconvex function (cf. [7, Th. 2]), i.e. H has a quasiconvex selection. Similarly, the second one means that H has a quasiconcave selection. ◊

Remark 5. It is proved in [3] that a multifunction $H : I \longrightarrow \mathrm{cc}(\mathbb{R})$, where $I \subset \mathbb{R}$ is an interval, has a quasiconvex and a quasiconcave selection iff it has a monotonic selection. Moreover, $H : I \longrightarrow \mathrm{cc}(\mathbb{R})$ has a monotonic selection iff

$$H\big(tx_1 + (1-t)x_2\big) \cap \operatorname{conv}\big(H(x_1) \cup H(x_2)\big) \neq \emptyset$$

for all $x_1, x_2 \in I$, $t \in [0,1]$. Analogous results are not true for multifunctions defined on $D \subset X$ with $\dim X > 1$.

Example 4. Let

$$A_1 = \big\{(x,y) : x \ge 0, y \ge 0\big\}, \quad A_2 = \big\{(x,y) : x < 0, y \ge 0\big\},$$
$$A_3 = \big\{(x,y) : x < 0, y < 0\big\}, \quad A_4 = \big\{(x,y) : x \ge 0, y < 0\big\}.$$

Consider the functions $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x,y)=\begin{cases}0, & (x,y)\in A_1\cup A_2\\ 1, & (x,y)\in A_3\\ 2, & (x,y)\in A_4\end{cases}, \quad g(x,y)=\begin{cases}0, & (x,y)\in A_2\\ 1, & (x,y)\in A_3\\ 2, & (x,y)\in A_1\cup A_4\end{cases}$$

and put

$$H(x,y)=\big[f(x,y),g(x,y)\big], \quad (x,y)\in\mathbb{R}^2.$$

It is easily seen that f is quasiconvex and g is quasiconcave. Of course, these functions are selections of H, and hence condition (15) is fulfilled. Suppose that h is a quasimonotonic selection of H. Then

$$h(x,y)=\begin{cases}0, & (x,y)\in A_2\\ 1, & (x,y)\in A_3\\ 2, & (x,y)\in A_4.\end{cases}$$

Consider the sets

$B=\{(x,y)\in\mathbb{R}^2 : h(x,y)\le 1\}$ and $C=\{(x,y)\in\mathbb{R}^2 : h(x,y)\ge 1\}$.

By the quasimonotonicity of h these sets are convex. Since $A_2\cup A_3\subset B$ and $A_4\cap B=\emptyset$, we have $A_1\cap B=\emptyset$. Similarly, since $A_3\cup A_4\subset C$ and $A_2\cap C=\emptyset$, we have $A_1\cap C=\emptyset$. But this is impossible, because $B\cup C=\mathbb{R}^2$. Thus H has no quasimonotonic selection.

6. Multifunctions with nondecreasing and nonincreasing selections

Finally we present a simple result connected with nondecreasing and nonincreasing selections.

Theorem 6. *Let $I\subset\mathbb{R}$ be an interval and $H : I\longrightarrow \mathrm{cc}(\mathbb{R})$. The following conditions are equivalent:*

(i) *H has a nonincreasing and a nondecreasing selection;*

(ii) *$H(x)\cap H(y)\neq\emptyset$ for all $x,y\in I$;*

(iii) *H has a constant selection.*

Proof. Let $H(x)=\big[f(x),g(x)\big]$, $x,y\in I$.

(i) $\Rightarrow$ (ii) Suppose, contrary to our claim, that $\big[f(x),g(x)\big]\cap$ $\cap\big[f(y),g(y)\big]=\emptyset$ for some $x,y\in I$. Then $f(x)>g(y)$ or $f(y)>$ $> g(x)$ and H would not have either nondecreasing or nonincreasing selection, which contradicts the assumption.

(ii) $\Rightarrow$ (iii) Condition (ii) implies that $f(x)\le g(y)$ for all $x,y\in I$. Hence

$$c := \sup\{f(x) : x \in I\} \leq \inf\{g(y) : y \in I\}$$

and the function $h \equiv c$ is a constant selection of H.

The implication (i) $\Rightarrow$ (ii) is obvious. $\Diamond$

Remark 6. Let us note that conditions (i) and (ii) are equivalent for arbitrary multifunction $H : A \longrightarrow \mathrm{cc}\,(\mathbb{R})$ defined on a non–empty set A. More general, it follows from the classical Helly's theorem, that a multifunction $H : A \longrightarrow cc(\mathbb{R}^n)$ has a constant selection iff

$$H(x_1) \cap \ldots \cap H(x_{n+1}) \neq \emptyset$$

for all $x_1, \ldots, x_{n+1} \in A$ (cf. also [2, Cor. 1]).

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