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# ON PROPERTIES OF QUADRATIC MATRICES

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**Abstract**: The aim of this paper is to derive new properties of quadratic matrices, i.e. matrices satisfying a quadratic equation $(A - pI)(A - qI) = 0$ where $p, q \in \mathbb{C}$ and $I$ denotes the identity matrix. We focus our attention to numerical range, singular values of quadratic matrices and the closest normal matrix to a quadratic matrix.

## 1. Introduction and notations

The purpose of this paper is to give a general characterization of quadratic matrices. The set of all $n$-by -$n$ matrices over $\mathbb{C}$ is denoted by $\mathcal{M}_n$. We say that $A \in \mathcal{M}_n$ is a quadratic matrix, if there exist $p, q \in \mathbb{C}$

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such that $(A-pI)(A-qI)=0$ where $I=I_n$ denotes the $n\times n$ identity matrix.

Such matrices find many applications in applied linear algebra. The set of quadratic matrices includes the set of projections ($A^2=A$), involutions ($A^2=I$), nilpotents ($A^2=0$) and elementary matrices ($A=I-uw^*$), eg. Householder reflections and elimination matrices (see [9]).

For $A\in\mathcal{M}_n$, let $\lambda(A)$ and $\sigma(A)$ denote, respectively, the spectrum and the set of singular values of $A$. The singular values of $A$ are the positive square roots of the eigenvalues of the Hermitian positive semi-definite matrix $A^*A$ (see eg. [4], [11], [18]). Here $A^*\in\mathcal{M}_n$ stands for the matrix formed by conjugating each element and taking the transpose. The singular values of $A$ can be found from the singular value decomposition SVD (see [4], [11]).

**Theorem 1.1** (SVD). *Every matrix $A\in\mathcal{M}_n$ of* rank $r$ *can be written as $A=U\Sigma V^*$, where $U,V\in\mathcal{M}_n$ are unitary and $\Sigma=\mathrm{diag}(\sigma_1,\ldots,\sigma_r,$ $0,\ldots,0)$ is a diagonal matrix with nonnegative main diagonal entries $\sigma_i$, called the singular values of $A$.*

Let $A^\dagger$ denote the pseudoinverse of $A\in\mathcal{M}_n$ defined as

$$A^\dagger=V\,\mathrm{diag}\left(\frac{1}{\sigma_1},\ldots,\frac{1}{\sigma_r},0,\ldots,0\right)U^*.$$

Then $X=A^\dagger$ is uniquely determined by the Moore–Penrose conditions: $AXA=A$, $XAX=X$, $AX=(AX)^*$ and $XA=(XA)^*$. For more details we refer the reader to [11].

We will consider the 2-norm (spectral norm) and Frobenius norm of $A$, i.e. $\|A\|_2=\sigma_1$ and $\|A\|_F=\sqrt{\sigma_1{}^2+\ldots+\sigma_r{}^2}$.

We investigate properties of quadratic matrices using known properties of projections. We summarize some basic properties of projections that are relevant to our discussion in the later sections.

A projection (idempotent) $Z\in\mathcal{M}_n$ is a matrix such that $Z^2=$ $=Z$, so $I-Z$ and $Z^*$ are also projections. It is diagonalizable and $\mathrm{rank}(Z)=\mathrm{tr}(Z)$. The minimal equation of a projection $Z$ is $\lambda^2=\lambda$, hence $Z$ has as only $+1$ and $0$ as eigenvalues.

In Section 2 we prove some inequalities for singular values of quadratic matrices using the Schur theorem (any complex matrix is unitarily similar to a triangular matrix). If $Z\in\mathcal{M}_n$ is a projection then there exists a unitary matrix $U\in\mathcal{M}_n$ such that

$$Z = U \begin{pmatrix} I_r & C \\ 0 & 0 \end{pmatrix} U^*, \tag{1}$$

where $r = \text{rank}(A)$ (see eg. [7]).

There is a deep overlap between projections and quadratic matrices. Notice that $A \in \mathcal{M}_n$ satisfies

$$(A - pI)(A - qI) = 0, \quad p \neq 0 \neq q \tag{2}$$

iff there exists a projection $Z \in \mathcal{M}_n$ such that

$$A = qI + (p - q)Z. \tag{3}$$

It is obvious that

$$Z = \frac{1}{p - q}(A - qI). \tag{4}$$

In case $p = q$ we have $(A - pI)^2 = 0$, so $A - pI$ is a nilpotent. From (1) and (3) we obtain the following theorem.

**Theorem 1.2** (Schur form). *For every $p, q \in \mathbb{C}$ a quadratic matrix $A \in \mathcal{M}_n$ such that $(A - pI)(A - qI) = 0$ can be written as*

$$A = URU^*, \quad R = \begin{pmatrix} pI_r & B \\ 0 & qI_{n-r} \end{pmatrix}, \tag{5}$$

*where $U \in \mathcal{M}_n$ is unitary.*

Th. 1.2 find many applications. Notice that if $p \neq q$ then there exists an involution $X \in \mathcal{M}_n$ containing the eigenvectors of $R$. Then $X^2 = I$ and

$$X = \begin{pmatrix} I_r & \frac{1}{p-q}B \\ 0 & -I_{n-r} \end{pmatrix}.$$

We see that $RX = XD$ where $D$ is a diagonal matrix:

$$D = \begin{pmatrix} pI_r & 0 \\ 0 & qI_{n-r} \end{pmatrix}.$$

Section 2 is devoted to the inverse eigenproblem of quadratic matrices. In Section 3 we show how to find a quadratic matrix with prescribed singular values. In Section 4 the Moore–Penrose inverse of a quadratic matrix is determined. Section 5 deals with the closest normal matrix $X$ to a quadratic matrix $A$ in the 2-norm and the Frobenius norm. Section 6 discusses the case of elementary matrices. In the last section a characterization of the numerical range of quadratic matrices is given.

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## 2. Eigenvalues of a product of two quadratic matrices

A quadratic matrix $A \in \mathcal{M}_n$ such that $(A - pI)(A - qI) = 0$ has the minimal equation $(\lambda - p)(\lambda - q) = 0$, hence has only $p$ and $q$ as eigenvalues. We are interested in eigenvalues of a product two quadratic matrices satisfying the same quadratic equation.

First we consider projections. In 1956 Afriat (see [1], [18]) proved the following theorem.

**Theorem 2.1.** *Let $A = A^* = A^2$ and $B = B^* = B^2$. If $\lambda$ is an eigenvalue of $AB$ , then $\lambda \in [0, 1]$.*

However, the Afriat theorem need not hold for arbitrary projections $A$ and $B$. Necessary and sufficient conditions for $n$ numbers in the interval $[0, 1]$ to form the spectrum of a product of two orthogonal projections were determined by Nelson and Neumann (see [17]).

Moreover, the following theorem holds.

**Theorem 2.2.** *For arbitrary nonzero $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ there exist quadratic matrices $A, B \in \mathcal{M}_{2n}$ satisfying the same quadratic equation*

$$(A - pI)(A - qI) = 0, \quad (B - pI)(B - qI) = 0$$

*such that $\lambda_1, \ldots, \lambda_n \in \lambda(AB)$.*

**Proof.** Let $A$ and $B$ be the direct sum of matrices $A_k$ and $B_k$:

$$A = A_1 \oplus A_2 \oplus \ldots \oplus A_n, \quad B = B_1 \oplus B_2 \oplus \ldots \oplus B_n,$$

where

$$A_k = \begin{pmatrix} p & x_k \\ 0 & q \end{pmatrix}, \quad B_k = \begin{pmatrix} p & 0 \\ 1 & q \end{pmatrix}.$$

We would like to determine $x_k \in \mathbb{C}$ such that $\lambda_k \in \lambda(A_k B_k)$.

An easy calculation gives

$$A_k B_k = \begin{pmatrix} p^2 + x_k & x_k q \\ q & q^2 \end{pmatrix},$$

hence $\lambda_k \in \lambda(A_k B_k)$ iff $(p^2 + x_k - \lambda_k)(q^2 - \lambda_k) - x_k q^2 = 0$. From this we get

$$x_k = \frac{(p^2 - \lambda_k)(q^2 - \lambda_k)}{\lambda_k}.$$

It is obvious that also $\frac{p^2 q^2}{\lambda_k} \in \lambda(A_k B_k)$, so

$$\lambda(AB) = \bigcup_{k=1}^{n} \lambda(A_k B_k) = \left\{ \lambda_1, \dots, \lambda_n, \frac{p^2q^2}{\lambda_1}, \dots, \frac{p^2q^2}{\lambda_n} \right\}.$$

This completes the proof. $\Diamond$

## 3. Singular values of quadratic matrices

We obtain a generalization of the following estimation of singular values of projections (see eg. [12]): *if $\sigma > 0$ is a singular value of a projection $Z \in \mathcal{M}_n$ then $\sigma \geq 1$.*

Assume that a quadratic matrix $A$ has the Schur form (5).

If $B = U_1 S U_2{}^*$ is an SVD and $Q = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}$ then $Q$ is unitary and $C = Q^* R Q = \begin{pmatrix} pI_r & S \\ 0 & qI_{n-r} \end{pmatrix}$.

It is evident that there exists a permutation matrix $P \in \mathcal{M}_n$ such that $P^T C P$ is a block diagonal matrix with blocks of size at most two. We may summarize these observations as

**Theorem 3.1.** *Suppose that $A \in \mathcal{M}_n$ satisfies a quadratic equation $(A - pI)(A - qI) = 0$ where $p, q \in \mathbb{C}$. Then $A$ is unitarily similar to a block diagonal matrix with blocks of size at most two.*

Another proof of this fact was given by A. Zalewska–Mitura and J. Zemánek (see [22]).

Notice that if $X \in \mathcal{M}_2$ has a form $X = \begin{pmatrix} \lambda_1 & s \\ 0 & \lambda_2 \end{pmatrix}$ where $| \lambda_1 | \geq | \lambda_2 |$, then $\sigma_1 \geq | \lambda_1 |$ and $\sigma_2 \leq | \lambda_2 |$. Here $\sigma_1 \geq \sigma_2$ are the singular values of $X$.

From this and Th. 3.1 we get the following theorem.

**Theorem 3.2.** *Suppose that $A \in \mathcal{M}_n$ satisfies $(A - pI)(A - qI) = 0$ where $p, q \in \mathbb{C}$. If $\sigma$ is a singular value of $A$ then*

$$(\sigma - | p |)\ (\sigma - | q |) \geq 0.$$

It is possible to construct a quadratic matrix with prescribed singular values.

**Theorem 3.3.** *There exists a matrix $A \in \mathcal{M}_{2n}$ satisfying a quadratic equation $(A - pI)(A - qI) = 0$ with given singular values $\sigma_1 \geq \ldots \geq \sigma_n > 0$ such that $(\sigma_k - | p |)\ \ (\sigma_k - | q |) > 0$ for $k = 1, \dots, n$.*

**Proof.** Let $A$ be the direct sum of matrices $A_k$:

$$A = A_1 \oplus A_2 \oplus \ldots \oplus A_n,$$

where

$$A_k = \begin{pmatrix} p & s_k \\ 0 & q \end{pmatrix}.$$

We would like to determine $s_k \in \mathbb{R}$ such that $\sigma_k \in \sigma(A_k)$. We have

$$A_k {A_k}^* = \begin{pmatrix} | p |^2 + {s_k}^2 & s_k \overline{q} \\ q s_k & | q |^2 \end{pmatrix}.$$

From this we get

$$s_k = \frac{\sqrt{({\sigma_k}^2 - | p |^2)({\sigma_k}^2 - | q |^2)}}{\sigma_k}.$$

It is clear that also $\frac{|pq|}{\sigma_k} \in \sigma(A_k)$, hence

$$\sigma(A) = \bigcup_{k=1}^{n} \sigma(A_k) = \left\{\sigma_1, \dots, \sigma_n, \frac{| pq |}{\sigma_1}, \dots, \frac{| pq |}{\sigma_n}\right\}.$$

This completes the proof. ◊

## 4. The Moore–Penrose inverse

Assume that $A \in \mathcal{M}_n$ is a quadratic matrix satisfying a quadratic equation $(A - pI)(A - qI) = 0$, $p, q \in \mathbb{R}$. Then $A$ is nonsingular iff $p \neq 0 \neq q$. From $A^{-1}(A^2 - (p+q)A + pqI) = 0$ we find that

$$(6) \qquad A^\dagger = A^{-1} = \frac{1}{pq}((p+q)I - A), \quad p \neq 0 \neq q.$$

From Th. 1.2 we have $A^\dagger = (URU^*)^\dagger = UR^\dagger U^*$, where $R$ has a form

$$(7) \qquad R = \begin{pmatrix} pI_r & B \\ 0 & qI_{n-r} \end{pmatrix}.$$

It is easy to check that

$$(8) \qquad R^\dagger = R^{-1} = \begin{pmatrix} \frac{1}{p} I_r & \frac{-1}{pq} B \\ 0 & \frac{1}{q} I_{n-r} \end{pmatrix}, \quad p \neq 0 \neq q.$$

If $p = q = 0$ then $R$ is a nilpotent and

$$R^\dagger = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & 0 \\ B^\dagger & 0 \end{pmatrix}.$$

Now we consider the case $q = 0$ and $p \neq 0$. Then using the formula for the Moore–Penrose inverse for partitioned matrices from [15] we get

$$(9)\quad R^{\dagger}=\begin{pmatrix} pI_r & B\\ 0 & 0\end{pmatrix}^{\dagger}=\begin{pmatrix} \frac{1}{p}(I_r+\frac{1}{|p|^2}BB^*)^{-1} & 0\\ \frac{1}{|p|^2}B^*(I_r+\frac{1}{|p|^2}BB^*)^{-1} & 0\end{pmatrix},\quad p\neq 0.$$

## 5. Nearest normal matrix

We seek normal approximants in the 2-norm and Frobenius norm to a quadratic matrix $A \in \mathcal{M}_n$ such that $(A-pI)(A-qI)=0$ where $p,q \in \mathbb{C}$ and $p \neq q$. We find a formula for nearest normal matrix in terms of $A$, $p$ and $q$.

We remind that a normal matrix $A \in \mathcal{M}_n$ is any matrix satisfying $AA^* = A^*A$. Normal matrices include Hermitian ($A^* = A$), skew–Hermitian ($A^* = -A$), unitary ($A^*A = I$) and real symmetric, skew–symmetric and orthogonal matrices.

Let $A \in \mathcal{M}_n$, and denote by $\nu_2(A)$ ($\nu_F(A)$) its distance from the set of normal matrices in the 2-norm (and Frobenius norm), see [10], [13], [20].

$$(10)\qquad \nu_2(A) = \inf\{\| A-N \|_2 : N \quad \text{is normal}\},$$

$$(11)\qquad \nu_F(A) = \inf\{\| A-N \|_F : N \quad \text{is normal}\}.$$

First, we present some results on projections. Assume that $Z \in \in \mathcal{M}_n$ is a projection (idempotent). It was proved by Phillips (see [19]) that the infimum $v_2(Z)$ is attained at $\frac{1}{2}(Z+Z^*)$, a Hermitian part of $Z$. The same holds for the Frobenius norm (see [3]). Bhatia, Horn and Kittaneh (see [3]) generalized a result of Phillips to the binormal operators with respect to every unitarily invariant norm. We use these results to exhibit nearest normal to a quadratic matrix. It is obvious that if $N \in \mathcal{M}_n$ is a nearest normal to $A \in \mathcal{M}_n$ in the 2-norm or Frobenius norm, then for any $\alpha, \beta \in \mathbb{C}$ the matrix $\alpha N + \beta I$ is a nearest normal to $\alpha A + \beta I$. Now we write a quadratic matrix $A \in \mathcal{M}_n$ such that $(A-pI)(A-qI)=0$ in a form $A = qI + (p-q)Z$, where $Z$ is a projection, and $Z = \frac{1}{p-q}(A-qI)$.

We have the following theorem (see [10] for a discussion on $2\times 2$ case).

**Theorem 5.1.** *Assume $A \in \mathcal{M}_n$ satisfies a quadratic equation $(A-pI)(A-qI)=0$ where $p,q \in \mathbb{C}$ and $p \neq q$. Then*

$$N = qI + \frac{p-q}{2}(Z + Z^*), \quad Z = \frac{1}{p-q}(A - qI) \tag{12}$$

*is a nearest normal to $A$ in the 2-norm and the Frobenius norm.*

## 6. Elementary matrices

The set of quadratic matrices includes the set of elementary matrices of a form $A = I - uw^*$, $u, w \in \mathbb{C}^n$.

They are frequently used in numerical linear algebra (see eg. [4], [9], [21]). A matrix $A = I - uw^*$ satisfies a quadratic equation $(A - pI)(A - qI) = 0$ where $p = 1 - w^*u$ and $q = 1$ (see [9]).

We would like to find the singular values of $A$. We write $A^*A$ in a form

$$A^*A = I - B, \quad B = pw^* + wp^*, \quad p = u - \frac{u^*u}{2}w.$$

We see that we need to determine the eigenvalues of a Hermitian matrix $B$. By simple computations we get the following result.

**Theorem 6.1.** *Assume that $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$ are the singular values of $A = I - uw^*$ where $u, w \in \mathbb{C}^n$ and $w^*u \in \mathbb{R}$. Then*

$$\sigma_2 = \ldots = \sigma_{n-1} = 1 \tag{13}$$

*and*

$$\sigma_1 = \frac{1}{2}\left(\sqrt{(u^*u)(w^*w)} + \sqrt{(u^*u)(w^*w) + 4 - 4(w^*u)}\right), \tag{14}$$

$$\sigma_n = \frac{|\, 1 - w^*u \,|}{\sigma_1}. \tag{15}$$

## 7. Numerical range

If $A \in \mathcal{M}_n$ is a given matrix, then the set

$$W(A) = \{x^*Ax : \; x \in \mathbb{C}^n, \; x^*x = 1\}$$

is called the *field of values* or *numerical range* of $A$.

In the following lemma we collect some basic properties of $W(A)$ (see [5], [6], [11], [16]).

**Lemma 7.1.** *Let $A \in \mathcal{M}_n$. Then*

(a) *$W(A)$ is compact and convex subset of the complex plane.*
(b) *$W(UAU^*) = W(A)$ for every unitary $U \in \mathcal{M}_n$.*

(c) $W(\alpha A+\beta I)=\beta+\alpha W(A)$ *for every* $\alpha,\beta\in\mathbb{C}$.

(d) *If* $A\in\mathcal{M}_2$ *has eigenvalues* $p,q\in\mathbb{C}$ *then the numerical range of* $A$ *is an elliptical disc with eigenvalues of* $A$ *as foci and a minor axis of the length* $\sqrt{\operatorname{tr}(A^*A)-|p|^2-|q|^2}$.

In 1993 J.I. Fuji and Y. Seo (see [8]) proved the following theorem.

**Theorem 7.1.** *If* $Y\in\mathcal{M}_n$ *is a nilpotent of a form* $Y=\left(\begin{smallmatrix}0&B\\0&0\end{smallmatrix}\right)$ *then the numerical range of* $Y$ *is a circular disc centered at* 0 *and the radius equal to* $\frac{1}{2}\|B\|_2$.

*If* $Z\in\mathcal{M}_n$ *is a projection of a form* $Z=\left(\begin{smallmatrix}I_r&B\\0&0\end{smallmatrix}\right)$ *then the numerical range of* $Z$ *is an elliptical disc:*

$$W(Z)=\left\{x+iy:\frac{(x-\frac{1}{2})^2}{a^2}+\frac{y^2}{b^2}\le 1\right\}$$

*where*

$$a=\frac{1}{2}\sqrt{1+\|B\|_2^2},\quad b=\frac{1}{2}\|B\|_2.$$

Using (2)–(4), Lemma 7.1, Th. 7.1 and applying the Schur form (5) we get the following theorem.

**Theorem 7.2.** *Assume that a quadratic matrix* $A\in\mathcal{M}_n$ *has the Schur form*

$$A=U\begin{pmatrix}pI_r&B\\0&qI_{n-r}\end{pmatrix}U^*$$

*where* $U$ *is unitary. Then the numerical range of* $A$ *is an elliptical disc with* $p,q$ *as foci and* $\|B\|_2$ *as the length of a minor axis.*

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