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UNIFORM APPROXIMATION FOR SUBLATTICES OF C*(X)

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Abstract: In this paper we give some characterizations for the uniform closure of a sub-lattice W of bounded real-valued continuous functions over a completely regular non-compact space X. Thus, we extend to $C^*(X)$ some known results for the compact case, like the Bonsall's approximation theorem for lattices or the Császár-Czipszer theorem for affine-lattices. In particular, we complete the study about the uniform closure of a lattice done in [6] by Garrido-Montalvo.

1. Introduction

Let K be a compact Haudorff space and let $W \subset C(K)$. It is well known that when W is a sublattice or a subalgebra of $C(K)$ the

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Kakutani-Stone theorem and the Weierstrass-Stone theorem characterize the uniform closure of W. Both theorems have many generalizations and extensions (see Choquet and Deny [4], Császár [5], Prolla [10], [11], ...). It could be said that how points are separated by the functions in W originates of the different theorems. On the other hand, although the same arguments are not sufficient for the general case, there are some versions of these theorems for the noncompact case. Thus, Hewitt [7] in 1947 gave a uniform density theorem for algebras containing all the real constant functions of $C^*(X)$ (the set of all bounded and real-valued continuous functions over X). In the proof he considered separation of zero-sets in place of separation of points. Similar results have been obtained by Mrowka [9], Brosowsky and Deutsch [3], Blasco and Moltó [1], Garrido and Montalvo [6].

In this paper, we present some new characterizations of the uniform closure of certain subfamilies $W \subset C^*(X)$ when X is a noncompact space. Here, we suppose either W or its uniform closure $\mathrm{cl}(W)$ is a lattice of $C^*(X)$ and we obtain different conditions according to $\mathrm{cl}(W)$ is a lattice, a lattice containing the constant functions or an affine lattice. Thus, we complete the results obtained in [6] for lattices.

2. Preliminaries

Let X be a completely regular Hausdorff space. As usual, βX denotes the Stone-Čech compactification of X and $C^*(X)$ the family of all real-valued bounded continuous functions defined on X endowed with the uniform norm. Recall that βX is the only (up to homeomorphisms) Hausdorff compactification of X such that each $f \in C^*(X)$ admits a (unique) extension $f^\beta \in C^*(\beta X)$. Thus, if $W \subset C^*(X)$ and $W^\beta := \{f^\beta : f \in W\}$, then W and W^β have similar algebraic properties. Moreover, $\mathrm{cl}_{C^*(\beta X)} W^\beta = (\mathrm{cl}_{C^*(X)} W)^\beta$, where $\mathrm{cl}_Y(A)$ denotes the closure of a set $A \subset Y$ with respect to the topology of Y. If there no confusion is possible, we will omit the underlying space, that is $\mathrm{cl}(A) := \mathrm{cl}_Y(A)$.

For $f \in C^*(X)$, the Lebesgue sets of f are defined by

$$L_a(f) := \{x \in X\colon f(x) \le a\}, \quad L^a(f) := \{x \in X\colon f(x) \ge a\}, \ (a \in \mathbb{R}).$$

Next, we list the separation properties which will be used. For a set $W \subset C^*(X)$ and a function $f : X \to \mathbb{R}$, we say that

1. W *S_1-separates* the Lebesgue sets of f if for $a, b \in \mathbb{R}$, $a < b$, there exists $g \in W$ such that $0 \leq g \leq 1$, $g(L_a(f)) = \{0\}$ and $g(L^b(f)) = \{1\}$.
2. W *S-separates* the Lebesgue sets of f if for each $a, b \in \mathbb{R}$, $a < b$, and $\delta > 0$, there exists $g \in W$ such that, $0 \leq g \leq 1$, $g(L_a(f)) \subset [0, \delta]$ and $g(L^b(f)) \subset [1-\delta, 1]$.
3. W *separates* the Lebesgue sets of f if for each $a, b \in \mathbb{R}$, $a < b$, there exists $g \in W$ such that $\mathrm{cl}_{\mathbb{R}}(g(L_a(f))) \cap \mathrm{cl}_{\mathbb{R}}(g(L^b(f))) = \emptyset$.

3. Main results

In the following we suppose either W or, more generally, $\mathrm{cl}(W)$ is a lattice (i.e., if $g, h \in \mathrm{cl}\, W$, then $g \vee h = \sup(g, h)$ and $g \wedge h = \inf(g, h)$ are in $\mathrm{cl}(W)$).

We shall obtain the first result by applying to W^β a known theorem of the compact case, the Bonsall theorem. Let us recall:

Theorem 3.1 [2, Bonsall]. *Let X be a Hausdorff compact space, $W \subset C^*(X)$ a lattice and $f \in C^*(X)$ a fixed function. Then $f \in \mathrm{cl}(W)$ if and only if, for each $x_1, x_2 \in X$ and $\varepsilon > 0$, there exists $g \in W$ such that*

$$f(x_1) \leq g(x_1) + \varepsilon\,, \quad f(x_2) \geq g(x_2) - \varepsilon.$$

Theorem 3.2. *Let X be a completely regular Hausdorff space, W a non-empty subset of $C^*(X)$ such that $\mathrm{cl}(W)$ is a lattice and $f \in C^*(X)$ a fixed function. The following assertions are equivalent:*

(i) $f \in \mathrm{cl}(W)$.

(ii) *For each pair of real numbers a, b and every $\varepsilon > 0$, there exists $g \in W$ such that*

$$L_a(f) \subset L_{a+\varepsilon}(g)\,, \quad L^b(f) \subset L^{b-\varepsilon}(g).$$

Proof. Assertions (ii) follow easily from (i). To see that (ii) implies (i), fix $p, q \in \beta X$, $\varepsilon > 0$ and let $0 < \delta < \varepsilon/2$. Then, by hypothesis, there exists $g \in W$ such that

$$f(x) \leq f^\beta(p) + \delta \quad \Rightarrow \quad g(x) \leq f^\beta(p) + 2\delta,$$
$$f(x) \geq f^\beta(q) - \delta \quad \Rightarrow \quad g(x) \geq f^\beta(q) - 2\delta,$$

from which it follows that

$$p \in \{y \in \beta X : f^\beta(y) < f^\beta(p) + \delta\} \subset \mathrm{cl}_{\beta X}\{x \in X : f(x) < f^\beta(p) + \delta\} \subset$$
$$\subset \mathrm{cl}_{\beta X}\{x \in X : g(x) \le f^\beta(p) + 2\delta\},$$

and then, $g^\beta(p) \le f^\beta(p) + 2\delta \le f^\beta(p) + \varepsilon$. Analogously, we obtain that $g^\beta(q) \ge f^\beta(q) - \varepsilon$ and, therefore, the Bonsall theorem gives $f^\beta \in$ $\in \mathrm{cl}(W^\beta)$. ◊

Remark 3.3. (1) We will show in Cor. 3.4 that if we include the hypothesis "cl(W) contains the constant functions" in Th. 3.2, then we can assume that $a < b$ in the condition (ii). But, in general, that is not true. For example, let W be the lattice of all functions $g \in C[0,1]$ such that $g(x) = mx$, for some $m \in \mathbb{R}$, and let $f \in C[0,1]$ be defined by

$$f(x) := \begin{cases} 2x & \text{si} \quad x \le 1/2 \\ 1 & \text{si} \quad x > 1/2. \end{cases}$$

Since W is uniformly closed, $f \notin \mathrm{cl}(W)$. But, for each pair $a, b \in \mathbb{R}$, $a < b$, there exists a function $g \in W$ such that

$$L_a(f) \subset L_a(g), \quad L^b(f) \subset L^b(g).$$

(2) There exist some cases (see Th. 3.5) in which we can remove ε from condition (ii) in Th. 3.2. But, in general, that is not possible, even with W being a linear lattice. For example, let W be the family of all real continuous functions g defined on $[0,1]$ such that there exist $m \in \mathbb{R}$ and $\delta \in (0, 1/4]$ satisfying

$$g(x) = mx \quad \text{if} \quad x \in \left[\frac{1}{2} - \delta, \frac{1}{2} + \delta\right] \cup [1 - \delta, 1],$$

and consider the function

$$f(x) := \begin{cases} 2x & \text{if } x \in [0, 1/2] \\ 4x - 1 & \text{if } x \in [1/2, 3/4] \\ 2 & \text{if } x \in [3/4, 1]. \end{cases}$$

As can easily be verified, W is a linear lattice and $f \in \mathrm{cl}(W)$. Also, since

$$L_1(f) = [0, 1/2] \quad L^2(f) = [3/4, 1],$$

if $g \in W$ is such that $L_1(f) \subset L_1(g)$, then $g(x) < 2$ in a neighborhood of 1. Hence, $L^2(f) \not\subset L^2(g)$.

Corollary 3.4. *Let $W \subset C^*(X)$ be such that* $\mathrm{cl}(W)$ *is a lattice containing the constant functions. Then, for a function $f \in C^*(X)$, the following assertions are equivalent:*

(i) $f \in \mathrm{cl}(W)$.

(ii) *For each* $a < b$ *and* $\varepsilon > 0$ *there exists* $g \in W$ *such that*

$$L_a(f) \subset L_{a+\varepsilon}(g), \quad L^b(f) \subset L^{b-\varepsilon}(g).$$

(iii) [6, Garrido and Montalvo] *For each* $a < b$ *and* $\varepsilon > 0$ *there exists* $g \in W$ *such that for* $x \in X$

$$x \in L_a(f) \Rightarrow |g(x) - a| < \varepsilon, \quad x \in L^b(f) \Rightarrow |g(x) - b| < \varepsilon.$$

Proof. From Th. 3.2 it follows that (i) implies (ii). Also, it was showed in [6, Th. 7] that (i) and (iii) are equivalent. Thus, it suffices to prove that (ii) implies (iii). Let a, b $(a < b)$ be real numbers and $0 < \varepsilon < b - a$. Choose $g_1 \in W$ such that $L_a(f) \subset L_{a+\varepsilon/2}(g_1)$ and $L^b(f) \subset L^{b-\varepsilon/2}(g_1)$. Notice that the function $(a \vee g_1) \wedge b \in \mathrm{cl}(W)$ satisfies

$$x \in L_a(f) \Rightarrow |(a \vee g_1(x)) \wedge b - a| \leq \varepsilon/2$$

$$x \in L^b(f) \Rightarrow |(a \vee g_1(x)) \wedge b - b| \leq \varepsilon/2.$$

Next, take $g \in W$ such that, for each $x \in X$, $|(a \vee g_1(x)) \wedge b - g(x)| \leq \varepsilon/2$. Then we have

$$x \in L_a(f) \Rightarrow |g(x) - a| \leq \varepsilon, \quad x \in L^b(f) \Rightarrow |g(x) - b| \leq \varepsilon. \qquad \Diamond$$

The following theorem is a consequence of the above result and Th. 8 of [6].

Theorem 3.5. *Let* $W \subset C^*(X)$ *be such that* $\mathrm{cl}(W)$ *is a lattice which contains all functions of the form* $ag + c$, $g \in W$, $a, c \in \mathbb{R}$ *(an affine lattice). Then the following assertions are equivalent.*

(i) $f \in \mathrm{cl}(W)$.

(ii) *For each* $a < b$ *and* $\varepsilon > 0$, *there exists* $g \in W$ *such that*

$$L_a(f) \subset L_{a+\varepsilon}(g), \quad L^b(f) \subset L^{b-\varepsilon}(g).$$

(iii) *For each* $a < b$, *there exists* $g \in W$ *such that*

$$L_a(f) \subset L_a(g), \quad L^b(f) \subset L^b(g).$$

(iv) *For each* $a < b$ *and* $\varepsilon > 0$, *there exists* $g \in W$ *such that*

$$|g(x) - a| < \varepsilon \quad \text{if } x \in L_a(f), \quad |g(x) - b| < \varepsilon \quad \text{if } x \in L^b(f).$$

(v) W *S-separates the Lebesgue sets of* f.

(vi) *For each* $a < b$, *there exists* $g \in W$ *such that*

$$\sup\{g(x) : x \in L_a(f)\} < \inf\{g(x) : x \in L^b(f)\}.$$

(vii) W *separates the Lebesgue sets of* f.

Proof. From Cor. 3.4 and Th. 8 of [6] applied to cl(W) it is easy to obtain that all the conditions but (iii) are equivalent. We must note that, since cl(W) is an affine lattice, W S-separates the Lebesgue sets of f if and only if cl(W) S^1-separates the Lebesgue sets of f.

Thus to finish, we shall prove that (vi) $\Leftrightarrow$ (iii). It is clear that (iii) implies (vi). For the converse, let $a < b$, $\varepsilon > 0$ and choose $g \in W$ such that

$$\alpha = \sup\{g(x): x \in L_a(f)\} < \inf\{g(x): x \in L^b(f)\} = \beta,$$

and let $\varphi \in C^*(\mathbb{R})$ be the straight line such that $\varphi(\alpha) = a - \varepsilon$ and $\varphi(\beta) = b + \varepsilon$. The map $\varphi \circ g \in \mathrm{cl}\,(W)$ satisfies $(\varphi \circ g)(x) \le \varphi(\alpha) = a - -\varepsilon$, if $x \in L_a(f)$. In the same way, $(\varphi \circ g)(x) \ge b + \varepsilon$, if $x \in L^b(f)$. Now, choose $h \in W$ such that $|\varphi \circ g - h| < \varepsilon$. Then $L_a(f) \subset L_a(h)$, $L^b(f) \subset \subset L^b(h)$. $\Diamond$

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