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STABILITY OF THE CAUCHY EQUATION IN ORDERED FIELDS

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Abstract: An analogue of Ulam's question on the stability of the Cauchy functional equation is treated for functions mapping into ordered fields.

1. Introduction

Let througout the paper $\mathbb{R}$ denote the set of the real numbers and $(S,+)$ denote an arbitrary groupoid. If $(G,+)$ is another groupoid, we shall call any homomorphism $\phi : S \to G$ an *additive* mapping. The proof of the following theorem is due to Hyers [2].

Theorem 1. *If S is an abelian semigroup, $f : S \to \mathbb{R}$, $0 < \delta \in \mathbb{R}$, and the inequality*

(1) $$|f(x+y) - f(x) - f(y)| \leq \delta$$

is satisfied for every $x, y \in S$, then there exists an additive mapping $g : S \to \mathbb{R}$ such that

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(2) $$|f(x) - g(x)| \le \delta$$

holds for every $x \in S$.

Let us note that Hyers formulated this statement for functions defined on a Banach space and mapping into another Banach space. However, as it is pointed out by Forti [1], Hyers' proof works for functions defined on an arbitrary abelian semigroup. It is convenient for our purposes to restrict ourselves to the case when the range is the set of the real numbers.

In fact, Hyers' result is a (partial) affirmative answer to Ulam's question [1]. As in Ulam's original question mappings from one metric semigroup into another were involved, several authors have investigated generalizations of Th. 1 into this direction. A somewhat different approach is represented, for instance, by [4], where most of the results are devoted to functions mapping into topological $\mathbb{Q}$-vector spaces.

It might be interesting to investigate the case when the range is equipped with an order relation instead of a topology. In this short note we consider functions mapping into ordered fields. Though one can try to reduce this situation to the previous one by generating an appropriate topology, the direct approach seems to be more powerful. In this case it is natural to consider inequality (1) with the (not real valued) modulus of the (so called) Cauchy difference, majorized by a positive element δ of the ordered field under consideration. Our terminology reflects the fact that additive mappings are the solutions of Caucy's functional equation $f(x+y) = f(x)+f(y)$. In this sense Hyers' theorem asserts the stability of the Cauchy equation in Banach spaces.

In what follows $(R, +, \cdot)$ will denote an ordered field. Several concepts of stability are usually associated with Hyers' theorem. We shall say that the Cauchy equation is *weakly stable* in (S, R), if for every function $f : S \to R$ that has a bounded Cauchy difference

$$(x, y) \mapsto f(x+y) - f(x) - f(y) \qquad \big((x, y) \in S \times S\big)$$

there exists an additive function $g : S \to R$ such that $f - g$ is bounded (in R). We say that the Cauchy equation in (S, R) is *stable in Ulam's sense*, if for every $0 < \varepsilon \in R$ there exists $0 < \delta \in R$ such that the following implication holds true: (**U**) if $f : S \to R$ satisfies

(3) $$|f(x+y) - f(x) - f(y)| \le \delta \quad \text{for every} \quad x, y \in S$$

(where, as usual, $|a| = \max\{a, -a\}$), then there exists an additive function $g : S \to R$ such that

$$(4) \qquad |f(x) - g(x)| \le \varepsilon \quad \text{for every} \quad x \in S.$$

We say that the Cauchy equation is *uniformly stable* in (S, R), if there exists $0 < \kappa \in R$ such that for every $0 < \delta \in R$ the implication (**U**) holds with $\varepsilon = \kappa\delta$. Finally, we say that the Cauchy equation is *finitely stable* in (S, R), if it is uniformly stable with some $\kappa \in \mathbb{Q}$ (as it is familiar, the smallest subfield of R is $\mathbb{Q}$, the set of the rational numbers). Obviously, if the Cauchy equation is finitely stable in (S, R), then it is uniformly stable as well. It is also trivial that if the Cauchy equation is uniformly stable in (S, R), then it is also stable in Ulam's sense and weakly stable in (S, R). Th. 1 states, under natural assumptions on S, that the Cauchy equation is finitely stable in $(S, \mathbb{R})$ with $\kappa = 1$. We shall prove, without any assumption on S, that the Cauchy equation is uniformly stable in (S, R) for every non-archimedean field R with every non-finite element $0 < \kappa \in R$. For ordered fields R that can be considered, in a certain sense, as extensions of $\mathbb{R}$, assuming that S satisfies the hypotheses of Th. 1, we prove that the Cauchy equation is finitely stable in (S, R). It is also presented that, for instance, the Cauchy equation is not weakly stable in $(\mathbb{Q}, \mathbb{Q})$.

2. Algebraic foundations

The following general and simple stability result concerning the Cauchy equation for vector-valued mappings is implicitly used in [4].

Lemma 1. *Let Y be a linear space over an arbitrary field K and Y_0 be a linear subspace of Y. If $f : S \to Y$ satisfies*

$$(5) \qquad f(x+y) - f(x) - f(y) \in Y_0 \quad \textit{for every} \quad x, y \in S,$$

then there exists an additive function $g : S \to Y$ such that

$$(6) \qquad f(x) - g(x) \in Y_0 \quad \textit{for every} \quad x \in S.$$

Proof. Let H_0 be a Hamel base of Y_0 and H be a Hamel base of Y such that $H_0 \subset H$. Let Y_1 denote the K-linear hull of $H \setminus H_0$. Then Y_0 and Y_1 are algebraically complementary subspaces of Y, that is

$$(7) \qquad Y_0 \cap Y_1 = \{0\} \quad \text{and} \quad Y_0 + Y_1 = Y.$$

Thus every $u \in Y$ has a unique representation

$$u = u_0 + u_1\,, \quad \text{where} \quad u_0 \in Y_0\,,\ u_1 \in Y_1\,.$$

Hence there exist well defined functions $f_0 : S \to Y_0$ and $f_1 : S \to Y_1$ such that $f = f_0 + f_1$. Due to the definitions of f_j and Y_j we have

(8) $f_j(x+y) - f_j(x) - f_j(y) \in Y_j$ for every $x, y \in S$ $(j = 0, 1)$.

On the other hand, (5), case $j = 0$ of (8), and $f_1 = f - f_0$ imply

(9) $f_1(x+y) - f_1(x) - f_1(y) \in Y_0$ for every $x, y \in S$.

From (8), (9), and (7) we obtain that f_1 is additive. Thus $g = f_1$ satisfies our statement. ◊

Now we introduce a few important concepts and structures in ordered fields and encounter a few properties of them. We begin with notions and notations concerning $u, w \in R$. Let

$$[u, w] = \{ t \in R \,|\, u \leq t \leq w \},$$

as usual. We say that w is u-infinitesimal if $n|w| \leq |u|$ for every $n \in \in \mathbb{N}$ (where $\mathbb{N}$ denote the set of the positive integers). We say that w is u-finite if there exists $n \in \mathbb{N}$ such that $|w| \leq n|u|$. We say that w is infinitesimal if it is 1-infinitesimal. We say that w is finite if it is 1-finite. Put

$$\mathcal{I}[u] = \{ w \in R \mid w \text{ is } u\text{-infinitesimal} \},$$

$$\mathcal{F}[u] = \{ w \in R \mid w \text{ is } u\text{-finite} \},$$

$\mathcal{I}_R = \mathcal{I}[1]$, and $\mathcal{F}_R = \mathcal{F}[1]$.

We call an ordered field R *archimedean* if $\mathcal{F}_R = R$. It is easily seen that R is archimedean if and only if $\mathcal{I}_R = \{0\}$.

It is well known (and easy to show) that $\mathcal{F}_R$ is a ring and $\mathcal{I}_R$ is a maximal ideal in $\mathcal{F}_R$. Hence the *standard part* of R, defined by $\mathrm{st}(R) = \mathcal{F}_R / \mathcal{I}_R$, is a field. Moreover, $\mathrm{st}(R)$ is an ordered field with the order inherited from R. It is also well known (cf., e.g., [3], Ch. 1, § 3) that $\mathrm{st}(R)$ is archimedean, hence it is isomorphic to a subfield of $\mathbb{R}$.

We call an ordered field R *quasi-real* if $\mathrm{st}(R)$ is isomorphic to $\mathbb{R}$. Since any field which is isomorphic to $\mathbb{R}$ provides a model of the axioms of $\mathbb{R}$, we can assume that in any quasi-real ordered field R we have $\mathrm{st}(R) = \mathbb{R}$.

3. Stability results

Let us observe that any ordered field R is a linear space over $\mathbb{Q}$. Moreover, $\mathcal{I}[u]$ and $\mathcal{F}[u]$ are $\mathbb{Q}$-linear subspaces of R for every $u \in R$. Therefore, applying Lemma 1, one immediately obtains the following result.

Corollary 1. *Let R be an arbitrary ordered field, $u \in R$, $I = \mathcal{I}[u]$ or $I = \mathcal{F}[u]$, and $f : S \to R$. If*

(10) $\qquad f(x+y)-f(x)-f(y)\in I$ *for every* $x,y\in S$,

then there exists an additive mapping $g: S\to R$ *such that*

(11) $\qquad f(x)-g(x)\in I$ *for every* $x\in S$.

Roughly speaking, if the Cauchy differences of f are infinitesimal (finite), then f is infinitesimally (finitely) close to an additive function.

Let us note that the notions, statements, and proofs of Cor. 1 can be extended to ordered linear spaces.

Now let us assume that R is non-archimedean. Then there exists $0<\kappa\in R\setminus\mathcal{F}_R$. If $0<\delta\in R$, then obviously δ is $\kappa\delta$-infinitesimal. Thus for a function $f: S\to R$ with Cauchy differences in $[-\delta,\delta]$ we can apply Cor. 1 with $I=\mathcal{I}[\kappa\delta]$. So we can establish another formal stability result.

Proposition 1. *If R is a non-archimedean ordered field, then the Cauchy equation is uniformly stable in (S,R).*

The following example shows that the archimedean case is different. Namely, the Cauchy equation is not weakly stable in $(\mathbb{Q},\mathbb{Q})$.

Example. Let $\xi\in\mathbb{R}\setminus\mathbb{Q}$ such that $0<\xi<1$ and define

$$f(x)=x-[(1-\xi)x] \qquad (x\in\mathbb{Q}),$$

where $[z]$ denotes the integer part of the real number z. Then $f:\mathbb{Q}\to\mathbb{Q}$ and

$$\xi x\le f(x)<\xi x+1 \quad \text{for every} \quad x\in\mathbb{Q},$$

hence

$$|f(x+y)-f(x)-f(y)|\le 2 \quad \text{for every} \quad x,y\in\mathbb{Q}.$$

On the other hand, the only additive mapping $g:\mathbb{Q}\to\mathbb{R}$ for which $f-g$ is bounded is given by $g(x)=\xi x$ $(x\in\mathbb{Q})$. Thus $g(1)=\xi\notin\mathbb{Q}$.

The following theorem yields that for every abelian semigroup S and quasi-real ordered field R the Cauchy equation is finitely stable in (S,R) with any rational number $\kappa>1$.

Theorem 2. *Suppose that S is an abelian semigroup, R is a quasi-real ordered field, $\delta\in R$, $\delta>0$, and $f: S\to R$ such that* (3) *holds. Then there exists an additive function $g: S\to R$ such that*

(12) $\qquad f(x)-g(x)\in[-\delta,\delta]+\mathcal{I}[\delta]$ *for every* $x\in S$

and thus

(13) $\qquad |f(x)-g(x)|\le\left(1+\dfrac{1}{n}\right)\delta$ *for every* $x\in S,\ n\in\mathbb{N}$.

Proof. Let us define $\tilde{f} : S \to R$ by $\tilde{f}(x) = f(x)/\delta$ $(x \in S)$. Then (3) implies

(14) $$|\tilde{f}(x+y) - \tilde{f}(x) - \tilde{f}(y)| \le 1 \quad \text{for every} \quad x, y \in S,$$

where 1 denotes the multiplicative unit of R. In particular, (14) implies that

(15) $$\tilde{f}(x+y) - \tilde{f}(x) - \tilde{f}(y) \in \mathcal{F}_R \quad \text{for every} \quad x, y \in S.$$

Thus, due to Cor. 1, there exists an additive function $f_2 : S \to R$ such that

(16) $$\tilde{f}(x) - f_2(x) \in \mathcal{F}_R \quad \text{for every} \quad x \in S.$$

Let $f_1 = \tilde{f} - f_2$. Then (16) yields $f_1 : S \to \mathcal{F}_R$. Applying (14) and the additivity of f_2 we obtain

(17) $$|f_1(x+y) - f_1(x) - f_1(y)| \le 1 \quad \text{for every} \quad x, y \in S.$$

Let us now define $F : S \to \mathcal{F}_R/\mathcal{I}_R$ by $F(x) = f_1(x) + \mathcal{I}_R$ $(x \in S)$. Then, applying (17) and using the inequality and the modulus in $\mathcal{F}_R/\mathcal{I}_R$ inherited from R, we get

(18) $$|F(x+y) - F(x) - F(y)| \le \mathbf{1} \quad \text{for every} \quad x, y \in S,$$

where $\mathbf{1} = 1 + \mathcal{I}_R$ denotes the multiplicative unit of $\mathcal{F}_R/\mathcal{I}_R = \mathbb{R}$. Due to Th. 1, there exists an additive function $G : S \to \mathcal{F}_R/\mathcal{I}_R$ such that

(19) $$|F(x) - G(x)| \le \mathbf{1} \quad \text{for every} \quad x \in S.$$

Consider a choice function $g_1 : S \to \mathcal{F}_R$ such that $g_1(x) \in G(x)$ $(x \in S)$. Since $\mathcal{I}_R$ is the additive unit of $\mathcal{F}_R/\mathcal{I}_R$, we have

(20) $$g_1(x+y) - g_1(x) - g_1(y) \in G(x+y) - G(x) - G(y) = \mathcal{I}_R$$

for all $x, y \in S$. Now we can apply Cor. 1, which states that there exists an additive function $g_0 : S \to R$ such that $g_1(x) - g_0(x) \in \mathcal{I}_R$ for every $x \in S$.

Define $\tilde{g} : S \to R$ by

$$\tilde{g}(x) = g_0(x) + f_2(x) \qquad (x \in S).$$

Then $\tilde{g}$ is additive and we have

$$\tilde{f}(x) - \tilde{g}(x) = (f_1(x) + f_2(x)) - (g_0(x) + f_2(x)) = f_1(x) - g_0(x) =$$
$$= (f_1(x) - g_1(x)) + (g_1(x) - g_0(x)) \in F(x) - G(x) + \mathcal{I}_R .$$

Due to (19) there exists $t \in R$ such that $|t| \le 1$ and $F(x) - G(x) = t + \mathcal{I}_R$. Hence

$$F(x) - G(x) + \mathcal{I}_R \subset [-1, 1] + \mathcal{I}_R + \mathcal{I}_R = [-1, 1] + \mathcal{I}_R,$$

therefore

$$\tilde{f}(x) - \tilde{g}(x) \in [-1, 1] + \mathcal{I}_R$$

for all $x \in S$.

Finally, put $g(x) = \delta\tilde{g}(x) \quad (x \in S)$. ◊

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