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A NOTE ON INTEGRAL INCLUSIONS IN BANACH SPACES

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Abstract: In this note, a fixed point theorem for condensing maps is used to investigate the existence of solutions of an integral inclusion in Banach spaces.

1. Introduction

In the past few years, several papers have been devoted to the study of integral equations by different authors under different conditions on the kernel (see for instance [4], [5], [6], [12], [13] and their references). However very few results are available for integral inclusions see [3], [8], and [14].

The fundamental tools used in the existence proofs of all above mentioned works are essentially fixed point arguments or iterative methods.

In this note, we shall be concerned with the existence of solutions of the integral inclusion:

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$$(1.1) \qquad y(t) \in \int_0^t K(t,s)F(s,y(s))ds + g(t) \quad \text{for} \quad t \in J := [0,T]$$

where $F : J \times E \longrightarrow E$ is a bounded, closed, convex multivalued map, $K : D \longrightarrow \mathbb{R}$, $D = \{(t,s) \in J \times J : t \geq s\}$, $g : J \longrightarrow E$ and E a real Banach space normed by $|.|$.

The method we are going to use is to reduce the integral inclusion (1.1) to the search for fixed points of a suitable multivalued map on the space $C(J,E)$ and we make use a fixed point theorem for condensing maps due to Martelli (see [11]).

2. Preliminaries

In this section, we introduce notations, definitions, and results which are used throughout this paper. $C(J,E)$ is the Banach space of continuous functions from J into E with norm

$$\|y\|_0 = \sup\{|y(t)| : t \in J\} \quad \text{for all} \quad y \in C(J,E).$$

Let $y : J \longrightarrow E$ be measurable function. By $\int_0^T y(s)ds$, we mean the Bochner integral of y, assuming it exists. A measurable function $y : J \longrightarrow E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. For properties of Bochner integral see [15]. $L^1(J,E)$ denotes the Banach space of functions $y : J \longrightarrow E$ Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^T |y(t)|dt \quad \text{for all} \quad y \in L^1(J,E).$$

$L^\infty(J,\mathbb{R})$ is the space of essentially bounded measurable functions $y : J \longrightarrow \mathbb{R}$.

Let $(X, \|.\|)$ be a Banach space. A multivalued map $G : X \longrightarrow X$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B)$ is bounded in X for any bounded set B of X (i.e. $\sup_{x\in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$). G is called upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set B of X containing $G(x_0)$, there exists an open neighbourhood A of x_0 such that $G(A) \subseteq B$.

G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \longrightarrow x_0,\ y_n \longrightarrow y_0,\ y_n \in Gx_n$ imply $y_0 \in Gx_0$).

G has a fixed point if there is $x \in X$ such that $x \in Gx$.

In the following $BCC(E)$ denotes the set of all bounded, closed, convex and nonempty subsets of E.

A multivalued map $G : J \longrightarrow BCC(X)$ is said to be measurable if for each $x \in X$ the distance between x and $G(t)$ is a measurable function on J.

An upper semi-continuous map $G : X \longrightarrow X$ is said to be condensing [2] if for any subset $B \subseteq X$ with $\alpha(B) \neq 0$, we have $\alpha(G(B)) < < \alpha(B)$, where α denotes the Kuratowski measure of noncompactness [2].

We remark that a completely continuous multivalued map is the easiest example of a condensing map. For more details on multivalued maps see [1], [7].

Let us list the following hypotheses:

(H1) $F : J \times E \longrightarrow BCC(E)$; $(t, y) \longmapsto F(t, y)$ is measurable with respect to t for each $y \in E$, u.s.c. with respect to y for each $t \in J$ and for each fixed $y \in C(J, E)$ the set

$$S_{F,y} = \{f_y \in L^1(J, E) :\ f_y(t) \in F(t, y(t)) \quad \text{for a.e.} \quad t \in J\}$$

is nonempty;

(H2) for each $t \in J$, $K(t, s)$ is measurable on $[0, t]$ and

$$K(t) = \text{ess sup}\{|K(t, s)|, \quad 0 \leq s \leq t\},$$

is bounded on J;

(H3) the map $t \longmapsto K_t$ is continuous from J to $L^\infty(J, \mathbb{R})$; here $K_t(s) = K(t, s)$;

(H4) $g : J \longrightarrow E$ is a continuous single valued-map;

(H5) there exist a continuous nondecreasing function $\psi : [0, \infty) \longrightarrow \longrightarrow (0, \infty)$ with $\int\limits_0^\infty \frac{du}{\psi(u)} = \infty$ and $p \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t, y)\| := \sup\{|v| : v \in F(t, y)\} \leq p(t)\psi(|y|) \qquad \text{for a.e. } t \in J \text{ and all } y \in E;$$

(H6) for each bounded set $B \subseteq C(J, E)$ and $y \in B$ the set

$$\left\{ \int_0^t K(t,s) f_y(s) ds + g(t) : f_y \in S_{F,y} \right\}$$

is relatively compact for each $t \in J$.

Remark. If dim $E < \infty$, then $S_{F,y} \neq \emptyset$ for any $y \in C(J, E)$ (see [10]).

By a solution of (1.1), we mean a function $y \in C(J, E)$ that satisfies the integral inclusion (1.1) on J.

The following lemmas are crucial in the proof of our main result:

Lemma 2.1. [10] *Let F be a multivalued map satisfying* (H1) *and let Γ be a linear continuous mapping from $L^1(J, E)$ to $C(J, E)$. Then the operator*

$$\Gamma \circ S_F : C(J,E) \longrightarrow BCC(C(J,E)), \; y \longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y}$$

is a closed graph operator in $C(J, E) \times C(J, E)$.

Lemma 2.2. (Lemma 1.5.3 [9]) *If $p \in L^1(J, \mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \longrightarrow \longrightarrow (0, \infty)$ is increasing with*

$$\int_0^\infty \frac{du}{\psi(u)} = \infty,$$

then the integral equation

$$z(t) = z_0 + \int_0^t p(s)\psi(z(s)), \quad \text{for} \quad t \in J,$$

has for each $z_0 \in \mathbb{R}_+$ a unique solution z. If $u \in C(J, E)$ satisfies the integral inequality

$$|u(t)| \leq z_0 + \int_0^t p(s)\psi(|u(s)|) ds \quad \text{on} \quad J,$$

then $|u| \leq z$.

Lemma 2.3. [11] *Let X be a Banach space and $N : X \longrightarrow BCC(X)$ a condensing map. If the set*

$$M := \{y \in X : \lambda y \in N(y) \quad \text{for some} \quad \lambda > 1\}$$

is bounded, then N has a fixed point.

3. Main result

Theorem 3.1. *Assume the hypotheses* (H1), (H2), (H3), (H4), (H5) *and* (H6) *are satisfied. Then the integral inclusion* (1.1) *has at least one solution.*

Proof. A solution of (1.1) is a fixed point for the multivalued map $N : C(J,E) \longrightarrow C(J,E)$ defined by

$$Ny := \left\{ h \in C(J,E) : h(t) = \int_0^t K(t,s)f_y(s)ds + g(t) \; ; f_y \in S_{F,y} \right\}$$

where

$$S_{F,y} = \left\{ f_y \in L^1(J,E) : f_y(t) \in F(t,y(t)) \quad \text{for a.e.} \quad t \in J \right\}.$$

We shall show that N is a completely continuous multivalued map, u.s.c. and has convex closed values. The proof will be given in several steps.

Step 1: Ny is convex for each $y \in C(J,E)$. Indeed, if h, $\overline{h}$ belong to Ny. Then there exist f_y, $\overline{f}_y \in S_{F,y}$ such that

$$h(t) = \int_0^t K(t,s)f_y(s)ds + g(t), \quad t \in J,$$

and

$$\overline{h}(t) = \int_0^t K(t,s)\overline{f}_y(s) + g(t), \quad t \in J.$$

Let $0 \le k \le 1$, then for $t \in J$ we have that

$$[kh + (1-k)\overline{h}](t) = \int_0^t K(t,s)[kf_y(s) + (1-k)\overline{f}_y(s)]ds + g(t).$$

Since $S_{F,y}$ is convex (because F is convex valued) then

$$kh + (1-k)\overline{h} \in Ny.$$

Step 2: N sends bounded sets into bounded sets in $C(J,E)$. Let $B_r = \{y \in C(J,E) : \|y\|_0 \le r\}$ be a bounded set in $C(J,E)$ and $y \in B_r$, then for each $h \in Ny$ there exists $f_y \in S_{F,y}$ such that

$$h(t) = \int_0^t K(t,s)f_y(s)ds + g(t), \quad t \in J.$$

Thus for each $t \in J$ we have

$$|h(t)| \leq \int_0^t |K(t,s)||f_y(s)|ds + |g(t)| \leq$$
$$\leq \|p\|_{L^1} \sup_{t\in J} K(t) \sup_{y\in[0,r]} \psi(y) + \sup_{t\in J} |g(t)| < \infty.$$

Step 3: N sends bounded sets in $C(J,E)$ into equicontinuous sets. Let $t_1,\ t_2 \in J,\ t_1 < t_2,\ B_r = \{y \in C(J,E) : \|y\|_0 \leq r\}$ be a bounded set in $C(J,E)$ and $y \in B_r$. Let $h \in Ny$, then there exists $f_y \in S_{F,y}$ such that

$$h(t) = \int_0^t K(t,s)f_y(s)ds + g(t), \quad t \in J.$$

Thus for each $t \in J$ we have that

$$|h(t_2) - h(t_1)| = \Big| \int_0^{t_2} K(t_2,s)f_y(s)ds - \int_0^{t_1} K(t_1,s)f_y(s)ds +$$
$$+ g(t_2) - g(t_1) \Big| =$$
$$= \Big| \int_0^{t_1} [K(t_2,s) - K(t_1,s)]f_y(s)ds +$$
$$+ \int_{t_1}^{t_2} K(t_2,s)f_y(s)ds + g(t_2) - g(t_1) \Big| \leq$$
$$\leq \|K(t_2,.) - K(t_1,.)\|_{L^\infty} \int_0^T |f_y(s)|ds +$$
$$+ \left(\sup_{t\in J} K(t)\right) \int_{t_1}^{t_2} |f_y(s)|ds + |g(t_2) - g(t_1)| \leq$$
$$\leq \|K(t_2,.) - K(t_1,.)\|_{L^\infty} \|p\|_{L^1} \sup_{y\in[0,r]} \psi(y) +$$
$$+ \sup_{t\in J} K(t)(t_2 - t_1)\|p\|_{L^1} \sup_{y\in[0,r]} \psi(y) + |g(t_2) - g(t_1)|.$$

As a consequence of Step 2, Step 3 and (H6) together with the Ascoli-Arzela theorem we can conclude that $N : C(J,E) \longrightarrow C(J,E)$ is a completely continuous multivalued map, and therefore, a condensing map.

Step 4: N has a closed graph. Let $y_n \longrightarrow y_0$, $h_n \in Ny_n$, $h_n \longrightarrow$ $\longrightarrow h_0$. We shall prove that $h_0 \in Ny_0$ i.e. there exists $v_0 \in S_{F,y_0}$ such that

$$h_0(t) = \int_0^t K(t,s)v_0(s)ds + g(t), \quad t \in J.$$

We have that

$$\|(h_n - g) - (h_0 - g)\|_0 \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Consider the linear continuous operator $\Gamma : L^1(J,E) \longrightarrow C(J,E)$ defined by

$$(\Gamma v)(t) := \int_0^t K(t,s)v(s)ds, \quad t \in J.$$

Clearly from the definition of Γ we have that

$$(h_n - g) \in \Gamma(S_{F,y_n}).$$

From Lemma 2.1, it follows that $\Gamma \circ S_{F,y}$ is a closed graph operator. This, besides to $y_n \longrightarrow y_0$ and Lemma 2.1, furnishes that

$$(h_0 - g) \in \Gamma(S_{F,y_0}),$$

i.e.

$$h_0(t) = \int_0^t K(t,s)v_0(s)ds + g(t), \quad t \in J,$$

for some $v_0 \in S_{F,y_0}$.

It remains now to prove that

$$M := \{y \in C(J,E) : \lambda y \in Ny, \ \lambda > 1\}$$

is bounded to conclude (by Lemma 2.3) that N has fixed points. For this, let $\lambda y \in Ny$ for some $\lambda > 1$. Then there exists $f_y \in S_{F,y}$ such that

$$y(t) = \lambda^{-1} \int_0^t K(t,s)f_y(s)ds + \lambda^{-1}g(t) \quad \text{for all} \quad t \in J.$$

In view of (H2), (H3), (H4) and (H5) we have for each $t \in J$

$$|y(t)| \leq \int_0^t |K(t,s)|p(s)\psi(|y(s)|)ds + \|g\|_0 \leq$$

$$\leq \sup_{t\in J} K(t) \int_0^t p(s)\psi(|y(s)|)ds + \|g\|_0.$$

As a consequence of Lemma 2.2, we obtain

$$\|y\|_0 \leq \|z\|_0,$$

where z is the unique solution on J of the integral equation

$$z(t) - \|g\|_0 = \sup_{t\in J} K(t) \int_0^t p(s)\psi(z(s))ds. \tag{3.1}$$

So M is bounded. Set $X := C(J, E)$. As a consequence of Lemma 2.3 we can conclude that the multivalued map N has a fixed point y which is a solution of (1.1). ◊

Remark. Hypothesis (H5) and Lemma 2.2 imply the existence and the uniqueness of the solution of (3.1).

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