

GEOMETRY OF REGULAR PLANE SETS AND CHEBYSHEV SYSTEM THEORY

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Abstract: There is given a comprehensive investigation of the bounded regular plane curves (curves meeting each straight line in at most two points). The obtained results are used to prove that for a three dimensional Chebyshev space of continuous functions on a closed interval the following conditions are equivalent: (i) The space contains elements having zero only at the left endpoint and elements having zero only at the right endpoint of the interval; (ii) The space contains a Chebyshev space of dimension two; (iii) The space extends with a point; (iv) The space extends with an interval; (v) The space extends to a periodic Chebyshev space.

Introduction and main results

Let Q be a topological space and suppose that $C(Q)$ is the vector space of continuous real valued functions defined on Q . The n -dimensional subspace L of $C(Q)$ is called a *Chebyshev space* if every nonzero function in L possesses at most $n - 1$ distinct zeros in Q . The existence in $C(Q)$ of a Chebyshev space of dimension $n \geq 2$, implies

strong conditions on the space Q [8], [7], which restrict it to be homeomorphic with a subset of S^1 , the one dimensional topological sphere. Hence, when Q is connected we can restrict ourselves to $Q = I$ with I an interval of the real line with endpoints 0 and 1. The case $Q = S^1$ is covered by the situation $I = [0, 1)$ and the condition that only elements in the subspace of $C(I)$ with the property $\lim_{t \rightarrow 1} \psi(t) = \psi(0)$ are considered. In this case we say that the Chebyshev space is *periodic*. A base of a Chebyshev space (of a periodic Chebyshev space) is called a *Chebyshev system* (a *periodic Chebyshev system*).

We say that a Chebyshev space (or a Chebyshev system) in $C(I)$ *extends with a point* if its elements can be defined in a point c outside to I such that the extended space (system) be a Chebyshev space (system) on $I \cup \{c\}$. It *extends with an interval* if there exists the interval $J \subset \mathbb{R}$, $I \subset J$, $I \neq J$ such that the functions of the space (of the system) can be extended continuously to J such that the extended space (the extended system) be a Chebyshev space (a Chebyshev system) too.

The isolated zero t_0 of $\varphi \in C(I)$ is called *nodal* if $t_0 \in \int I$ and φ change sign at t_0 , or if $t_0 \in I \setminus \int I$. It is called *nonnodal* if $t_0 \in \int I$ and φ does not change sign at t_0 ([5], p. 22). These notions are starting points in various results regarding zeros of the functions in a Chebyshev space. We mention here the very classical ones summarized in

Theorem 0. *If L is an n -dimensional Chebyshev space in $C(I)$, $\varphi \in L \setminus \{0\}$, $s_1, \dots, s_q \in \int I$ are distinct nonnodal zeros of φ , $t_1, \dots, t_p \in I$ are distinct nodal zeros of φ , then $2q + p \leq n - 1$. For arbitrary distinct points as above with $2q + p = n - 1$ there exists a function $\varphi \in L$ with nonnodal zeros at s_1, \dots, s_q , nodal zeros at t_1, \dots, t_p and $\varphi(t) \neq 0$ elsewhere in I .*

(See e.g. the First and the Second theorem of S.N. Bernstein in [1], or the results in the first chapters of [5].)

If $2q + p < n - 1$, then there exist Chebyshev spaces of dimension n and some points as in Th. 0, for which there does not exist a function with nonnodal zeros at s_1, \dots, s_q and nodal zeros at t_1, \dots, t_p (see e.g. [1] and [12]). Abakumov and Domrachev give a geometric characterization of those Chebyshev spaces for which there exists functions with arbitrary repartitions of zeros, i.e. functions with nonnodal zeros at any distinct $s_1, \dots, s_q \in \int I$, and nodal zeros at any distinct $t_1, \dots, t_p \in I$, whenever $2q + p \leq n - 1$. These spaces will be called Chebyshev spaces with *arbitrary distributions of zeros* (or spaces with the *adz* property).

The purpose of this paper is to show that for $n = 3$ these problems

have a common geometric background. For some results for general n we mention the papers [9], [10], the paper of Zalik and Zwick [15], the results of Abakumov and Domrachev [1], and the course of Vinogradov [14]. The main result can be stated as follows:

Theorem 1. *Let $\{\varphi_1, \varphi_2, \varphi_3\} \subset C[0, 1]$ be a Chebyshev system. Then the following conditions are equivalent:*

- (i) : $\text{sp}\{\varphi_1, \varphi_2, \varphi_3\}$ has the adz property.
- (ii) There exist functions in $\text{sp}\{\varphi_1, \varphi_2, \varphi_3\}$ vanishing only in 0 and functions vanishing only in 1.
- (iii) The space $\text{sp}\{\varphi_1, \varphi_2, \varphi_3\}$ contains a subspace of dimension two, which is a Chebyshev space.
- (iv) The Chebyshev system $\{\varphi_1, \varphi_2, \varphi_3\}$ extends with a point.
- (v) The Chebyshev system $\{\varphi_1, \varphi_2, \varphi_3\}$ extends with an interval.
- (vi) The Chebyshev system $\{\varphi_1, \varphi_2, \varphi_3\}$ extends to a periodic Chebyshev system on $[0, 2)$.

The equivalence (i) \Leftrightarrow (ii) follows in fact from Th. 0. The equivalence (iii) \Leftrightarrow (iv) is the consequence of a more general result in [9]. The key of the proof of the whole theorem is the proof of the implication (ii) \Rightarrow (iv). The preparatory geometric results in doing this use some techniques in [1] and [14].

A negative form of this theorem is also worth mentioning. The principal statements in this line can be summarized as:

Theorem 2. *For the Chebyshev system $\{\varphi_1, \varphi_2, \varphi_3\}$ in $C[0, 1]$ the following assertions (i), (ii) and (iii) are equivalent:*

- (i) At least one of the following two conditions holds:
 - (a) Each function of $\text{sp}\{\varphi_1, \varphi_2, \varphi_3\}$ which vanishes at 0, vanishes in a point of $(0, 1]$.
 - (b) Each function of $\text{sp}\{\varphi_1, \varphi_2, \varphi_3\}$ which vanishes at 1, vanishes in a point of $[0, 1)$.
- (ii) The Chebyshev space $\text{sp}\{\varphi_1, \varphi_2, \varphi_3\}$ contains no Chebyshev space of dimension two.
- (iii) The Chebyshev system $\{\varphi_1, \varphi_2, \varphi_3\}$ cannot be extended with a point.

1. Regular plane sets

The set $M \subset \mathbb{R}^2$ is said *regular* if every line $d \subset \mathbb{R}^2$ intersects M in at most two points.

The investigations on regular sets in the n -dimensional space (a set B in \mathbb{R}^n is regular if every hyperplane intersects it in at most n points) lie at the confluence of topology and approximation theory (see e.g. [4], [2], [3] and respectively [8], [7]). From the various local and global characterization theorems about these sets it follows for instance that if such a set is compact and connected or it is locally connected and connected or locally compact and connected, then it is homeomorphic with a subset of S^1 , the circle. Hence, when speaking on connected regular sets we restrict ourselves to *regular curves*, i.e., curves which have parametrizations by continuous coordinate functions from I to \mathbb{R} where I is an interval in \mathbb{R} with the endpoints 0 and 1. We can assume also that such a parametrization, say $\Psi = (\psi_1, \psi_2)$, is injective vector function on $\int I$. In this case we can assert, using a result in [13] (see also [6] p. 59, Problem 12) that ψ_1 and ψ_2 are of bounded variation on each compact interval in I . Thus C is rectifiable.

To simplify the exposition *we shall consider only regular curves contained in bounded domains of \mathbb{R}^2* . Hence speaking on regular curves we assume always this restriction.

Summing our above considerations and agreements we can suppose that a regular curve C :

- (a) is a rectifiable closed curve in \mathbb{R}^2 without selfintersections, or
- (b) is a rectifiable nonclosed curve without selfintersections contained in a bounded subset of \mathbb{R}^2 ,

In the case (a) we can parametrize C by $\Psi = (\psi_1, \psi_2)$ with $\psi_i \in C[0, 1]$, $\psi_i(0) = \psi_i(1)$, $i = 1, 2$ and Ψ is injective on $[0, 1]$.

In the case (b) we can parametrize C by $\Psi = (\psi_1, \psi_2)$ with $\psi_i \in C(I)$, $i = 1, 2$ and Ψ injective on I , where I is an interval in \mathbb{R} with the endpoints 0 and 1.

It will be seen later (see 3, section 3) that in the case (b) we can consider that C has the endpoints A and B with $:\lim_{t \rightarrow 0} \Psi(t) = A$, $:\lim_{t \rightarrow 1} \Psi(t) = B$. Hence, when consider nonclosed regular curves we shall use these notations. A point in C different from its endpoints A and B will be said an *interior point of C* . When C is closed, every point of its is interior.

We shall present without proofs some simple geometric consequences of the above definition. Their proofs are standard and since we have to do with plane sets, it is in fact a simplified version of the machinery used in [10], [1] and [13]. (A detailed version of proofs can be get in our preprint [11].)

1.1. Let \mathcal{C} be a regular curve with the parametrization $\Psi = (\psi_1, \psi_2) : I \rightarrow \mathcal{C}$. Then in every point $\Psi(t) \in \mathcal{C}$ the convex hull $\text{co}\mathcal{C}$ of \mathcal{C} has a supporting line d . If $\Psi(t)$ is an interior point of \mathcal{C} then d does not meet \mathcal{C} in any other point. We call every supporting line to $\text{co}\mathcal{C}$ through points of \mathcal{C} supporting line to \mathcal{C} .

1.2. Let \mathcal{C} be a regular curve with the parametrization Ψ . Then for each point $\Psi(t_0)$ on \mathcal{C} the line $\Psi(t_0)\Psi(t)$ possesses a limit position when t tends monotonically to t_0 . The line defined this way is called *lateral tangent* of \mathcal{C} at $\Psi(t_0)$. The lateral tangent is a supporting line to \mathcal{C} . When $t_0 \in \int I$, we have two lateral tangents at $\Psi(t_0)$ which can coincide. The curve \mathcal{C} is placed in one of the four angles these two lateral tangents determine. Every point of \mathcal{C} different from $\Psi(t_0)$ is in the interior of this angle. If $\Psi(t_0)$ is one of the endpoints of \mathcal{C} , then there exists only one lateral tangent in it.

Let $\Psi(t_0) \in \mathcal{C}$. The closed angle containing \mathcal{C} determined by the lateral tangents at $\Psi(t_0)$ is called the *tangent cone* of \mathcal{C} at $\Psi(t_0)$. If $\Psi(t_0)$ is an endpoint of \mathcal{C} , then this cone is always a closed semispace.

Denote by $\Omega(\mathcal{C})$ the intersection of all the tangent cones of \mathcal{C} . Then $\Omega(\mathcal{C})$ is a closed convex set containing \mathcal{C} , hence $\text{co}\mathcal{C} \subset \Omega(\mathcal{C})$.

A set $M \subset \mathbb{R}^2$ is called a *maximal regular set*, if it is regular and it is not a proper subset of any other regular set.

\mathcal{C} is a *maximal regular curve* if it is a regular curve and is also a maximal regular set.

1.3. If \mathcal{C} is a maximal regular curve then

$$\Omega(\mathcal{C}) = \text{co}\mathcal{C}.$$

1.4. If $\Omega(\mathcal{C}) \neq \text{co}\mathcal{C}$, then \mathcal{C} cannot be a maximal regular curve.

1.5. If \mathcal{C} is a regular curve such that one of its endpoints, say A , is not contained in \mathcal{C} , then $\mathcal{C} \cup \{A\}$ is a regular curve.

1.6. If \mathcal{C} is a regular curve with the property that there exists a direction v such that every line parallel with v can intersect \mathcal{C} at most once, then \mathcal{C} cannot be a maximal regular curve.

1.7. A regular curve with the endpoints A and B , contained in a strip determined by two parallel lines through A and B cannot be maximal.

1.8. The regular curve \mathcal{C} is maximal if it holds one of the following two conditions:

(a) \mathcal{C} is a closed curve.

(b) \mathcal{C} is not a closed curve, it contains its endpoints, and at least one of the lateral tangents in the endpoints of \mathcal{C} contains the other

endpoint.

Observe that there exist two types of nonclosed maximal regular curves. For the first type the tangent line in one of the endpoints, say A contains the other endpoint B , but it is not tangent to the curve in this point (Fig. 1 (α)). For the second type the curve possesses a common tangent line at the endpoints A and B (Fig. 1 (β)).

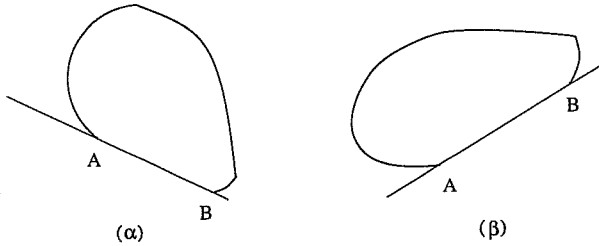


Fig. 1.

1.9. If C is a nonclosed regular curve containing its endpoints A and B which is not maximal, then it must hold one of the following cases:

- (a) The lateral tangents d_1 and d_2 at A , respectively at B are different and parallel (see Fig. 2 (α)).
- (b) The lateral tangents d_1 and d_2 have a common point $P \notin \{A, B\}$ and C is contained in the triangle ABC (see Fig. 2 (β)).
- (c) The lateral tangents d_1 and d_2 have a common point $P \notin \{A, B\}$ and C is contained in the closure of the complement with respect to the angle APB of the triangle APB (see Fig. 2 (γ)).

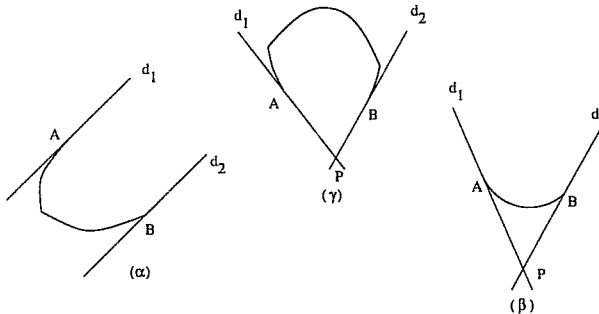


Fig. 2

1.10. From 1.3, 1.8 and 1.9 it follows that C is a maximal regular curve if and only if

$$\Omega(C) = \text{co } C.$$

1.11. Suppose that C is a maximal regular curve which is not

closed. Let A and B be the endpoints of C . Then at least one of the following two conditions holds:

- (a) Each line through A meets C in a point different from A .
- (b) Each line through B meets C in a point different from B .

1.12. Suppose that C is a regular curve which is not maximal.

(a) If C contains its endpoints then it can be extended by an arc to a closed regular curve.

(b) If C does not contain both its endpoints, then one of the following two alternatives hold:

(b1) C can be extended with one or with two points to a maximal regular curve.

(b2) C can be extended with an arc to a closed regular curve.

2. Vectorial independent sets in the space

The set $M \subset \mathbb{R}^3$ is called *vectorial independent* if every three distinct elements every of its three are linearly independent vectors.

Consider in \mathbb{R}^3 the Cartesian coordinate system $Oxyz$.

2.1. If $M \subset H_1 = \{(1, y, z) : y, z \in \mathbb{R}\}$, then M is vectorial independent if and only if it is a regular set in the plane H_1 .

Proof. The set M is vectorial independent if and only if each plane through the origin of $Oxyz$ intersects it in at most two points. Let H be an arbitrary plane of this type. Then $H \cap M = (H \cap H_1) \cap M$ since M is in H_1 . But $d = H \cap H_1$ is a line in H_1 and when H varies, it runs over the lines in this plane, wherefrom our assertion is immediate. \diamond

2.2. Suppose that the curve C is in the plane H_1 defined in 2.1. Then C is a maximal regular curve in H_1 if and only if it is a maximal vectorial independent set in \mathbb{R}^3 in the sense that it is not a proper subset of any vectorial independent set of this space.

Proof. Suppose that C is a maximal regular curve in the plane H_1 , but it is not a maximal vectorial independent set in \mathbb{R}^3 , that is, there exists a vector $P \notin C$ such that $C \cup \{P\}$ is vectorial independent.

Suppose first that $P = (0, y_0, z_0)$. Each plane through O and P can meet C once. Since $H_0 = \{(0, y, z) : y, z \in \mathbb{R}\}$ is parallel with H_1 , a plane H through O meets H_0 and H_1 in parallel lines. Suppose that $P \in H$. Then the intersection line of H and H_1 will be parallel with the direction of the segment OP . Since H can meet C only once, we arrive to the conclusion that each line in H_1 parallel with OP can meet C only

once. But then \mathcal{C} cannot be a maximal regular curve in the plane H_1 by 1.6.

If $P = (x_0, y_0, z_0)$ with $x_0 \neq 0$, then let Q be the intersection with H_1 of the line through O engendered by P . If it were be $Q \in \mathcal{C}$, then P and some point of \mathcal{C} would be on a line through O , which is impossible. Hence we must have $Q \notin \mathcal{C}$. Since $\mathcal{C} \cup \{Q\} \subset H_1$, and this set is vectorial independent in \mathbb{R}^3 , it must be a regular set in H_1 by 2.1. Thus \mathcal{C} cannot be a maximal regular curve in H_1 .

Suppose now that $\mathcal{C} \subset H_1$ is a maximal vectorial independent set in the space \mathbb{R}^3 . Then by 2.1, it must be a maximal regular curve in the plane H_1 . \diamond

3. Chebyshev systems of three functions

Let us consider the Chebyshev system of bounded functions $\{\varphi_1, \varphi_2, \varphi_3\} \subset C(I)$, where I is an interval with the endpoints 0 and 1 in \mathbb{R} .

We distinguish the case when I is semiclosed, e.g., $I = [0, 1)$ and $\lim_{t \rightarrow 1} \varphi_i(t) = \varphi_i(0)$, $i = 1, 2, 3$. In this case (and in the similar case when the role of 0 and 1 is inverted), we can identify our system with a Chebyshev system on S^1 , the circle, and in this case it is called a *periodic Chebyshev system*.

It is well known that the Chebyshev (or Haar) system $\{\varphi_1, \varphi_2, \varphi_3\} \subset C(I)$ can be characterized by the fact that the matrix

$$\begin{pmatrix} \varphi_1(t_1) & \varphi_2(t_1) & \varphi_3(t_1) \\ \varphi_1(t_2) & \varphi_2(t_2) & \varphi_3(t_2) \\ \varphi_1(t_3) & \varphi_2(t_3) & \varphi_3(t_3) \end{pmatrix}$$

is nonsingular whenever t_1, t_2, t_3 are distinct points of I .

Let us consider the space \mathbb{R}^3 endowed with the Cartesian reference system $Oxyz$. The above characterization of the Chebyshev system can be transposed in the following assertion:

3.1. *The functions $\varphi_1, \varphi_2, \varphi_3 \in C(I)$ form a Chebyshev system if and only if the function $\Phi = (\varphi_1, \varphi_2, \varphi_3) : I \rightarrow \mathbb{R}^3$ is injective and $\Phi(I)$ is a vectorial independent set in the space \mathbb{R}^3 .*

This assertion together with 2.1. gives:

3.2. *The function system $\{1, \psi_1, \psi_2\} \in C(I)$ is a Chebyshev system if and only if $\Psi = (1, \psi_1, \psi_2) : I \rightarrow \mathbb{R}^3$ is an injective function and $\Psi(I)$ is a regular curve in the plane $H_1 = \{(1, y, z) : y, z \in \mathbb{R}\}$ of the space \mathbb{R}^3 .*

Conversely, if C is an arbitrary regular curve in the plane H_1 , then for each continuous injective parametrization $\Psi = (1, \psi_1, \psi_2)$ of it, $\{1, \psi_1, \psi_2\}$ is a Chebyshev system.

The above functional characterization of regular curves yields the following result:

3.3. If C is a regular curve in a bounded domain of the plane H_1 of \mathbb{R}^3 defined in 2.1, and if $\Psi = (1, \psi_1, \psi_2)$ is its injective parametrization with $\psi_i \in C(I)$, $i = 1, 2$ then $A = \lim_{t \rightarrow 0} \Psi(t)$ and $B = \lim_{t \rightarrow 1} \Psi(t)$ exist. The points A and B are called the endpoints of the curve C .

Proof. According to 3.2, $\{1, \psi_1, \psi_2\}$ is a Chebyshev system, and since C is a bounded set there exist the real numbers $a_i = \liminf_{t \rightarrow 0} \psi_i(t)$ and $b_i = \limsup_{t \rightarrow 0} \psi_i(t)$ ($i = 1, 2$). If it were for some i $a_i < b_i$, then the function

$$\psi(t) = \psi_i(t) - \frac{a_i + b_i}{2}$$

would have an infinite number of zeros when $t \rightarrow 0$. But this is impossible since $\psi \in \text{sp}\{1, \psi_1, \psi_2\}$. \diamond

In accordance with the terminology introduced in section 1, we shall consider only regular curves which are bounded. This corresponds with the condition that only Chebyshev systems of bounded functions are considered (see the beginning of this section).

3.4. The Chebyshev system $\{1, \psi_1, \psi_2\}$ can be extended with a point if and only if the regular curve $C = \Psi(I)$, ($\Psi = (1, \psi_1, \psi_2)$) in the plane H_1 defined in 3.2, is not maximal.

Proof. If the Chebyshev system $\{1, \psi_1, \psi_2\}$ can be extended to $c \notin I$, then if we put $P = \Psi(c)$, the set $C \cup \{P\}$, $P \notin C$ is vectorial independent in the space \mathbb{R}^3 . Thus by 2.2, C cannot be a maximal regular curve.

Conversely, if C is not a maximal regular curve then for a point $P \in H_1$ such that $C \cup \{P\}$ is a regular set, by putting $\Psi(c) = P$ for some point $c \notin I$, we get that $C \cup \{P\}$ is vectorial independent by 2.1, and hence by 3.1, the system $\{1, \psi_1, \psi_2\}$ so extended is a Chebyshev system on $I \cup \{c\}$. \diamond

3.5. Let $\{1, \psi_1, \psi_2\} \subset C(I)$ be a Chebyshev system with ψ_1, ψ_2 bounded functions, $\Psi = (1, \psi_1, \psi_2)$.

(a) If $I = (0, 1)$ and

$$\lim_{t \rightarrow 0} \Psi(t) = \lim_{t \rightarrow 1} \Psi(t),$$

then $\{1, \psi_1, \psi_2\}$ can be extended to $[0, 1]$ as a periodic Chebyshev system.

(b) If $I = (0, 1)$ and

$$\lim_{t \rightarrow 0} \Psi(t) \neq \lim_{t \rightarrow 1} \Psi(t),$$

then $\{1, \psi_1, \psi_2\}$ can be extended to a Chebyshev system in the space $C[0, 1]$.

(c) If $I = [0, 1]$, then $\{1, \psi_1, \psi_2\}$ is whether a periodic Chebyshev system, or it can be extended to a Chebyshev system in the space $C[0, 1]$.

Proof. (a) In this case the regular curve $C = \Psi(I)$ does not contain its endpoints $A = \lim_{t \rightarrow 0} \Psi(t)$ and $B = \lim_{t \rightarrow 1} \Psi(t)$. According to 1.5, $C \cup \{A\}$ is a regular curve. It is closed, since $A = B$. Put $\Psi(0) = A$. Then $\Psi : [0, 1] \rightarrow \mathbb{R}^3$ is injective and continuous, hence $\{1, \psi_1, \psi_2\}$ extended this way is a periodic Chebyshev system.

(b) We can apply this case twice the assertion 1.5 to conclude that the curve $C \cup A \cup B$ is a regular curve. Now putting $\Psi(0) = A$ and $\Psi(1) = B$, $\Psi : [0, 1] \rightarrow \mathbb{R}^3$ will be continuous, injective, hence $\{1, \psi_1, \psi_2\}$ in this way extended will be a Chebyshev system in the space $C[0, 1]$.

(c) The proof of this case is contained in fact in the proofs of (a) and (b). \diamond

3.6. Let $\{1, \psi, \psi_2\} \subset C(I)$ be a Chebyshev system with ψ_1, ψ_2 bounded functions. Suppose that it can be extended with a point.

(a) If $I = [0, 1]$, then $\{1, \psi_1, \psi_2\}$ can be extended to a periodic Chebyshev system on $[0, 2]$.

(b) If $I = (0, 1)$ we have one of the following variants:

(b1) The system can be extended to a periodic Chebyshev system on $[0, 1]$.

(b2) The system can be extended to a Chebyshev system in the space $C[0, 1]$ which cannot be extended further with any point.

(b3) The system can be extended to a periodic Chebyshev system on $[0, 2]$.

(c) If $I = [0, 1)$ or $I = (0, 1]$, then we have the alternatives:

(c1) The system can be extended to a Chebyshev system in the space $C[0, 1]$, which cannot be extended further with any point.

(c2) The system can be extended to a periodic Chebyshev system on $[0, 2]$.

Proof. (a) The regular curve $C = \Psi([0, 1])$ contains its endpoints A and B and is not maximal by 3.4. In this case we are within the condition of the assertion 1.12 (a) and hence there exists a curve C_1 with the endpoints A and B such that $C \cup C_1$ is a closed regular curve Γ . Let us take a continuous injective parametrization $\Upsilon = (1, v_1, v_2) : [1, 2] \rightarrow \mathbb{R}^3$, of C_1 with $\Upsilon(1) = B$ and $\Upsilon(2) = A$. Let be

$$\xi_i = \begin{cases} \psi_i(t), & \text{if } t \in [0, 1], \\ v_i(t), & \text{if } t \in (0, 2), \end{cases} \quad i = 1, 2.$$

Then $(1, \xi_1, \xi_2)$ will be a continuous parametrization of Γ , which is injective on $[0, 2)$. Since Γ is a closed regular curve, we have by 3.2 that $\{1, \xi_1, \xi_2\}$ is a periodic Chebyshev system on $[0, 2)$.

(b) The case (b1) follows from 3.5 (a). From 3.5 (b) we have that if 3.5 (a) does not hold, then the system can be extended to a Chebyshev system on $[0, 1]$. If it cannot be extended with any point, we have the situation (b2). If it extends with a point, we use (a) to conclude (b3).

(c) The proof of this case is contained in fact in the proof of (b). \diamond

Proof of Theorem 1. By Th. 0, there exists a function in $\text{sp}\{\varphi_1, \varphi_2, \varphi_3\}$ which has a nonnodal zero at $1/3$, is positive elsewhere in $[0, 1]$, and there exists a function with a nonnodal zero at $2/3$ which is positive elsewhere in $[0, 1]$. By summing these two functions, we obtain a positive function in $\text{sp}\{\varphi_1, \varphi_2, \varphi_3\}$. Hence we can suppose, passing if necessary to another basis, that the function φ_1 is strictly positive. Thus $\psi_1 = \varphi_2/\varphi_1$ and $\psi_2 = \varphi_3/\varphi_1$ are continuous functions on $[0, 1]$. Let us see that $\{1, \psi_1, \psi_2\}$ is a Chebyshev system. Take $\psi \in \text{sp}\{1, \psi_1, \psi_2\} \setminus \{0\}$ arbitrarily. Then $\varphi_1\psi \in \text{sp}\{\varphi_1, \varphi_2, \varphi_3\} \setminus \{0\}$ and the number of zeros of ψ and of $\varphi_1\psi$ is the same, since φ_1 is strictly positive. Hence ψ can have at most two zeros, which proves our claim.

Let us suppose that $\{1, \psi_1, \psi_2\}$ has been extended to a periodic Chebyshev system on $[0, 2)$. We extend φ_1 to a continuous positive function on $[0, 2]$ with the property that $\varphi_1(2) = \varphi_1(0)$. Let us define φ_2 and φ_3 on the interval $[1, 2)$ by putting $\varphi_2 = \varphi_1\psi_1$, $\varphi_3 = \varphi_1\psi_2$. Then $\{\varphi_1, \varphi_1\psi_1, \varphi_1\psi_2\} = \{\varphi_1, \varphi_2, \varphi_3\}$ will be a periodic Chebyshev system on $[0, 2)$.

By similar reasonings it follows that all the properties of the Chebyshev system $\{\varphi_1, \varphi_2, \varphi_3\}$ which occur in Th. 1, have their correspondents for the Chebyshev system $\{1, \psi_1, \psi_2\}$ defined above, and conversely. Hence it is sufficient to prove Th. 1 for the Chebyshev system $\{1, \psi_1, \psi_2\}$ which we shall do in what follows.

The implications (vi) \Rightarrow (v) \Rightarrow (iv) are obvious.

(iv) \Rightarrow (iii). Suppose that $\{1, \psi_1, \psi_2\}$ extends to $[0, 1] \cup \{c\}$ with $c \notin [0, 1]$. Consider the elements ψ of the extended space having the property $\psi(c) = 0$. These form a two dimensional subspace of $\text{sp}\{1, \psi_1, \psi_2\}$. Each nonzero element ψ of this subspace can vanish at most once in $[0, 1]$. Hence, restricted to $[0, 1]$ it will be a Chebyshev space of dimension two.

(iii) \Rightarrow (i). If $\text{sp}\{1, \psi_1, \psi_2\}$ possesses a subspace L_0 of dimension two, which is a Chebyshev space, then for each $t_0 \in [0, 1]$, there exists an element of L_0 having a nodal zero at t_0 and is different from zero on $[0, 1] \setminus \{t_0\}$. We have seen above that every Chebyshev space of three functions contains elements which differ from zero throughout. From Th. 0 it follows that $\text{sp}\{1, \psi_1, \psi_2\}$ contains elements with nonnodal zero at any $t_0 \in (0, 1)$. In conclusion, the Chebyshev space $\text{sp}\{1, \psi_1, \psi_2\}$ possesses the *adz* property.

(i) \Rightarrow (ii) is obvious.

(ii) \Rightarrow (iv). Assume the contrary: the Chebyshev system $\{l, \psi_1, \psi_2\}$ cannot be extended with any point. Then by 3.4, $\mathcal{C} = \Psi([0, 1])$ where $\Psi = (1, \psi_1, \psi_2)$, is a maximal regular curve in the plane $H_1 = \{(1, y, z) : y, z \in \mathbb{R}\}$ of \mathbb{R}^3 . According to 1.11 at least one of the following condition holds:

(a) Each line through $A = \Psi(0)$ in H_1 meets \mathcal{C} in a point $\Psi(t)$ with $t \in (0, 1]$.

(b) Each line through $B = \Psi(1)$ in H_1 meets \mathcal{C} in a point $\Psi(t)$ with $t \in [0, 1)$.

Suppose (a) holds and let be $\psi \in \text{sp}\{1, \psi_1, \psi_2\}$ a function which vanishes at 0, $\psi = c_1 + c_2\psi_1 + c_3\psi_2$. Then the plane H through 0 with the normal vector $c = (c_1, c_2, c_3)$ contains $A = \Psi(0)$. Thus $H \cap H_1$ is a line through A , which by (a) meets \mathcal{C} in a point $\Psi(t)$ with $t \in (0, 1]$. But then $\Psi(t) \in H$ and consequently $\langle c, \Psi(t) \rangle = c_1 + c_2\psi_1(t) + c_3\psi_2(t) = 0$. We have concluded this way that a function $\psi \in \text{sp}\{1, \psi_1, \psi_2\}$ which vanishes at 0, vanishes in another point in $(0, 1]$. But this contradicts (ii).

The proof of (iv) \Rightarrow (vi), is contained in the assertion 3.6 (a).

This completes the proof of Th. 1. \diamond

Remark. Observe that in the statement of Th. 1 condition (i) (the requirement of the *adz* property) can be weakened as

(i') The Chebyshev space $\text{sp}\{\varphi_1, \varphi_2, \varphi_3\}$ contains elements vanishing only at 0 and contains elements vanishing only at 1.

Proof of Theorem 2. The same reasoning as that at the beginning of the proof of Th. 1 shows that it suffices to consider Chebyshev systems of form $\{1, \psi_1, \psi_2\}$.

We have shown in the proof of (iv) \Rightarrow (iii) of Th. 1, that if $\{1, \psi_1, \psi_2\}$ extends with a point, then the space $\text{sp}\{1, \psi_1, \psi_2\}$ possesses a two dimensional Chebyshev subspace, Wherefrom we have the proof of (ii) \Rightarrow (iii) of Th. 2.

If the Chebyshev space $\text{sp}\{1, \psi_1, \psi_2\}$ possesses a Chebyshev subspace of dimension two, then (i) of Th. 2 cannot hold. This shows (i) \Rightarrow (ii).

Finally, if (i) of Th. 2 does not hold, by (a) there exists a function $c_1 + c_2\psi_1 + c_3\psi_2$ vanishing only at 0. Let H be the hyperplane through O in \mathbb{R}^3 with the normal $c = (c_1, c_2, c_3)$. The line $d_1 = H \cap H_1$ meets the regular curve $C = \Psi([0, 1])$ of the plane $H_1 = \{(1, y, z) : y, z \in \mathbb{R}\}$ only at $A = \Psi(0)$. We can get by (b) similarly a line d_2 in the plane H_1 , which meets C only at $B = \Psi(1)$. We are then in one of the situations (α), (β) or (γ) of Fig. 2 and thus by 1.8 the regular curve C cannot be maximal. But then, by 3.4, (iii) cannot hold. This proves (iii) \Rightarrow (i) and completes the proof of Th. 2. \diamond

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