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ON A CLASS OF PARTIAL t-SPREADS OF $PG(n,q)$ ARISING FROM SCROLLS

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Abstract: Here, using algebraic geometry (scrolls over projective curves) we construct a class of partial t-spreads of $PG(n,q)$, study their properties and give general criteria to say when two t-spreads arising in this way are the same (e.g. if they contains a large number of the same points of $PG(n,q)$).

In this paper we will consider the following "Recognition and Uniqueness Problem" for subsets of a projective n-dimensional space $PG(n,q)$ over the finite field $GF(q)$ with q elements. Fix a set O and assume to have a family of "rules" or "laws" $\{O_m\}_{m\in O}$; each rule O_m

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gives a subset A_m of $PG(n,q)$. We may have $A_m = A_{m'}$ even if $m \neq m'$. Suppose that you fix $S \subseteq PG(n,q)$. First, it is obviously interesting to have necessary conditions that S must have to be one of the subsets A_m. If you know that $S = A_m$ for some m, show that there is no $m' \neq m$ with $S = A_{m'}$ (or describe all m' with $S = A_m = A_{m'}$). To make sense of this we must fix an interesting class of "rules". In this paper we will give a family of "rules" coming from Algebraic Geometry ($GF(q)$-points of scrolls over algebraic curves embedded in $PG(n,q)$), but motivated by the following concept in Finite Geometry. Recall (see [13, p. 29]) that a partial t-spread of $PG(2t+1,q)$ is a set of mutually disjoint t-dimensional subspaces; a partial t-spread is said to be maximal if it is not properly contained in another partial t-spread; a partial t-spread if it covers $PG(2t+1,q)$. Each A_m will be a partial t-spread of $PG(n,q)$. This class is a generalization of an important example ($\mathbf{P}^1 \times \mathbf{P}^t$ with the Segre embedding) considered in [28] and which gives a maximal t-spread of $PG(2t+1,q)$. For a different generalization (i.e. $\mathbf{P}^r \times \mathbf{P}^t$ with the Segre embedding which gives a t-spread of $PG((r+1)(t+1)-1,q)$) see [11]. Take again a family of "rules" $\{O_m\}_{m \in O}$ and the associated family of sets $\{S_m\}$. In the case in which we are interested in, i.e. the case of partial t-spreads, each O_m is equipped with a partition (say $A_m = \cup A_{mk}$) and it is essential the data of the partition to say that we have a partial t-spread. As we will see at the end of the paper (Section 6), if we know that one A_m is a partial t-spread, in some very important cases we will be able to say something non trivial on the partition.

Since this paper from now on is essentially "Algebraic Geometry", we will use the usual notations of that topic rather than the ones of Finite Geometries which was the topic which motivated this research. We hope that this would be a good topic and a motivation for further joint research involving specialists in these subjects. For a "pure" algebraic geometer as background on the motivations and t-spreads, see [13, pp. 1–29], and (also for the state of the art on t-spreads) [11].

Motivated by the previous discussion on the "Recognition and Uniqueness Problem" in the setting of our examples of partial t-spreads coming from scrolls (see Remark 1.3, the discussion in (2.1) and Th. 2.8), we discuss two related but different problems:

(a) What is the maximal number $N(m,d',d'')$ of common points of two different integral non degenerate curves C, D in $\mathbf{P}^m$ if we fix $\deg(C) := d'$ and $\deg(D) := d''$?

(b) Set $\mathbf{S}(m,d',d'') := \{t \in \mathbb{N}$: there exists two integral non degenerate curves C, D in $\mathbf{P}^m$ with $\deg(C) = d'$, $\deg(D) = d''$ and $\mathrm{card}(C \cap D) = t\}$. Describe $\mathbf{S}(m,d',d'')$.

What is the minimal number $n(m,d',d'')$ of points in $\mathbf{P}^m$ (in terms of m, d', d'') such that if C, D are integral non degenerate curves with $\deg(C) \leq d'$, $\deg(D) \leq d''$ and $\mathrm{card}(D \cap C) \geq n(m,d',d'')$, then $C = D$?

Problem (a) was completely solved (in any characteristic) in [14] in the case $m = 3$ and independently in [16]; the result (see [14, Th. 1]) is that $N(3,d',d'') = (d'-1)(d''-1)+1$ and that if $\deg(C \cap D) = N(3,d',d'')$, then $C \cup D$ is contained in a quadric surface which is smooth if $\min(d',d'') \geq 4$; if $\mathrm{card}(C \cap D) = N(3,d',d'')$, then C and D are smooth and rational ([14, Remark 16]). Furthermore (see [14, Remark 17]) there are gaps in the possible values of $\mathrm{card}(C \cap D)$. Furthermore ([14, Cor. 13]) in $\mathbf{P}^m$ we have an upper bound $(d'-m+1)d''$ for C integral and non degenerate and C' only assumed to be reduced and with no line or C as irreducible component. In [17] it is proven in characteristic 0 the upper bound $N(m,d',d'') \leq (d'-m+2)(d''-m+2)+m-2$ for every $m \geq 3$. Obviously $N(m,d',d'')$ is the maximal element of $\mathbf{S}(m,d',d'')$. We believe that for most integers m, d' and d'' the set $\mathbf{S}(m,d',d'')$ has many gaps below $N(m,d',d'')$. However we do not have any non trivial result on problem (b).

However, for the applications to fibrations by linear spaces (i.e. for the morphisms from curves to a Grassmannian), we are interested in problems (a) and (b) when the set of curves is restricted to the set of all curves contained in a suitable Grassmannian $G(t+1,n+1)$, say, embedded in a projective space $\mathbf{P}^m$, $m = (n+1)!/(n-t)!(t+1)!)$, by its Plücker embedding. We will see in Section 2 that for problem (a) this makes no difference. Indeed we will prove (see Th. 2.8) that $G(t+1,n+1)$ contains the rational surfaces scroll on which we found the pairs of non degenerate integral curves with the maximal number of common points.

Now we describe the structure of the paper. In the first section we introduce several standard notations and several standard notions which will be used in the entire paper. We tried our best to give for these notions as much standard and easy references as we know. Then we describe the scrolls whose embedding gives the partial t-spread. At the end of the section we give a well known cohomological lemma which will be used heavily in Sections 2 and 5 but which the non expert reader may safely skip. Then in Section 2 we study problem (a) of the introduction in our setting, stressing the positive characteristic case. In Section 3 we discuss the monodromy problem for the generic hyperplane section of $C \cup C'$ with C and C' integral non degenerate curves (see the beginning of Section 3 for references for the definitions and motivations). The stress is on the positive characteristic case, but even in characteristic 0, as far as we know, the results were unknown (see Ths. 3.1 and 3.3). Using Ths. 3.1 and 3.3 we may attach several uniformity problems for the generic hyperplane section of $C \cup C'$ (see Ths. 3.5 and 3.6). An important application of 3.6 is a key "convexity result" (see Th. 3.7) for the Hilbert function of the general hyperplane section of $C \cup C'$. For an important application and motivation for the "convexity result" 3.7, see Section 5, Th. 5.11. We believe that these monodromy problems are very nice and very useful. In Section 4 we discuss the notion of Frobenius non classical curve $C \subset \mathbf{P}^n$ introduced in [24] and show what happens for the curves embedded in our scroll. Then in the same section we generalize the notion of plane strange curve (see [8]) for curves contained in our scroll (see Def. 4.1 of ruling strange). We link these two concepts. The main results of this section are Prop. 4.3 and Th. 4.4. Then in Section 5 we discuss cohomological properties (see Def. 5.1) of our scroll. Using them we will be able to find in Section 5 several cases in which Th. 4.4 can be applied and a few properties of our partial t-spreads (e.g. when 3 colinear points of the partial t-spread must be contained in the same t-plane). These properties are related to a strong form of the Recognition and Uniqueness Problem, in the sense that knowing only the union of the t-planes of the t-spread and that the t-spread arises from a suitable scroll and has "geometric origin", we may (at least partially) reconstruct each t-plane (see Section 6).

We stress again the importance of monodromy and Galois groups for these topics and that in positive characteristic the picture is not complete (see [5] for the standard of the art). The questions still open seems to require even more than in [5] specialists (or motivated users) of finite group theory and this would be in itself a good topic for joint work for some algebraic geometers and some "finite geometers".

1. First in (1.1) and (1.2) we introduce several notations, conventions, and standard concepts on scrolls (with references).

(1.1) Let $\mathbf{K}$ be the algebraically closed field; if $\mathrm{char}(\mathbf{K}) > 0$, set $p := \mathrm{char}(\mathbf{K})$; every scheme will be defined over $\mathbf{K}$ and we will stress explicitely when a scheme is defined over a subfield (e.g. a finite field $GF(q)$, $q = p^e$) of $\mathbf{K}$. The Grassmannian $G(r,n)$ is the set of dimension r vector subspaces of $\mathbf{K}^n$ (hence for instance $G(2,4)$ will denote the Grassmannian of lines in $\mathbf{P}^3$). If B is a closed subscheme of a scheme A, let $\boldsymbol{I}_{B,A}$ be the ideal sheaf of B in A. Π and $\mathbf{P}$ will denote always a projective space whose dimension is either clear when we use the convention or not important. We will not distinguish between a (Cartier) divisor and the associated line bundle.

(1.2) As general references for the following notions and notations, see for instance [15], [23, pp. 162–169] and for the case $r = 2$ (i.e. ruled surfaces) [23, Ch. V, §2] or [20, Ch. IV, §2]. Let C be a smooth curve of genus g and E a rank r algebraic vector bundle on C; let $\mathbf{P}(E)$ be the projectivization of E and $\pi : \mathbf{P}(E) \to C$ the natural projection (with fibers isomorphic to $\mathbf{P}^{r-1}$); we have $\mathrm{Pic}(\mathbf{P}(E)) = \pi^*(\mathrm{Pic}(C)) \oplus \oplus \mathbb{Z}[H]$ with H any line bundle whose restriction to one fiber (hence to all fibers) of π is $\boldsymbol{O}(1)$, i.e. has degree 1. In particular if $g = 0$, we have $\mathrm{Pic}(\mathbf{P}(E)) \cong \mathbb{Z}[H] \oplus \mathbb{Z}[F]$. Recall ([23, Cor. V.2.13]) that if $g = 0$, every rank r vector bundle is the direct sum of r line bundles. Assume $r = 2$, i.e. $\mathbf{P}(E)$ a ruled surface; let $-e \in \mathbb{Z}$ be the minimal self intersection of a section of π; by a theorem of Nagata ([23, V.2, Ex. 2.5]) we have $e \geq -g$; if $g = 0$ than any $e \geq 0$ determines uniquely a rational ruled surface F_e (a Segre–Hirzebruch surface). From now on, assume $r = 2$ and $g = 0$. We will use additive or multiplicative notation both for divisors and line bundles. As free generators of $\mathrm{Pic}(F_e)$ we will take a section h of π with minimal self-intersection $-e$ (and the section is unique if $e > 0$, i.e. if F_e is not a quadric) and a fiber f; thus $h^2 = -e$, $h \cdot f = 1$ and $f^2 = 0$; the linear system $|ah + cf|$ is non empty if and only if $a \geq 0$ and $c \geq ae$; this linear system is spanned if and only if either $a = c = 0$ or $e = 0$, $a \geq 0$, $c \geq 0$, or $e > 0$, $a > 0$, $c \geq ae$; it is very ample if and only if $a > 0$ and $c > ae$; if $|h + af|$ is very ample, we will call $S(a,b)$, $b := a - e$, the image of F_e into the projective space $\mathbf{P}^{a+b+1}$; $S(a,b)$ has degree a+b and it is a minimal degree surface of

$\mathbf{P}^{a+b+1}$; if $e > 0$ and $b = 0$ this linear system induces a morphism w which contracts exactly the minimal section h and whose image is a degree a surface (again denoted by $S(a, 0)$ in $\mathbf{P}^{a+1}$) which is a cone over a rational normal curve of $\mathbf{P}^a$ and has as vertex the point $w(h)$; outside h the morphism w is an embedding.

Remark 1.3. Note that the vector bundles over $\mathbf{P}^1$ are defined over the prime field (and even over the integers) since they are direct sum of line bundles uniquely determined by their degree and defined over the prime field. In general, if instead of $\mathbf{P}^1$ we have any smooth complete curve C defined over a finite field $GF(q)$ and we have a rank $t+1$ vector bundle E on C defined over $GF(q)$ then the abstract dimension $t+1$ scroll $\mathbf{P}(E)$ is defined over $GF(q)$. Assume given an embedding $\mathbf{i}$ of $\mathbf{P}(E)$ into a projective space $\mathbf{P}^n$ with $\mathbf{i}$ defined over an extension $GF(q')$ of $GF(q)$. Then the points of $\mathbf{i}(\mathbf{P}(E))$ defined over $GF(q')$ are the union of the points of a partial t-spread of $PG(n, q')$, the t-planes (or more precisely their points over $GF(q')$) being the embedded t-planes of the scroll fibration $\mathbf{P}(E) \to C$.

Remark 1.4. Let $A \subset \mathbf{P}^m$ be a smooth scroll. Since A contains disjoint linear spaces of dimension $\dim(A) - 1$ we have $2\dim(A) \leq m$.

Now we conclude the first section with a cohomological well known lemma which the reader may skip on a first reading.

Lemma 1.5. *Let $A \subset \mathbf{P}^N$ be a smooth scroll over a curve C. Let $\pi : A \to C$ be the projection. Set $a := \dim(A)$ and assume $a \geq 2$. Then we have $R^i\pi_*(\mathbf{O}_A(b)) = 0$ if either $i = 0$ and $b < 0$ or $1 \leq i \leq a-2$ or $i = a-1$ and $b \geq a-1$ or $i \geq a$.*

Proof. This follows from the cohomology of $\mathbf{P}^{a-1}$ (see e.g. [23, Ch. 3, §5]) and a Theorem on changing basis for the cohomology (see e.g. [23, Th. III.12.11]). ◊

2. In this section we study problem (a) of the introduction.

(2.1) Here is the connection between Remark 1.3, Th. 2.8, the results on $N(m, d', d'')$ and the motivation coming from partial t-spreads. A pair (C, E) with C smooth curve and E rank r vector bundle on C spanned by its global sections induces a morphism $h_E : C \to G(r, n)$. If we take another such pair (C', E'), what is the maximal number of common points of $h_E(C)$ and $h_{E'}(C')$ if $(C, E) \neq (C', E')$ (and hence over a suitable extension of the finite field the two partial t-spread associated are different)? The link between the problem of maximal number of common points in a Grassmannian and in a projective space (its Plücker embedding) is given by Th. 2.8.

Remark 2.2. Let (C, C') be a pair of integral curves of $\mathbf{P}^m$. If C is non degenerate but C' is contained in a hyperplane, then $\text{card}(C \cap C') \leq$ $\leq \deg(C)$ by Bézout theorem. We have $\text{card}(C \cap C') \leq \deg(C)$ even if C is degenerate, but $C \cup C'$ spans $\mathbf{P}^m$ because $C \cap H \neq C$ implies $\text{card}(C \cap H) \leq \deg(C)$. Hence we see that to find $N(m, d', d'')$ or good upper bounds for $N(m, d', d'')$ we may assume that both C and C' are non degenerate. This is the content of the next two statements. The key statement (see Prop. 2.3) is stated and proved in [2] in characteristic 0 and here in positive characteristic for reflexive curves (see [8] or [27] for the notion of reflexive curve) as Th. 2.5. Let $T \subset \mathbf{P}^m$ be a curve (perhaps reducible) spanning $\mathbf{P}^m$; a general hyperplane section of T is in linear general position (terminology of [2]) or in lineary general position (terminology of [22] and [15]) or, with the old terminology of [20, p. 249], in generic position, if for a general hyperplane H of $\mathbf{P}^m$ and every integer x with $1 \leq x \leq m$ every subsets S of $T \cap H$ with $\text{card}(S) = x$ spans a linear space of dimension $x - 1$.

Proposition 2.3 ([2, middle part of §1]). *Assume characteristic* 0. *Let C and C' be integral non degenerate curves in $\mathbf{P}^m$. Then a general hyperplane section of $C \cup C'$ is in linear general position.*

This result was claimed parenthetically also in [12, p. 30], but no outline of any proof was given there since it was inserted just to pose to the reader as interesting question the search of stronger results.

Let $g(m, d)$ be the maximal arithmetic genus of an integral non degenerate curve of degree d in $\mathbf{P}^m$. For the values $g(m, d)$ for all m and d, see [20] or [22, Ch. III] or [1, Ch. III, §2] or the introduction of [12]. By Prop. 2.3 and the proof in [20, p. 249 and pp. 527–533] or [22, Ch. III] or [1, Ch. III, §2] we obtain the following result.

Theorem 2.4. *Assume characteristic* 0. *Then $N(m, d', d'') \leq$ $\leq g(m, d' + d'')$ and if C and C' are integral non degenerate curves with* $\text{card}(C \cap C') = g(m, d' + d'')$, *then C and C' are smooth and rational and $C \cup C'$ is contained in a minimal degree surface S which is not a Veronese surface, i.e. either in a linearly normal smooth rational surface scroll $S(a, b)$, $a + b + 1 = m$, or in a cone $S(m - 1, 0)$ over a rational normal curve of $\mathbf{P}^{m-1}$.*

Proof. By the quoted references it is sufficient to check that S is not the Veronese surface V in $\mathbf{P}^5$. Since the only smooth rational curves of the plane are the lines and the conics, we see that their double embedding as curves in V do not span $\mathbf{P}^5$. $\Diamond$

Now we will extend Th. 2.4 under the assumption that both C and C' are reflexive integral non degenerate curves. Since the case m = 3 is covered completely by [14], it is sufficient to extend Prop. 2.3 to this positive characteristic situation for $m \geq 4$. This will be done proving the following result.

Theorem 2.5. *Fix an integer $m \geq 4$. Let C and C' be integral reflexive non degenerate curves in $\mathbf{P}^m$. Then a general hyperplane section of $C \cup \cup C'$ is in linear general position.*

Proof. (a) We will use induction on m. However, in part (a) of the proof we will analyze the case of space curves from our point of view. Hence here we assume $m = 3$. Assume that for a general plane H, there is at least a trisecant line D to $(C \cup C') \cap H$. Since both C and C' are reflexive and H is general, we have $\text{card}(C \cap D) \leq 2$ and $\text{card}(C' \cap D) \leq \leq 2$ ([6, §7] or [27, Cor. 2.2]). If a general secant line to C (resp. C') is secant to C' (resp. C), projecting $C \cup C'$ into a plane from a general $P \in C$ (resp. $P \in C'$) we find $d' - 1 = d''/2$ (resp. $d'' - 1 = d'/2$). Since these inequalities cannot be satisfied simultaneously with $d'' \geq 3$, we have $\text{card}((C \cup C') \cap D) = 3$. Just to fix the notations assume $\text{card}(C \cap \cap D) = 2$ and $\text{card}(C' \cap D) = 1$. Set $A := \{P \in C' \cap H$: there is a line $T \subset H$ with $P \in T$, $\text{card}(C \cap T) = 2$ and $\text{card}(C' \cap T) = 1\}$ (resp. $A' := \{P \in C' \cap H$: there is no line $T \subset H$ with $P \in T$, $\text{card}(C \cap T) = 2$ and $\text{card}(C' \cap T) = 1\}$) and $B := \{\text{pairs}(Q, Q') \in (C \cap H) \times (C \cap H)$, $Q \neq Q'$: there is a line T with $\{Q, Q'\} \subset T$ and $T \cap C' \neq \emptyset\}$ (resp. $B' := \{\text{pairs}(Q, Q') \in (C \cap H) \times (C \cap H)$, $Q \neq Q'$: there is no line T with $\{Q, Q'\} \subset T$ and $T \cap C' \neq \emptyset\}$). Since C' is reflexive and we assumed $A \neq \emptyset$, we have $A' = \emptyset$. Since C is reflexive and we assumed $B \neq \emptyset$, we have $B' = \emptyset$. Since a line is contained only in ∞^1 planes, there are ∞^2 secant lines to C meeting C'. Hence for a general $x \in C'$ the projection of C from x into a plane is $2:1$; call $C(x)$ its image; we have $\deg(C(x)) = d'/2$. Since $B' = \emptyset$ we see that $C(x)$ is the image of C' under the projection from x. Since a general secant line to C' is not 3-secant by the reflexivity of C', $C(x)$ is birational to C' and we have $d'/2 = d'' - 1$.

(b) Assume $m \geq 4$ and that m is the first integer for which the statement of this theorem fails. Taking a general projection into $\mathbf{P}^4$ we see that we have $m = 4$; we will use this observation only to simplify the notations. Since from a general point of $\mathbf{P}^4$ there is a hyperplane containing it, taking a general projection into $\mathbf{P}^3$ we obtain $d'/2 = d'' - 1$ or $d''/2 = d' - 1$. Just to fix the notations we assume $d' \geq d''$, hence $d'/2 = d'' - 1$. Furthermore, the analysis in part (a) and the generality of the point of projection into $\mathbf{P}^3$, show that for a general hyperplane M of $\mathbf{P}^4$ there is a line $D \subset M$ secant to C and meeting C'. Part (a) shows also that for a general $z \in C'$ the images of C and C' into $\mathbf{P}^3$ are the same curve. Hence for a general line T secant to C' the images of C and C' by the projection from T into a plane are the same curve; call it $C(T)$. Since C' is reflexive, we have $\deg(C(T)) = d''-2$. Since a general such T is contained in a general hyperplane M of $\mathbf{P}^4$ we see that there is a plane $M' \subset M$ containing T and with exactly 2 points of C' and 3 points of C, one of them the point of intersection of two lines secant to C and meeting the 2 points of $T \cap C'$. Thus $\deg(C(T)) = d'/3$. Since $d' = 2d'' - 2$, we have $d'' = 4$ and $d' = 6$. Hence C' is a rational normal curve. If we were interested only in the case $m \geq 5$, working also in $\mathbf{P}^5$ we would obtain another numerical restriction (i.e. $d'/4 = d'' - 3$) and hence a numerical contradiction. For the case $m = 4$, note that the line T must be contained in every quadric hypersurface containing C. Since C is cut out by quadrics, we have a contradiction. $\diamondsuit$

For a more general result with a different proof, see the case $w = 1$ of Th. 3.3.

Remark 2.6. The proof of Th. 2.5 shows that we have $\text{card}(C \cap C') \leq g(m, d', d'') - p_a(C) - p_a(C')$ and that, again, if we have equality, $C \cup C'$ is contained in a minimal degree surface of $\mathbf{P}^m$.

(2.7) Now we will describe the class of examples of pair of curves with a large number of common points contained in the scroll $S(a, b)$ for $b > 0$. We have $S(a, b) \cong F_{a-b}$, i.e. here $e = a - b$. With the conventions used in this paper we have $\boldsymbol{O}_{S(a,b)}(1) = (h + af)$ because $\deg(S(a, b)) = a + b = (h + af)^2$. For all integers $x > a - b$, $y > 1$, $c > 0$ we have $\deg(h + xf) = x + a - e = x + b$, $\deg(yh + (ye + c)f) = c + ay$, $(h + xf) \cdot (yh + (ye + c)f) = c + xy$ and the linear systems $|h + xf|$ and $|yh + (ye + c)f|$ are very ample. Hence by Bertini theorem we may find smooth curves $C \in |h + xf|$, $C' \in |yh + (ey + c)f|$ with exactly $c + xy$ common points; in particular for $c = 1$, varying a and b with $a + b = N - 1$ and $a \geq b > 0$ we find (in any characteristic) pairs of smooth curves with a number of common points near the maximal value of $N(m, d', d'')$ given in the statement of Th. 2.4 and such that the pairs of their degree $(\deg(C), \deg(C'))$ cover a huge band in $\mathbb{N}^2$. Furthermore we have a good description of all pairs (C, C') with $N(m, d', d'')$ common points (e.g. both C and C' are smooth and rational) and by the structure of the linear systems on the surfaces F_e listed in (1.2), as in the case $m = 3$ considered in [14, Remark 17], there are gaps for the possible triples $(d', d'', N(m, d', d''))$ and for the values of $\mathrm{card}(C \cap C')$. A similar description holds if $b = 0$, i.e. if $S(a, b)$ is a cone. The details are left to the reader.

A major result is the following theorem which shows the problem on the maximal number of common points of two projective spaces is equivalent to a weak form of the Recognition and Uniqueness Problem for our class of examples coming from scrolls.

Theorem 2.8. *For all integers r, n, a, b with $0 < r < n$, $0 \leq b \leq$ $\leq a$, $a + b + 2 = n!/(r!(n-r)!)$ the Grassmannian $G(r, n)$ with its Plücker embedding into $\Pi := \mathbf{P}^N$, $N = n!/(r!(n-r)!) - 1$, contains the rational normal scroll $S(a, b)$; here we use the usual convention that for $b = 0$ $S(a, 0)$ is the cone ever a rational normal curve of a hyperplane of Π, i.e. the contraction of the section of F_a with negative self-intersection induced by the morphism associated to $|h + af|$. Furthermore we will find such an embedding, say $\boldsymbol{i} : \mathbf{P}^1 \to G(r, n)$ with $\mathbf{P} := \boldsymbol{i}(\mathbf{P}^1)$, which is induced by the restriction $R := Q|\mathbf{P}$ to $\mathbf{P}$ of the universal rank r quotient bundle of $G(r, n)$ to a rank r vector bundle, say $R := \boldsymbol{O}_{\mathbf{P}}(b(1)) \oplus \cdots \oplus \boldsymbol{O}_{\mathbf{P}}(b(t)) \oplus \boldsymbol{O}_{\mathbf{P}}(a(1)) \oplus \cdots \oplus \boldsymbol{O}_{\mathbf{P}}(a(r-t))$, such that the restriction map induces an isomorphism between the spaces of global sections; we will find a surjection $R \to \boldsymbol{O}_{\mathbf{P}}(a) \oplus \boldsymbol{O}_{\mathbf{P}}(b)$ which induces an isomorphism on global sections. An embedding with these properties is uniquely determined up to a translation by an element of* $\mathrm{Aut}^0(G(r, n))$.

Proof. First, we will explain the second statement of the theorem. By the universal property of the Grassmannian every morphism $f : \mathbf{P}^1 \to$ $\to G(r, n)$ is given by the choice of a rank r vector bundle, E, on $\mathbf{P}^1$ and by a subspace $V \subseteq H^0(\mathbf{P}^1, R)$ with V spanning R and $\dim(V) = n$; furthermore, we have $f^*(Q) \cong E$. The second statement of the theorem says that we may find an embedding corresponding to a vector bundle R with a further property. This additional property is essential for our proof of the first statement of the theorem and it has a geometric meaning because it says that $\mathbf{i}(\mathbf{P}^1)$ is contained in a scroll isomorphic to F_{a-b}. Concerning the last statement, recall that if $2r \neq n$, then $\mathrm{Aut}(G(r, n)) \cong GL(n, \mathbf{K})$ and hence $\mathrm{Aut}(G(r, n)) = \mathrm{Aut}^0(G(r, n))$, while if $2r = n$ then $\mathrm{Aut}(G(r,n))$ has two connected components, each of them isomorphic to $GL(n,\mathbf{K})$ and hence $\mathrm{Aut}^0(G(r,n)) \cong GL(n,\mathbf{K})$ even in this case. Let V be a vector space of dimension n and A (resp. B) complementary vector subspaces of V with $\dim(A) = a + 1$, $\dim(B) = b+1$ and $A + B = V$. Consider the surjections of the trivial sheaf $\mathbf{P}^1 \times A \to \boldsymbol{O}_{\mathbf{P}^1}(a)$ and the trivial sheaf $\mathbf{P}^1 \times B \to \boldsymbol{O}_{\mathbf{P}^1}(b)$ induced by any choice of isomorphisms $A \cong H^0(\mathbf{P}^1, \boldsymbol{O}_{\mathbf{P}^1}(a))$ and $B \cong H^0(\mathbf{P}^1, \boldsymbol{O}_{\mathbf{P}^1}(b))$. Since $A + B = V$ and $A \cap B = \{0\}$ we obtain a surjection $\phi : \mathbf{P}^1 \times V \to \boldsymbol{O}_{\mathbf{P}^1}(a) \oplus \boldsymbol{O}_{\mathbf{P}^1}(b)$ including an isomorphism of global sections. By definition the surjection ϕ induces an embedding of the Serge-Hirzebruch surface F_{a-b} into Π as a minimal degree rational normal scroll $S(a,b)$ (with the usual convention for the cone $S(N-1,0)$). Fix an integer t with $b+1 \le t < r$ and integers $b(1) \ge b(2) \ge \ldots \ge b(t) \ge 0$, $a(1) \ge a(2) \ge \ldots \ge a(r-t) \ge 0$ with $\sum_{1\le j\le t} b(j) = b - t + 1$, $\sum_{1\le i\le r-t} a(i) = a - r + t + 1$. Note that $h^0(\mathbf{P}^1, \boldsymbol{O}_{\mathbf{P}^1}(b(1)) \oplus \ldots \oplus \boldsymbol{O}_{\mathbf{P}^1}(b(t))) = h^0(\mathbf{P}^1, \boldsymbol{O}_{\mathbf{P}^1}(b))$ and $h^0(\mathbf{P}^1, \boldsymbol{O}_{\mathbf{P}^1}(a(1)) \oplus \ldots \oplus \boldsymbol{O}_{\mathbf{P}^1}(a(r-t))) = h^0(\mathbf{P}^1, \boldsymbol{O}_{\mathbf{P}^1}(a))$; choose isomorphisms $i_b : H^0(\mathbf{P}^1, \boldsymbol{O}_{\mathbf{P}^1}(b(1)) \oplus \ldots \oplus \boldsymbol{O}_{\mathbf{P}^1}(b(t))) \to H^0(\mathbf{P}^1, \boldsymbol{O}_{\mathbf{P}^1}(b))$ and $i_a : H^0(\mathbf{P}^1, \boldsymbol{O}_{\mathbf{P}^1}(a(1)) \oplus \ldots \oplus \boldsymbol{O}_{\mathbf{P}^1}(a(r-t))) \to H^0(\mathbf{P}^1, \boldsymbol{O}_{\mathbf{P}^1}(a))$ and surjections $\boldsymbol{O}_{\mathbf{P}^1}(b(1)) \oplus \ldots \oplus \boldsymbol{O}_{\mathbf{P}^1}(b(t)) \to \boldsymbol{O}_{\mathbf{P}^1}(b)$, $\boldsymbol{O}_{\mathbf{P}^1}(a(1)) \oplus \ldots \oplus \boldsymbol{O}_{\mathbf{P}^1}(a(r-t)) \to (a))$ inducing the isomorphisms i_b and i_a on global sections. Let $\pi : F_{a-b} \to \mathbf{P}^1$ be the projection. Set $R : \boldsymbol{O}_{\mathbf{P}^1}(b(1)) \oplus \ldots \oplus \boldsymbol{O}_{\mathbf{P}^1}(b(t)) \oplus \boldsymbol{O}_{\mathbf{P}^1}(a(1)) \oplus \ldots \oplus \boldsymbol{O}_{\mathbf{P}^1}(a(r-t))$ and $M := \pi^*(R)$. By the projection formula we have $\pi_*(M) = R \otimes \pi_*(\boldsymbol{O}_{F_{a-b}})$; hence (by Lemma 1.5) we have $\pi_*(M) = R$ and an isomorphism $H^0(F_{a-b}, M) \cong H^0(\mathbf{P}^1, R)$. Since $\mathrm{rank}(M) = r$ this isomorphism and the previous construction on $\mathbf{P}^1$ induces an embedding of F_{a-b} into $G(r,n)$ (with the convention of cones if $b = 0$) which (for the Plücker embedding $G(r,n) \subset \Pi$) has as image a minimal degree scroll $S(a,b)$, proving the first and the second assertion. The last assertion follows from the universal property of the Grassmannian. $\Diamond$

3. Here we consider the monodromy problem and a kind of possible "uniformity" of the generic hyperplane section of the curve $C \cup C' \subset \mathbf{P}^m$ (see Def. 3.4). For general background and definitions see [21] (topological approach over $\mathbb{C}$) and for the positive characteristic case see [6], [27], [5]). For the main motivation and application ("Castelnuovo's Theory"), see [20, pages 527–533], [1, Ch. III, §2] (characteristic 0) and [2], [27, §2] (positive characteristic). One of the main points of this paper is the monodromy type results given in Ths. 3.1 and 3.3. Then these results are applied to study "the uniformity" (see the same references as for the monodromy problem, in particular for the applications to Castelnuovo's theory). Our main results on uniformity are Ths. 3.5 and 3.6 which give Th. 3.7, a key convexity result for the Hilbert function). We stress that there are several papers with tools from commutative algebra and/or projective geometry and general algebraic geometry which use suitable notions of uniformity (for instance the very active area of research on "fat points").

Theorem 3.1. *Let C' and C'' be integral non degenerate curves in $\mathbf{P}^m$; set $C := C' \cup C''$; set $a' := \deg(C')$ and $a'' := \deg(C'')$; let H (resp. H', resp. H'') be the monodromy group of the generic hyperplane section of C (resp. C', resp. C''). Assume that the pair (H', H'') is either $(S_{a'}, S_{a''})$ or $(S_{a'}, A_{a''})$ or $(A_{a'}, S_{a''})$ or $(A_{a'}, A_{a''})$. Then either $H = H' \times H''$ or $H = H'$ (and then $d' = d''$ and either $H' = H''$ or $H' = S_{d'}$ and $H'' = A_{d''}$) or $H = H''$ (and then $d' = d''$ and either $H' = H''$ or $H'' = S_{d''}$ and $H' = A_{d'}$).*

Proof. The group G acts as permutation group on the generic hyperplane section of C. By the definition of monodromy group for the generic hyperplane section we have two surjections $s' : G \to G'$ and $s'' : G \to G''$. Set $N' := \ker(s')$ and $N'' := \ker(s'')$. Since $C \neq C'$ and C, C' are integral, by the definition of monodromy group for the generic hyperplane section we have $N' \cap N'' = \{0\}$. Since the alternating groups involved are simple, we see that if $\mathrm{card}(N') + \mathrm{card}(N'') \geq 4$, we have $H \cong H' \times H''$. The remaining assertions follow at once. ◊

Remark 3.2. With the notations of 3.1, we have $H' = S_{a'}$ (and similarly for (A'', H'')) if either $\mathrm{char}(\mathbf{K}) = 0$ ([21] or [1]) or A' is reflexive ([6, Th. at p. 906] or [27, Cor. 2.2]) or $p > 2$, $m \geq 4$, A' is smooth of genus ≥ 2 ([5, Th. 3.1]). Using the classification of multiply transitive finite permutation groups (see [10, §5] or [27, Th. 2.4] (with the obvious slip of M_{22} among the 3-transitive permutation groups)) it was proved in [27] that we have $H' = A_{a'}$ or $H' = S_{a'}$ if either $m \geq 6$ or $m = 4, 5$ and $d' > 25$ or $m \geq 5$ and $p > 5$ or $m = 4$ and $p > 7$.

Theorem 3.3. *Let A' and A'' be integral non degenerate reflexive curves in $\mathbf{P}^m$. Set $A := A' \cup A''$, $a' := \deg(A')$, $a'' := \deg(A'')$. Then the monodromy group G of the generic hyperplane section Γ of A is*

$S_{a'} \times S_{a''}$.

Proof. By [6, Th. at p. 906] or [27, Cor. 2.2] the monodromy groups for A' and A'' are the full symmetric groups. The proof of 3.1 shows that to prove 3.3 it is sufficient to show that G contains a non trivial permutation fixing $\Gamma \cap A'$ and a non trivial permutation fixing $\Gamma \cap A''$. The proof given in [6, §5] or [27] for the monodromy group of A'', shows that G contains a non trivial permutation fixing $\Gamma \cap A'$ if there is a hyperplane U which is transversal to A' and intersects A'' at $a'' - 1$ smooth points, at $a'' - 2$ of them transversally and at the other point, P, with simple tangency. The second condition on U holds for the general hyperplane tangent to A'' by the reflexivity of A''. Since both A' and A'' are reflexive, by biduality they have different dual varieties (and both are hypersurfaces because A' and A'' are curves). Hence also the first condition is satisfied by the general hyperplane U tangent to A''. ◊

Definition 3.4. Fix an integer $w \geq 1$. A set of points $A \subset \Pi := \mathbf{P}^s$ is said to be in *w-uniform position* if for every $B \subseteq A$ and every integer v with $1 \leq v \leq w$ we have $h^0(\Pi, \boldsymbol{I}_{B,\Pi}(v)) = \max\{(s+v)!/(s!v!) - - \mathrm{card}(B), 0\}$.

Theorem 3.5. *Let C, C' be integral non degenerate curves in $\mathbf{P}^{s+1}$; set $a := \deg(C)$, $b := \deg(C')$. Fix an integer $w \geq 1$. Let $\Gamma := (C \cup C') \cap \cap H$ be a general hyperplane section of $C \cup C'$. Set $x := (s+w)!/(s!w!)$. Let G' (resp. G'') be the monodromy group of the generic hyperplane section of C (resp. C'). Assume $A_a \subseteq G'$ and $A_b \subseteq G''$. Assume $\min\{a, b\} \geq x + 2$. Then Γ is in w-uniform position.*

Proof. Let $B \subseteq \Gamma$ be minimal such that

$$h^0(H, \boldsymbol{I}_{B,H}(w)) \cdot h^1(H, \boldsymbol{I}_{B,H}(w)) \neq 0.$$

By the minimality of B we have $3 \leq \mathrm{card}(B) \leq x$. By Th. 3.1 and the definition of alternating group, if $\mathrm{card}(B \cap C) \geq 2$ we have $C \cap \cap \Gamma \subseteq B$ (and the same for C'). Since $\min\{a, b\} \geq x + 2$, we have a contradiction. ◊

The proof of Th. 3.5 gives also the following key convexity result ([22, Cor. 3.5] and [12, Lemma 1.5]).

Theorem 3.6. *Fix integers n, m with $n \geq m \geq 1$. Let C, C' be integral non degenerate curves in $\mathbf{P}^{s+1}$; set $a := \deg(C)$, $b := \deg(C')$. Let $\Gamma := := (C \cup C'0 \cap H$ be a general hyperplane section of $C \cup C'$. Assume that Γ is in w-uniform position. Set $x := (s+w)!/(s!w!)$. Let G' (resp. G'') be the monodromy group of the generic hyperplane section of C (resp.*

C'). Assume $A_a \subseteq G'$ and $A_b \subseteq G''$. Assume $(s+n)!/(s!n!)+2 \leq$ $\leq \min\{a,b\}$. Then for the Hilbert function $h_\Gamma : \mathbb{N} \to \mathbb{N}$ of Γ we have $h_\Gamma(n+m) \geq \min\{\deg(C)+\deg(C'), h_\Gamma(n)+h_\Gamma(m)-1\}$.

We note that for the application (see Th. 5.11) we need only the case $m = 2$ of the key convexity result (proved in any way) plus the assumption that the monodromy group contains the product of the two alternating groups.

4. For the notion of strange curve, see for instance [8] or [27]. Fix a power q of p; let $C \subset \mathbf{P}^n$ be an integral non-degenerate curve; take homogeneous coordinates $X_0, \ldots, X_n$ of $\mathbf{P}^n$ and see $x_i := X_i/X_0$ as a rational function on C; for every integer $i \geq 0$ and every rational function f on C let $D^i_X(f)$ be the Hasse derivative of f of order i with respect to the separating variable x on C; for any n non-negative integers $a(0), \ldots, a(n-1)$ and any $n+1$ rational functions $f_0, \ldots, f_n$ on C, let $\mathrm{Det}(a(0), \ldots, a(n-1); f_0, \ldots, f_n))$ be the $(n+1) \times (n+1)$ matrix $(a_{ij})_{0\leq i\leq n, 0\leq j\leq n}$ with $a_{1j} := f_j^q$ and $a_{i+1j} := D^{a(i)}_{f_j}$ for $i \geq 1$; by [29], Prop. 2.1, there are integers $v_0, \ldots, v_{n-1}$ with $0 = v_0 < v_1 < \cdots <$ $< v_{n-1}$ such that $\mathrm{Det}(v_0, \ldots; v_{n-1}; 1, x_1, \ldots, x_n) \neq 0$ among these sets of integers we will take the minimal one in the lexicographic order; these integers $v_0, \ldots, v_{n-1}$ are called the Frobenius sequence of C; C is called Frobenius classical if $v_i = i-1$ for every i. The notion of Frobenius classical curve was introduced, motivated, applied and studied in [24]. The following definition is a natural generalization to the case of curves on a ruled surface of the notion of strange plane curve.

Definition 4.1. Let C be a curve contained in a smooth scroll S over a curve D; let $\pi : S \to D$ be the projection. C is said to be *ruling strange* if $\pi|D_{\mathrm{reg}}$ has everywhere differential 0.

Remark 4.2. (a) By the definition it is obvious that in characteristic 0 a ruling strange curve of the scrollar fibration $\pi : S \to D$ is contained in a fiber of π.

(b) If S is embedded in a projective space Π, by definition a ruling strange curve C is a curve such that there is a projective space A, $\dim(A) = \dim(S) - 1$, which is a fiber of the ruling of π and such that for every smooth point P of C the embedded tangent line T_PC is contained in A.

(c) Obviously in Def. 4.1 and part (b) of this Remark instead of "every smooth point of C" we may take "*a* general smooth point of C".

The notion of ruling strange curve is related to the work in [7, §2] on curves contained in a ruled surface.

The proof of the following key result will be just a modification of the proof of [24, Prop. 2].

Proposition 4.3. *Let D be an integral ruling strange curve on the scroll $S \subset \mathbf{P}^n$. Then D is Frobenius classical at the first step, i.e. with the terminology of [24] D has $v_1 = 1$.*

Proof. Assume that D is not Frobenius classical at the first step. Take $P \in D_{\mathrm{reg}}$ such that P is not defined over $GF(q^t)$ with $t = \deg(D) + 1$, so that its orbit under the Frobenius F_q of $GF(q)$ has at least $\deg(D) +$ $+ 1$ elements. Since D is ruling strange and not Frobenius classical at the first step this orbit is contained in the ruling linear space A of S containing D. Since $\mathrm{card}(A \cap D) > \deg(D)$, we have $D \subseteq A$ by Bézout theorem. ◊

Theorem 4.4. *Let $A \subset \mathbf{P}^m$ be a scroll of any dimension $< m$. Assume that A is set theoretically the intersection of hypersurfaces of degree $\leq t$. Assume $p > t$. Then every integral curve D not ruling strange is reflexive.*

Proof. Assume that D is not reflexive. Then for every smooth point P of D we have length $(T_P D \cap D) \geq p > t$. Hence by the assumption on A and Bézout theorem, we have $T_P D \subset A$. Taking P general, we see that D is ruling strange. ◊

Now we give here (see 4.5, 4.6 and 4.7) 3 corollaries of Th. 4.4. For other corollaries, see Section 5.

Corollary 4.5. *Let $A \subset \mathbf{P}^m$ be a linearly normal scroll of any dimension $< m$ and with genus 0. Assume $p > 2$. Then every integral curve $D \subset A$ not ruling strange is reflexive.*

Proof. It is well-known (see e.g. [15]) that A is set theoretically the intersection of quadric hypersurfaces (and the same reference prove much more). ◊

The next result is interesting because the scrollar examples of partial t-spreads may arise even from embeddings coming from non complete linear systems without base points. Note also that in several situations just a morphism, even not arising from a complete very ample linear system, from a curve to $G(t+1, n+1)$ induces a partial t-spread, say over $GF(q)$, i.e. sends the t-planes which are fibers of the dimension $t+1$ scroll into t-planes of $PG(n, q)$ and the images of two such t-planes, if different, are disjoint.

Corollary 4.6. *Let $B \subset \mathbf{P}^m$ be a variety which is the projection of a linearly normal smooth scroll U on a curve of genus 0 of $\mathbf{P}^{m+1}$. Assume $m \geq 3$ and $p > 3$. Then every integral curve $D \subset B$ not ruling strange is reflexive.*

Proof. Note that $\deg(B) \leq \deg(U) = m - \dim(B) + 2$. Hence by [25, Th. 2] B is the intersection of hypersurfaces of degree ≤ 3. ◊

Theorem 4.7. *Let $S \subset \mathbf{P}^n$ be a smooth scroll over a curve of genus $q >$ > 0. Assume that the homogeneous ideal of every smooth curve section of S is generated by forms of degree $\leq x$. Then every line T containing at least $x + 1$ points of S is contained in a linear space of the ruling of S.*

Proof. Since $b > 0$ every line (as any curve with a genus 0 curve as normalization) contained in S is contained in a fiber of the ruling of S. Now fix a line T intersecting S only at a finite number of points. By the assumption on x it is sufficient to check that for a general linear subspace W of A with $\text{codim}(W) = \dim(S) - 1$ and $T \subset W$, the curve section $C := W \cap S$ is smooth. By Bertini theorem C is smooth outside the finitely many points $T \cap S$. Fix $P \in T \cap S$. To find a curve section C smooth at $P \in (T \cap S)$ it is sufficient to take as W a linear space not containing $T_P S$. Since $\text{card}(T \cap S)$ is finite, we obtain the thesis. ◊

Remark 4.8. In Th. 4.7 it is sufficient to assume that C is set-theoretically cut out by forms of degree $\leq x$.

5. In this section we give a few cohomological criteria which give other cases to which Th. 4.4 may be applied. For a general discussion of the postulation and index of regularity (in the sense of [26, p. 100]) of embeddings of scrolls over curves, see [9].

Definition 5.1. Fix integers $n \geq 3$, $t \geq 2$, $r \geq 2$. Set $\Pi := \mathbf{P}^n$. A smooth scroll $F \subset \Pi$ over a curve with $\dim(F) = r$ is called of *cohomological level* $\leq t$ if $h^1 \boldsymbol{O}_F(t-2)) = h^1(\Pi, \boldsymbol{I}_{F,\Pi}(t-1)) = 0$.

In a first reading the reader may skip the following cohomological result.

Proposition 5.2. *The homogeneous ideal of a smooth scroll $F \subset \Pi$ of cohomological level $\leq t$ is generated by forms of degree $\leq t$.*

Proof. By Castelnuovo-Mumford regularity theorem (see e.g. the original source, [Mu, p. 100]) it is sufficient to prove that $H^i(\Pi, \boldsymbol{I}_F(t - i -$ $- 1)) = 0$ for every $i > 0$. By the exact sequence

$$0 \to \boldsymbol{I}_{F,\Pi}(j) \to \boldsymbol{O}_{\Pi}(j) \to \boldsymbol{O}_F(j) \to 0$$

and the vanishing of $H^i(\Pi, \boldsymbol{O}_\Pi(j))$ for every i, j with $1 \le i < \dim(\Pi)$, and of $H^{\dim(\Pi)}(\Pi, \boldsymbol{O}_\Pi(j))$ and $j \ge -\dim(\Pi)$, we have $H^{u+1}(\Pi, \boldsymbol{I}_F(j)) \cong$ $\cong H^u(F, \boldsymbol{O}_F(j))$ for every $u \ge 1$ and every j. By the Leray spectral sequence of π, Lemma 1.5 and the assumption "$t \ge 2$" (hence $t - - r > -r$), we have $H^m(F, \boldsymbol{O}_F(t-m-1))$ for every $m \ge 2$. Hence $H^i(\Pi, \boldsymbol{I}_F(t-i-1)) = 0$ for every $i \ge 3$. We have $H^2(\Pi, \boldsymbol{I}_F(t-3)) = = H^1(\Pi, \boldsymbol{I}_F(t-2)) = 0$ by the definition of cohomological level. ◊

Definition 5.3. Let $B \subset \mathbf{P}^N$ be an integral non degenerate variety. We will call the integer $\delta(C) := h^0(B, \boldsymbol{O}_B(1)) - N - 1$ the *defect* of linear normality of B.

The proof of Prop. 5.2 gives without any change the following useful result whose union with Th. 4.7 gives another application of Th. 4.4.

Proposition 5.4. *Let $B \subset \mathbf{P}^N$ be a smooth scroll over a curve D of genus $b \ge 0$. Set $r := \dim(B)$. Let C be a smooth curve linear section of B (hence $C \cong D$ as abstract curves). Then $\delta(B) \le \delta(C) + (r-1)b$.*

Remark 5.5. In [3, Th. 0.1] and [4, Th. 3.1] there are two criteria (respectively one for linearly normal embeddings and one much weaker for non linearly normal embeddings) on the integers (d, g) such that the homogeneous ideal of a general degree d embedding $C \subset \mathbf{P}^n$ of a general curve C of genus g is generated by quadrics.

Remark 5.6. Let $C \subset \mathbf{P}^m$ be a smooth non degenerate genus b linearly normal curve of degree d. By [19, Th. 3.3] the homogeneous ideal of C is generated by quadrics if $d \ge 2b + 2$. By Castelnuovo-Mumford regularity theorem ([26, p. 100]) the homogeneous ideal of C is generated by forms of degree ≤ 3 in all cases in which in [18] it is proved that C is projectively normal.

Theorem 5.7. *Let $C \subset \mathbf{P}^m$ be a smooth non degenerate curve of genus b. Let $\delta(C) := h^0(C, \boldsymbol{O}_C(1)) - m - 1$ be the defect of linear normality of C, i.e. assume that C is an isomorphic projection from a linearly normal curve $X \subset \mathbf{P}^{m+\delta(C)}$. Assume that $h^1(A, \boldsymbol{I}_{X,\mathbf{P}^{m+\delta(C)}}(w)) = = h^1(C, \boldsymbol{O}_C(w-1)) = 0$. Let $\Delta(C)$ be the first integer $t \ge w$ such that $(m+t)!(m!t!) > \deg(X) \cdot t + 1 - b$. Then $h^1(U, \boldsymbol{I}_{C,\mathbf{P}^m}(t)) = 0$ for every $t \ge \Delta(C)$ and the homogeneous ideal of C is generated by forms of degree $\le \Delta(C) + 1$.*

Proof. By Castelnuovo-Mumford regularity theorem (see [26, p. 100]) it is sufficient to prove that $h^1(U, \boldsymbol{I}_{C,A}(\Delta(C))) = 0$. Since $h^1(X, \boldsymbol{O}_X(w -$

$-1)) = h^1(C, \mathcal{O}_C(w-1)) = 0$, by Castelnuovo-Mumford regularity theorem we have $h^1(A, \mathcal{I}_{X,A}(t)) = 0$ for every $t \geq w$. Let $B \cong \mathbf{P}^{\delta(C)-1}$ be the linear subspace of A such that C is an isomorphic projection of X from B. Fix a general divisor D of X in the linear system corresponding to $H^0(X, \mathcal{O}_X(\Delta(C)))$. Since the projection $X \to C$ from B is an isomorphism it is sufficient to show that every such D is cut out by a hypersurface of A which is a cone with vertex containing B. The linear subspace M'' of $M := H^0(A, \mathcal{O}_A(\Delta(C)))$ formed by all sections vanishing on D has codimension $\deg(D)+1-b = \Delta(C)\deg(X)+1-b$. The linear subspace M' of M formed by the equations of the cones with vertex B has dimension $(m+\Delta(C))!/(m!\Delta(C)!)$. Hence $M' \cap M'' \neq \{0\}$, as wanted. $\diamondsuit$

To apply results on linearly normal embeddings of curves (as for instance the ones quoted in Remark 5.6) to non linearly normal embeddings of curves, we use the following Prop. 5.8; then we may use Prop. 5.4 and Th. 4.7 to obtain other applications of Th. 4.4.

Proposition 5.8. *Fix positive integers u, and w. Let $C \subset \mathbf{P}^m$ be a smooth curve of degree ≥ 2 which is an isomorphic projection of a curve $X \subset \mathbf{P}^{m+u}$ which is set theoretically cut out by forms of degree $\leq w$. Then there is no line $T \subset \mathbf{P}^m$ such that* $\text{card}(T \cap C) > w^{u+1}$.

Proof. Assume the existence of such line T. Then there is a linear space $U \subset \mathbf{P}^{m+u}$ mapped to T by the linear projection mapping X to C with $\dim(U) = u+1$ and with $\text{card}(U \cap X) > w^{u+1}$. Since $C \neq T$, U does not contain X. Since $X \cap U$ is the intersection of forms of degree $\leq w$, we have $\text{card}(U \cap X) \leq w^{u+1}$, contradiction. $\diamondsuit$

Remark 5.9. By [24, Prop. 1], every reflexive curve is Frobenius classical at the first step.

Remark 5.10. By Remark 5.9 and the applications of Th. 4.4 we have the Frobenius classicity at the first step of all non ruling strange curves in a huge number of cases.

Our last result shows that the positive characteristic monodromy results (as Th. 3.1) have applications to interesting geometric situations.

Theorem 5.11. *Let C, C' be integral non degenerate curves in $\mathbf{P}^{s+1}$; set $a := \deg(C)$, $b := \deg(C')$. Let $\Gamma := (C \cup C') \cap H$ be a general hyperplane section of $C \cup C'$. Assume that Γ is in 2-uniform position. Let G' (resp. G'') be the monodromy group of the generic hyperplane section of C (resp. C'). Assume $A_a \subseteq G'$ and $A_b \subseteq G''$. For any integer x define $m_0(x)$ and $\varepsilon_0(x)$ by the relatives $x - 1 = m_0(x)s + \varepsilon_0(x)$ and $0 \leq \varepsilon_0(x) \leq s-1$; set $\pi_0(x, s+1 := m_0(x)(m_0(x)-1)s/2 + m_0(x)\varepsilon_0(x)$.*

Set $m_1 := [(a+b-1)/(s+1)]$, $\varepsilon_1 := a+b-m_1(s+1)-1$, $\mu_1 := 1$ *if* $\varepsilon_1 = s$, $\mu_1 := 0$ *if* $\varepsilon_1 \neq s$ *and* $\pi_1(a+b, s+1) := m_1(m_1-1)(s+1)/2+$ $+m_1(\varepsilon_1+1)+\mu_1$. *Then:*

(a) *If* $a+b \geq 2s+5$ *and* $h_\Gamma(2) = 2s+2$, *then* Γ *lies on an elliptic normal curve* $E \subset H$ *of degree* $s+1$ *cut out by all quadrics containing* Γ.

(b) *In our positive characteristic situation if* $a+b \geq 2s+5$ *then* Th. 3.15 *and* Cors. 3.16, 317, 318 *of* [22] *are true; if* $p_a(C \cup C' > \pi_1(a+$ $+b, s+1)$ *then* $C \cup C'$ *lies on a surface of degree* s*; furthermore the set of pairs of integers* $(a+b, p_a(C \cup C'))$ *arising in this way is completely described and for any such pair the corresponding Hilbert scheme is described (e.g. the number of its irreducible components) with the only further assumption that* $s \geq 9$ *if* $p_a(C \cup C') = \pi_1(a+b, s+1)$*;*

(c) *In our positive characteristic situation* Th. 2.3, Th. 2.5 *and* Cors. 2.6, 2.7, 2.8 *of* [12] *are true; in particular if* $h_\Gamma(2) \geq 2s+1+\delta$ *for some integer* δ *with* $0 < \delta \leq a+b-2s-1$, *then* $p_a(C \cup C') \leq \pi_0(a+$ $+b-\delta, s+1)$.

To check Th. 5.11 we need just to show that the proofs of part (a) (i.e. the extension to our positive characteristic situation of [22, Prop. 3.2]) and of the references listed in parts (b) and (c) work under our assumption with no change. For part (a) part of its proof in [22] (e.g. [22, Lemma 3.21]) does not consider Γ and are characteristic free, while the long analysis on pages 109–115 of [22] uses only 2-uniformity, the assumption on G (with the observation that at page 104 the non existence of the reducible quadrics follows again from the fact that C and C' are integral and non degenerate). To check part (b) it is sufficient to have part (a) and just the case $m = 2$ of the [22, 3.19]; this case uses only the key convexity result for $m = 2$ and the 2-uniformity (or the assumption on G). To check part (c), just note that [12, Th. 2.3] needs only the key convexity result for $m = 2$ (hence our Th. 3.7 may be applied) listed there as eq. 2.4, that [Ci, Th. 2.5] is true for the same reason and that everything else follows (using the key convexity result only for $m = 2$, 2-uniformity and the assumption on G) immediately using the statement of Th. 2.5 in [12].

6. Here we discuss the connections between our results (and in particular the one in Section 4 and Section 5) and the Uniqueness part of the motivating problem of our research. Of course, if a partial t-spread does not satisfy some of these properties, we have proved that it is not in our class. Fix a prime p and a power q of p; let $\mathbf{K}$ be the algebraic closure of $GF(q)$. Fix integers n, t and x with $1 \leq t \leq$ $\leq n$ and $x \geq 0$. Fix a linear subspace A of $PG(n,q)$ with $\dim(A) =$ $= t$ and see its $(q^{t+1}-1)/(q-1)$ points as $GF(q)$-points of $\mathbf{P}^n(\mathbf{K})$. A very important cohomological invariant of A in $\mathbf{P}^n(\mathbf{K})$ is given by the integer $a(n,t;q;x) := (n+x)!/n!x!) - h^0(\mathbf{P}^n, \boldsymbol{I}_A(x))$, i.e. by the number of conditions that A imposes to the degree x hypersurfaces containing it (or, better, the entire family $\{a(n,t;q;x)\}_{x\in\mathbf{N}}$). Fix a set $S \subseteq PG(n,q)$ which is the union of the t-planes of a partial t-spread arising from an embedding of a scroll over a smooth curve C of genus $g \geq 2$. Assume that this scroll is set-theoretically (over $\mathbf{K}$) the intersection of hypersurfaces of degree $\leq x$; we gave in Section 5 a few ways to find upper bounds for x for the scrolls over a curve. Suppose you know the existence of a suitable $B \subseteq (S \cap A)$; set $u := \mathrm{card}(B)$. Is A forced to be contained in S? And if $A \subseteq S$, is A one of the t-planes of our partial t-spread? Since B is contained in A, it imposes at most $a(n,t;q;x)$ conditions to hypersurfaces of degree x. If it imposes $a(n,t;q;x)$ conditions, then every degree x hypersurface containing B contains A. Hence $A \subseteq S$. Now just start with any $A \subseteq S$; let A' be the t-dimensional subspace (over $\mathbf{K}$) of $\mathbf{P}^n(\mathbf{K})$ generated by A. Note that for every power q' of p with $q' \geq q$, the embedding over $GF(q)$ of our scroll induces a unique embedding of it over $GF(q')$; call $S(q') \subseteq$ $\subseteq PG(n,q')$ the image of its $GF(q')$-points. We assume that our linear space A has a "geometric origin" as our scrollar fibration, i.e. for every q' we assume the existence of a t-plane $A(q')$ of $PG(n,q')$ such that if $q' \leq q''$, then $A(q')$ is the set of $GF(q')$-points of $A(q'')$. If there is such a geometric family, then $A' \subseteq S'$ and A is one of our t-planes if and only if A' is a fiber of the scrollar fibration $\pi : S' \to C$. If $t \geq$ ≥ 2 every such A' is contained in a fiber of π because every morphism (over $\mathbf{K}$) of $\mathbf{P}^t$, $t \geq 2$, into a lower dimensional variety is constant. Hence we may assume $t = 1$. If (as we will assume) A' in not a fiber, then $\pi(A') = C$. Thus $g = 0$, i.e. the abstract scroll is one of the Segre-Hirzebruch surfaces described in 1.2. Since it has an embedding containing a line which is not a fiber, the discussion in 1.2 shows that for some $N \geq n$ S' is an isomorphic projection of $S(N-1,1)$ into $\mathbf{P}^n$. Hence $e = N - 2 \geq n - 2$. We know only one situation in which the same assertions can be made without assuming that A has "geometric origin". We need to show that A' is contained in S'. This is true if the intersection of every t-plane of $\mathbf{P}^n(\mathbf{K})$ not contained in a fiber of π has less than $(q^{t+1}-1)/(q-1)$ points. For instance if $t=1$ it is sufficient to know that S' is set-theoretically (over $\mathbf{K}$) cut out by hypersurfaces of degree $< (q^{t+1}-1)/(q-1)$. This type of cohomological information is exactly the content of Section 5.

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