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NOTE ON CANONICAL FORMS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract: We introduce a new type of canonical forms for certain classes of linear differential equations with the delayed argument. These special forms may serve for investigations of asymptotic behaviour of solutions of the studied equations and enable us to generalize some asymptotic results concerning these equations.

1. Introduction

In this paper we are going to discuss some aspects of the transformation theory of functional differential equations. The origin of this theory goes back to paper [7], where T. Kato and J. B. Mcleod used the logarithmic substitution to convert equation

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$$y'(x) = p\,y(x) + q\,y(\lambda x), \quad x \in [0, \infty)$$

into an equation with a constant delay. This idea has been generalized by M. L. Heard [6] and F. Neuman [8], [9]. These authors introduced a change of the independent variable $t = \varphi(x)$ to convert equation

$$y'(x) = p(x)y(x) + q(x)y(\tau(x)), \quad x \in I = [x_0, \infty) \tag{1.1}$$

into an equation with a constant deviation. This problem leads to finding a solution $\varphi(x) \in C^1(I)$ of Abel equation

$$\varphi(\tau(x)) = \varphi(x) - 1, \quad x \in I \tag{1.2}$$

such that $\varphi'(x) > 0$ for every $x \in I$. Moreover, a change of the dependent variable $z(x) = \exp\left\{ -\int\limits_{x_0}^{x} p(s)\,\mathrm{d}s \right\} y(x)$ introduced in [9] enables us to transform equation (1.1) into an equation with the vanishing coefficient of $z(x)$. Summarizing these results we can recall the following statement.

Theorem 1. *Let $p(x)$, $q(x) \in C^0(I)$, $\tau(x) \in C^1(I)$, $\tau(x) < x$ and $\tau'(x) > 0$ for every $x \in I$. Then equation* (1.1) *can be transformed into*

$$\dot{z}(t) = r(t)z(t-1), \quad t \in J. \tag{1.3}$$

Proof. The proof was given in [9]. ◊

Equations of type (1.3) have been much studied and therefore may serve as canonical forms for a wide class of equations (1.1). Consequently, this transformation approach enables us to generalize some results concerning qualitative properties of solutions of (1.1).

Our aim is to propose alternative canonical forms for a special class of equations (1.1). We show that these forms can be used with the success especially in investigations of asymptotic properties of some equations (1.1). Particularly, we derive asymptotic bounds of solutions of certain equations (1.1) with an unbounded delay, i.e., with an unbounded function $r(x) = x - \tau(x)$.

Throughout this paper we shall assume that $\tau(x) \in C^1(I)$ is an increasing function such that $\tau(x) < x$ for every $x \in I$ (i.e., we consider equations (1.1) with a delayed argument). By the symbol τ^n $(n \in \mathbb{Z})$ we understand the n-th iterate of τ (for $n > 0$) or the -n-th iterate of the inverse function τ^{-1} (for $n < 0$) and put $\tau^0 =$id. Finally, we denote $I_{-1} = [\tau(x_0), \infty)$.

2. A change of variables

We start off with the study of Abel equation (1.2).

Proposition 1. *Let $\tau(x) \in C^r(I)$ and $\tau'(x) > 0$ for every $x \in I$. Then for any initial function $\varphi_0(x) \in C^r([\tau(x_0), x_0])$ such that $\varphi_0'(x) > 0$ for every $x \in [\tau(x_0), x_0]$ and*

$$\varphi_0^{(s)}(\tau(x_0)) = [\varphi_0(x_0) - 1]^{(s)}, \quad s = 0, 1, \ldots, r$$

there exists a unique solution $\varphi(x) \in C^r(I_{-1})$ of (1.2) such that $\varphi'(x) > 0$ for every $x \in I_{-1}$ and

$$\varphi(x) = \varphi_0(x)$$

for every $x \in [\tau(x_0), x_0]$. This solution is given by the formula

$$\begin{aligned} &\varphi(x) = \varphi_0(\tau^n(x)) + n, \\ &\tau^{-n+1}(x_0) \le x \le \tau^{-n}(x_0), \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{2.1}$$

Proof. The existence of such a solution $\varphi(x)$ can be proved by steps (cf. [8], Th. 1). ◊

Remark. Assuming that $\tau(x) \in C^1(I)$ is an increasing function such that $\tau(x) < x$ for every $x \in I$ we can similarly show the existence of an increasing solution $\varphi(x) \in C^1(I_{-1})$ of (1.2).

Proposition 2. *In addition to assumptions of* Prop. 1 *we suppose that $\tau'(x) \le \lambda < 1$ for every $x \in I$. Then a solution $\varphi(x) \in C^1(I_{-1})$ of equation* (1.2) *given by* (2.1) *satisfies the relation $\varphi' \circ \varphi^{-1}(t) = O(\lambda^t)$ as $t \to \infty$.*

Proof. Let $\varphi(x)$ be a solution of (1.2) given by (2.1) and put $\psi(x) = \lambda^{-\varphi(x)}$. Then $\psi(x)$ defines a solution of the equation

$$\psi(\tau(x)) = \lambda\psi(x), \quad x \in I. \tag{2.2}$$

Differentiating (2.2) we obtain

$$\psi'(x) = \frac{\tau'(x)}{\lambda}\psi'(\tau(x)), \quad x \in I,$$

hence $\psi'(x)$ is bounded. Then we can easily deduce that

$$\varphi' \circ \varphi^{-1}(t) = \frac{\lambda^t}{\ln \lambda^{-1}}\psi'(\varphi^{-1}(t)) = O(\lambda^t) \quad \text{as } t \to \infty. \qquad \Diamond$$

Now let $\varphi(x) \in C^1(I_{-1})$ be an increasing solution of (1.2) and denote $h = \varphi^{-1}$ on $J_{-1} = \varphi(I_{-1})$. Hence, $h(t) \in C^1(J_{-1})$ is an increasing solution of the functional equation

$$h(t-1)=\tau(h(t)),\quad t\in J=\varphi(I).$$

Then we consider the difference equation

$$\alpha(t)=\alpha(t-1)+\ln\frac{q(h(t))}{-p(h(t))},\quad t\in J. \tag{2.3}$$

For

$$\gamma(t)=\frac{\mathrm{d}}{\mathrm{d}t}\left[\ln\frac{q(h(t))}{-p(h(t))}\right], \tag{2.4}$$

where the symbol ln denotes the principal branch of the corresponding logarithm, it holds

Proposition 3. *Let $p(x)$, $q(x)$, $\tau(x)\in C^1(I)$, $p(x)$, $q(x)\neq 0$ for every $x\in I$, $\varphi(x)\in C^1(I_{-1})$ be an increasing solution of (1.2) and let $h=\varphi^{-1}$ on $J_{-1}=\varphi(I_{-1})$. Assume that $|\gamma(t)|$ is nonincreasing on J and $\int_{\varphi(x_0)}^{\infty}|\gamma(s)|\,\mathrm{d}s$ converges. Then there exists a (possibly complex valued) solution $\alpha(t)\in C^1(J_{-1})$ of (2.3) such that $\dot\alpha(t)$ is bounded.*

Proof. The existence of a solution $\alpha(t)\in C^1(J_{-1})$ of (2.3) can be easily proved by steps. We show that $\dot\alpha(t)$ is bounded. Function $\dot\alpha(t)$ is a solution of the equation

$$\dot\alpha(t)=\dot\alpha(t-1)+\gamma(t),\quad t\in J.$$

To prove the boundedness of $\dot\alpha(t)$ we denote $t_0=\varphi(x_0)$, $t_{-1}=t_0-1$, $t_i=t_0+i$, $L_i=[t_{i-1},t_i]$ and $M_i=\sup\{|\dot\alpha(t)|: t\in L_i\}$, $i=0,1,\dots$. Choose any $t\in L_{i+1}$. Then

$$\dot\alpha(t)\le M_i+|\gamma(t_i)|,$$

i.e.,

$$M_{i+1}\le M_i+|\gamma(t_i)|,\quad i=0,1,\dots.$$

Repeating this we can deduce that

$$M_{i+1}\le M_0+\sum_{j=0}^{i}|\gamma(t_j)|,\quad i=0,1,\dots.$$

Letting $i\to\infty$ we can easily verify that the infinite series $\sum_{j=0}^{\infty}|\gamma(t_j)|$ converges by the use of Cauchy integral criterion. This proves the boundedness of $(M_i)_{i=1}^{\infty}$ as $i\to\infty$. ◊

Prop. 1, Prop. 2 and Prop. 3 yield

Theorem 2. *Let $p(x)$, $q(x)$, $\tau(x)\in C^1(I)$, $p(x)$, $q(x)\neq 0$, $0<\tau'(x)\le\lambda<1$ for every $x\in I$. Further, let $\varphi(x)\in C^1(I_{-1})$, $\varphi'(x)>0$ on*

I_{-1}, be a solution of (1.2) given by (2.1) and let $h = \varphi^{-1}$ on $J_{-1} = \varphi(I_{-1})$. Assume that $|\gamma(t)|$ is nonincreasing on J and $\int\limits_{\varphi(x_0)}^{\infty} |\gamma(s)|\,\mathrm{d}s$ converges, where $\gamma(t)$ is given by (2.4). Then equation (1.1) can be transformed into an equation

$$(2.5)\quad v(t)\dot{z}(t) = [p(h(t)) - w(t)]z(t) - p(h(t))z(t-1), \quad t \in J = \varphi(I),$$

where $v(t)$, $w(t) = O(\lambda^t)$ as $t \to \infty$.

Proof. Let $\alpha(t) \in C^1(J_{-1})$ be a solution of (2.3) with a bounded derivative on J_{-1} (see Prop. 3). We introduce a change of variables $z(t) = \exp\{-\alpha(t)\}y(h(t))$. It is easy to verify that equation (1.1) becomes

$$\dot{z}(t) = [p(h(t))\dot{h}(t) - \dot{\alpha}(t)]z(t) - p(h(t))\dot{h}(t)z(t-1), \quad t \in J.$$

Put $v(t) = \frac{1}{\dot{h}(t)}$ and $w(t) = \frac{\dot{\alpha}(t)}{\dot{h}(t)}$. Then the asymptotic relations $v(t)$, $w(t) = O(\lambda^t)$ as $t \to \infty$ follow from Prop. 2 with the respect to $\frac{1}{\dot{h}(t)} = \varphi' \circ \varphi^{-1}(t)$. ◊

Remark. Due to the relation $w(t) = O(\lambda^t)$ as $t \to \infty$, $0 < \lambda < 1$, we expect that the behaviour at infinity of solutions of equation (2.5) may be close to the behaviour of solutions of the equation

$$(2.6)\qquad \dot{z}(t) = \beta(t)[z(t) - z(t-1)], \quad t \in J$$

Equation (2.6) has been subject of numerous investigations (for results, methods and references see, e.g., F. V. Atkinson and J. R. Haddock [1], J. Diblík [4], [5], S. N. Zhang [10] and others). Consequently, we can extend some results and proof techniques concerning equation (2.6) also to equation (2.5) and thus obtain new asymptotic results for equation (1.1).

3. Applications

To demonstrate this transformation approach we consider equation (1.1) with $\frac{q(x)}{-p(x)} = K \neq 0$ for every $x \in I$. Then the difference equation (2.3) has the simple form, namely

$$\alpha(t) = \alpha(t-1) + \ln K, \quad t \in J$$

and admits the function $\alpha(t) = (\ln K)t$ as the required solution.

First we state

Proposition 4. *Let $\tau(x) \in C^2(I)$ be such that $\tau''(x) \leq 0$ for every*

$x \in I$. Then for any initial function $\varphi_0(x) \in C^2([\tau(x_0), x_0])$ such that $\varphi_0''(x) \leq 0$ for every $x \in [\tau(x_0), x_0]$ and

$$\varphi_0^{(s)}(\tau(x_0)) = [\varphi_0(x_0) - 1]^{(s)}, \quad s = 0, 1, 2$$

there exists a unique solution $\varphi(x) \in C^2(I_{-1})$ of (1.2) such that $\varphi''(x) \leq$ ≤ 0 for every $x \in I_{-1}$ and

$$\varphi(x) = \varphi_0(x)$$

for every $x \in [\tau(x_0), x_0]$. This solution is given by (2.1).

Proof. The proof is obvious. ◊

Then we have

Theorem 3. *Consider equation (1.1), where $p(x)$, $q(x) \in C^1(I)$, $\tau(x) \in C^2(I)$, $p(x) < 0$ and $p(x)$ is nonincreasing on I, $q(x) \neq 0$, $\frac{q(x)}{-p(x)} = K \neq 0$ and $0 < \tau'(x) \leq \lambda < 1$, $\tau''(x) \leq 0$ for every $x \in I$. If $\varphi(x) \in C^2(I)$, $\varphi'(x) > 0$, $\varphi''(x) \leq 0$ on I, is a solution of (1.2) given by (2.1), then*

$$y(x) = O\left(\exp\{(\ln|K|)\varphi(x)\}\right) \quad \text{as } x \to \infty, \tag{3.1}$$

for every solution $y(x)$ of (1.1).

Proof. We put $\alpha^* := \ln K$, $\alpha_r^* := \operatorname{Re} \alpha^* = \ln|K|$ and let $h =$ $= \varphi^{-1}$ on $J_{-1} = \varphi(I_{-1})$. We introduce the transformation $z(t) =$ $= \exp\{-\alpha^* t\} y(h(t))$ in (1.1) to obtain equation (2.5), where $v(t) = \frac{1}{\dot{h}(t)}$ and $w(t) = \frac{\alpha^*}{\dot{h}(t)}$. This equation can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[z(t)\exp\left\{\alpha^* t - \int\limits_{x_0}^{h(t)} p(u)\,\mathrm{d}u\right\}\right] =$$

$$= -p(h(t))\dot{h}(t)\exp\left\{\alpha^* t - \int\limits_{x_0}^{h(t)} p(u)\,\mathrm{d}u\right\} z(t-1).$$

Let $t_0 \geq \varphi(x_0)$ be such that ${\alpha^*}_r - p(h(t))\dot{h}(t) > 0$ for all $t \geq t_0$. Put $t_{-1} = t_0 - 1$, $t_i = t_0 + i$, $L_i = [t_{i-1}, t_i]$ and $M_i = \sup\{|z(t)| : t \in L_i\}$, $i = 0, 1, \ldots$. Let $t \in L_{i+1}$. Then integrating the last relation over $[t_i, t]$ we obtain

$$z(t)=\exp\left\{\alpha^*(t_i-t)+\int_{h(t_i)}^{h(t)}p(u)\,\mathrm{d}u\right\}z(t_i)+\exp\left\{\int_{x_0}^{h(t)}p(u)\,\mathrm{d}u-\alpha^*t\right\}\times$$

$$\times\int_{t_i}^{t}\left(-p(h(s))\dot{h}(s)\exp\left\{\alpha^*s-\int_{x_0}^{h(s)}p(u)\,\mathrm{d}u\right\}z(s-1)\right)\mathrm{d}s.$$

Now $z(t)$ can be estimated as

(3.2)

$$|z(t)|\le M_i\exp\left\{\alpha_r^*(t_i-t)+\int_{h(t_i)}^{h(t)}p(u)\,\mathrm{d}u\right\}+$$

$$+M_i\exp\left\{\int_{x_0}^{h(t)}p(u)\,\mathrm{d}u-\alpha_r^*t\right\}\times$$

$$\times\int_{t_i}^{t}\left(-p(h(s))\dot{h}(s)\exp\left\{\alpha_r^*s-\int_{x_0}^{h(s)}p(u)\,\mathrm{d}u\right\}\right)\mathrm{d}s.$$

Further, we have

$$\int_{t_i}^{t}\left(-p(h(s))\dot{h}(s)\exp\left\{\alpha_r^*s-\int_{x_0}^{h(s)}p(u)\,\mathrm{d}u\right\}\right)\mathrm{d}s\le$$

$$\le\left[\exp\left\{\alpha_r^*s-\int_{x_0}^{h(s)}p(u)\,\mathrm{d}u\right\}\right]_{t_i}^{t}+$$

$$+\bar{\alpha_r^*}\int_{t_i}^{t}\left(\exp\left\{\alpha_r^*s-\int_{x_0}^{h(s)}p(u)\,\mathrm{d}u\right\}\right)\mathrm{d}s,$$

where $\bar{\alpha_r^*}=\max(-\alpha_r^*,0)$. The integration by parts enables us to estimate the last term as

$$\bar{\alpha_r^*}\int\limits_{t_i}^{t}\left(\exp\left\{\alpha_r^* s-\int\limits_{x_0}^{h(s)}p(u)\,\mathrm{d}u\right\}\right)\,\mathrm{d}s\leq$$

$$\leq\left[\exp\left\{\alpha_r^* s-\int\limits_{x_0}^{h(s)}p(u)\,\mathrm{d}u\right\}\frac{\bar{\alpha_r^*}}{\alpha_r^*-p(h(s))\dot{h}(s)}\right]_{t_i}^{t}-$$

$$-\exp\left\{\alpha_r^* t-\int\limits_{x_0}^{h(t)}p(u)\,\mathrm{d}u\right\}\left[\frac{\bar{\alpha_r^*}}{\alpha_r^*-p(h(s))\dot{h}(s)}\right]_{t_i}^{t}=$$

$$=\left[\exp\left\{\alpha_r^* s-\int\limits_{x_0}^{h(s)}p(u)\,\mathrm{d}u\right\}\right]_{t_i}^{t}\frac{\bar{\alpha_r^*}}{\alpha_r^*-p(h(t_i))\dot{h}(t_i)}$$

by use of Prop. 4. Substituting this back into (3.2) we get

$$|z(t)|\leq M_i\left(1+\frac{\bar{\alpha_r^*}}{\alpha_r^*-p(h(t_i))\dot{h}(t_i)}\right),\quad t\in L_{i+1}.$$

Since $t\in L_{i+1}$ was arbitrary, we have

$$M_{i+1}\leq M_i\left(1+\frac{\bar{\alpha_r^*}}{\alpha_r^*-p(h(t_i))\dot{h}(t_i)}\right)\leq$$

$$\leq M_0\prod_{j=0}^{i}\left(1+\frac{\bar{\alpha_r^*}}{\alpha_r^*-p(h(t_j))\dot{h}(t_j)}\right),\quad i=0,1,\dots.$$

Using Prop. 2 and the assumptions $p(x)<0$ and $p(x)$ is nonincreasing on I we get that this product converges as $i\to\infty$, hence $z(t)$ is bounded. The statement is proved. ◊

Estimate (3.1) yields the upper bound of all solutions of equation (1.1). The result of the following statement can be viewed as the lower bound of all nontrivial solutions of (1.1).

Theorem 4. *Consider equation* (1.1), *where* $p(x)$, $q(x)$, $\tau(x)\in C^\infty(I)$, $p(x)$, $q(x)\neq 0$, $\frac{q(x)}{-p(x)}=K\neq 0$ *and* $\tau'(x)>0$ *for all* $x\in I$. *Let* $\varphi(x)\in$ $\in C^\infty(I_{-1})$, $\varphi'(x)>0$ *on* I_{-1}, *be a solution of* (1.2) *given by* (2.1), $h=\varphi^{-1}$ *on* $J_{-1}=\varphi(I_{-1})$ *and assume that there exist constants* $L>0$, $\rho>1$ *such that*

$$\left|\left(\frac{1}{-p(h(t))\dot{h}(t)}\right)^{(m)}\right| \le \frac{L^{m+1}m^m}{t^{m+\rho}}, \quad t \ge t_0 = \varphi(x_0), \quad m = 0, 1, \ldots.$$

Then no solution $y(x)$ of (1.1) *except the trivial one satisfies*

$$y(x) = o\left(\exp\{(\ln|K|)\varphi(x)\}\right) \quad \textit{as } x \to \infty.$$

Proof. Substituting $z(t) = \exp\{-\alpha^* t\} y(h(t))$, where $\alpha^* = \ln K$, in (1.1) we get equation (2.5) with $v(t) = \frac{1}{\dot{h}(t)}$ and $w(t) = \frac{\alpha^*}{\dot{h}(t)}$. Put

$$r(t) = \frac{v(t)}{-p(h(t))} = \frac{1}{-p(h(t))\dot{h}(t)}$$

and $s(t) = 1 - \alpha^* r(t)$. Then equation (2.5) becomes

$$r(t)\dot{z}(t) = -s(t)z(t) + z(t-1), \quad t \ge \varphi(x_0). \tag{3.3}$$

This equation was deeply discussed by N. G. de Bruijn in [2]. The author proved, among others, that equation (3.3) has no nontrivial solution $z(t)$ tending to zero provided

$$|r^{(m)}(t)| < \frac{L^{m+1}m^m}{t^{m+\rho}}, \quad |(s(t)-1)^{(m)}| < \frac{L^{m+1}m^m}{t^{m+\rho}}$$

for suitable constants $L > 0$, $\rho > 1$ and all $t \ge t_0 = \varphi(x_0)$, $m = 0, 1, \ldots$. Our statement follows immediately from this result. ◊

Remark. Asymptotic relations $v(t)$, $w(t) = O(\lambda^t)$ as $t \to \infty$, $0 < < \lambda < 1$, imposed on coefficients of canonical equation (2.5) can be often weakened to $v(t)$, $w(t) = O(g(t))$ as $t \to \infty$, where $\int^{\infty} |g(s)|\, ds$ converges. Thus we can extend class of equations (1.1) that can be converted into its canonical form (2.5) (for a similar situation see [6] and [3]).

Example 1. Consider the equation

$$y'(x) = p\,x\,y(x) + q\,x\,y(\lambda x), \quad x \in [1, \infty), \tag{3.4}$$

where $p < 0$, $q \ne 0$, $0 < \lambda < 1$ are real constants. Abel equation (1.2) becomes

$$\varphi(\lambda x) = \varphi(x) - 1, \quad x \in [1, \infty),$$

and admits $\varphi(x) = \frac{\ln x}{\ln \lambda^{-1}}$ as the solution with the required properties.

Equation (2.3) has the form

$$\alpha(t) = \alpha(t-1) + \ln \frac{q\,\lambda^{-t}}{-p\,\lambda^{-t}}$$

with the solution $\alpha(t) = \left(\ln \frac{q}{-p}\right) t$. Th. 3 and Th. 4 then imply that

$$y(x) = O\left(x^{\omega}\right), \quad \omega = \frac{\ln \frac{|q|}{-p}}{\ln \lambda^{-1}} \quad \text{as } x \to \infty$$

for every solution $y(x)$ of (3.4) and, moreover, only the trivial solution $y(x)$ of (3.4) satisfies

$$y(x) = o\left(x^{\omega}\right), \quad \omega = \frac{\ln \frac{|q|}{-p}}{\ln \lambda^{-1}} \quad \text{as } x \to \infty.$$

Example 2. We investigate the asymptotic behaviour of the equation

$$y'(x) = p\,y(x) + q\,y(x^{\gamma}), \quad x \in [2, \infty), \tag{3.5}$$

where $p < 0$, $q \neq 0$, $0 < \gamma < 1$ are real constants. Substituting $\tau(x) = x^{\gamma}$ into (1.2) we obtain Abel equation in the form

$$\varphi(x^{\gamma}) = \varphi(x) - 1.$$

This equation has the function $\varphi(x) = \frac{\ln \ln x}{\ln \gamma^{-1}}$ as the required solution. Further, equation (2.3) obviously admits the same solution as in Example 1, namely $\alpha(t) = \left(\ln \frac{q}{-p}\right) t$. Then we can deduce from our asymptotic results that

$$y(x) = O\left((\ln x)^{\omega}\right), \quad \omega = \frac{\ln \frac{|q|}{-p}}{\ln \gamma^{-1}} \quad \text{as } x \to \infty$$

for every solution $y(x)$ of (3.5). Moreover, no solution $y(x)$ of (3.5) except the trivial one satisfies

$$y(x) = o\left((\ln x)^{\omega}\right), \quad \omega = \frac{\ln \frac{|q|}{-p}}{\ln \gamma^{-1}} \quad \text{as } x \to \infty.$$

We note that these results coincide with the asymptotic formula derived in [6] for certain equations (1.1) with constant coefficients.

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