Mathematica Pannonica
11/1 (2000), 17 – 28

A NEW CLASS OF QUASI-UNIFORM SPACES

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Dedicated to Professor Gino Tironi on his 60th birthday

Received: October 1999

MSC 1991: 54 E 15, 54 E 50, 54 D 35

Keywords: Fitting quasi-uniform space, quiet, bicomplete, bicompletion, totally bounded, uniform space, uniformly R_0, half-complete, (strongly) D-complete.

Abstract: We introduce and study the notion of a fitting quasi-uniform space. Every quiet quasi-uniform space is fitting. We show that any bicompletion of a fitting quasi-uniform space is fitting and deduce that every fitting totally bounded quasi-uniformity is a uniformity. We also characterize those T_1 quasi-uniform spaces whose bicompletion is T_1. Finally, we discuss other kinds of completeness on fitting quasi-uniform spaces.

1. Introduction and preliminaries

In [9] Doitchinov introduced the notion of a quiet quasi-uniform space and obtained a consistent theory of completion for these spaces (see also [10]). By using Doitchinov's completion, Fletcher and Hunsaker proved in [11] the interesting result that every quiet totally

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The author acknowledges the support of the DGES, grant PB98-0564.

bounded quasi-uniform space is a uniform space (see [15] for an alternative proof). In [7] Deák generalized quietness in several directions and extended in this way parts of the theory of quiet quasi-uniform spaces, although at cost of losing the good property, cited above, that every quiet totally bounded quasi-uniformity is a uniformity (see [17] Section 6).

In this paper we introduce and study the notion of a fitting quasi-uniform space. Every quiet quasi-uniform space is fitting and each nonregular quasi-metric space (X,d) such that the supremum metric $d \vee d^{-1}$ is the discrete metric on X provides an example of a fitting quasi-uniform space which is not quiet. We show that fittingness is preserved by bicompletion of each quasi-uniform space and deduce that every fitting totally bounded quasi-uniform space is a uniform space. Finally, other completeness properties on fitting quasi-uniform spaces are discussed. In many cases, these properties are obtained in the more general context of uniformly weakly regular quasi-uniform spaces, concept introduced in [1], which permits us to generalize and extend several known results.

The letter $\mathbb{N}$ will denote the set of positive integers and $\mathbb{R}$ the set of real numbers.

Our basic references for quasi-uniform and quasi-metric spaces are [13] and [16].

We recall some pertinent concepts.

If $\mathcal{U}$ is a *quasi-uniformity* on a set X, then $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$ is also a quasi-uniformity on X called the conjugate of $\mathcal{U}$, and

$$\mathcal{T}(U) = \{A \subseteq X : \text{ for each } x \in A \text{ there is } U \in \mathcal{U} \text{ such that } U(x) \subseteq A\}$$

is the topology generated by $\mathcal{U}$, where as usual $U(x) = \{y \in X : : (x,y) \in U\}$ for all $U \in \mathcal{U}$. The coarsest uniformity on X finer than the quasi-uniformity $\mathcal{U}$ will be denoted by $\mathcal{U}^s$, i.e. $\mathcal{U}^s = \mathcal{U} \vee U^{-1}$. If $U \in \mathcal{U}$, the element $U \cap U^{-1}$ of $\mathcal{U}^s$ will be denoted by U^s.

A quasi-uniform space $(X,\mathcal{U})$ is called *bicomplete* if $(X,\mathcal{U}^s)$ is a complete uniform space. In this case we say that $\mathcal{U}$ is a bicomplete quasi-uniformity.

A *bicompletion* of a quasi-uniform space $(X,\mathcal{U})$ is a bicomplete quasi-uniform space $(Y,\mathcal{V})$ such that $(X,\mathcal{U})$ is quasi-unimorphic to a $\mathcal{T}(\mathcal{V}^s)$-dense subset of Y. It was proved in [3] and in [24] (see also [13]) that every quasi-uniform space admits a bicompletion. Moreover $(Y,\mathcal{V}^{-1})$ is a bicompletion of $(X,\mathcal{U}^{-1})$. In addition, if $(X,\mathcal{U})$

is a T_0 quasi-uniform space, then $(X,\mathcal{U})$ has a unique (up to quasi-isomorphism) T_0 bicompletion $(\widetilde{X},\widetilde{\mathcal{U}})$. In this case $(\widetilde{X},\widetilde{\mathcal{U}})$ is called the bicompletion of (the T_0 quasi-uniform space) $(X,\mathcal{U})$.

A *quasi-pseudometric* on a set X is a nonnegative real-valued function d on $X \times X$ such that for all $x, y, z \in X : d(x,x) = 0$ and $d(x,y) \leq d(x,z) + d(z,y)$.

A *quasi-metric* on X is a quasi-pseudometric d on X such that $d(x,y) = 0$ if and only if $x = y$.

Each quasi-pseudometric d on X generates a topology $\mathcal{T}(d)$ on X, where the basic open sets of $\mathcal{T}(d)$ are the d-balls $B_d(x,r) = \{y \in X : : d(x,y) < r\}$ for all $x \in X$ and $r > 0$. Note that if d is a quasi-metric, then $\mathcal{T}(d)$ is a T_1 topology.

Each quasi-(pseudo)metric d on X induces a (pseudo)metric d^s on X defined by $d^s(x,y) = \max\{d(x,y), d^{-1}(x,y)\}$ for all $x, y \in X$, where d^{-1} is the conjugate quasi-(pseudo)metric of $d : d^{-1}(x,y) = d(y,x)$ for all $x, y \in X$.

A quasi-pseudometric d on X induces a quasi-uniformity $\mathcal{U}_d$ on X with basic entourages of the form $\{(x,y) : d(x,y) < 2^{-n}\}$, $n \in \mathbb{N}$.

2. Bicompletion of fitting quasi-uniform spaces

In order to obtain a consistent theory of quasi-uniform completion based on a notion of completeness which provided $\mathcal{T}(U)$-convergence of Cauchy filters, Doitchinov introduced the concept of a quiet quasi-uniform space ([9], [10]).

A filter $\mathcal{F}$ on a quasi-uniform space $(X,\mathcal{U})$ is said to be *D-Cauchy* ([9], [12]) if there is a so-called co-filter $\mathcal{G}$ of $\mathcal{F}$ such that $(\mathcal{G},\mathcal{F}) \to 0$, where $(\mathcal{G},\mathcal{F}) \to 0$ if for each $U \in \mathcal{U}$ there are $G \in \mathcal{G}$ and $F \in \mathcal{F}$ such that $G \times F \subseteq U$.

A quasi-uniform space $(X,\mathcal{U})$ is called *quiet* ([9], [12]) if for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that whenever $\mathcal{F}$ and $\mathcal{G}$ are filters on X satisfying $(G,\mathcal{F}) \to 0$ and x and y are points of X satisfying $V(x) \in \mathcal{F}$ and $V^{-1}(y) \in \mathcal{G}$, then $(x,y) \in U$.

In this case, we say that V is quiet for U and $\mathcal{U}$ is called a *quiet quasi-uniformity*.

The Sorgenfrey line and the Kofner plane are interesting examples of nonmetrizable spaces which admit compatible quasi-metrics such that the induced quasi-uniformities are quiet.

Clearly, each uniform space is quiet. It is also well known that each quiet quasi-uniform space $(X, \mathcal{U})$ is regular, i.e. $\mathcal{T}(U)$ is a regular topology, and that a quasi-uniform space $(X, \mathcal{U})$ is quiet if and only if $(X, \mathcal{U}^{-1})$ is quiet. Thus, in a certain sense, quietness is a symmetric property.

Motivated by the existence of many interesting examples of non-regular quasi-uniform spaces having useful symmetry properties (see Ex. 1 below), we introduce the following generalization of the notion of a quiet quasi-uniform space.

Definition 1. A quasi-uniform space $(X, \mathcal{U})$ is called *fitting* if for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that whenever $\mathcal{F}$ and $\mathcal{G}$ are filters on X satisfying $(G, \mathcal{F}) \to 0$ and x and y are points of X satisfying $V^s(x) \in \mathcal{F}$ and $V^s(y) \in \mathcal{G}$, then $(x, y) \in U$.

In this case, we say that V is *fitting* for U and $\mathcal{U}$ is called a *fitting quasi-uniformity.*

Remark 1. (a) It is easily seen that every fitting quasi-uniform space is R_0, and, hence, every fitting T_0 quasi-uniform space is T_1. Furthermore, a quasi-uniform space $(X, \mathcal{U})$ is fitting if and only if $(X, \mathcal{U}^{-1})$ is fitting.

(b) Similarly to the quiet case, fittingness is a hereditary and productive property.

Example 1. Let (X, d) be a quasi-metric space such that d^s is the discrete metric on X. We show that then $\mathcal{U}_d$ is a fitting quasi-uniformity on X.

Indeed, given $k \in \mathbb{N}$ put $U_k = \{(x, y) \in X \times X : d(x, y) < 2^{-k}\}$. We shall prove that U_k is fitting for U_k : Let $(\mathcal{G}, \mathcal{F}) \to 0$ and $x, y \in X$ such that $U_k^s(x) \in \mathcal{F}$ and $U_k^s(y) \in \mathcal{G}$. Since d^s is the discrete metric on X, $U_k^s(x) = \{x\}$ and $U_k^s(y) = \{y\}$. Therefore the filters $\mathcal{F}$ and $\mathcal{G}$ are generated by $\{x\}$ and $\{y\}$, respectively. Thus $d(y, x) = 0$, so $x = y$.

Next we give an example of a fitting quasi-uniform space $(X, \mathcal{U})$ such that both $\mathcal{T}(U)$ and $\mathcal{T}(U^{-1})$ are the discrete topology on X but $\mathcal{U}$ is not a quiet quasi-uniformity.

Example 2. Let d be the function defined on $\mathbb{N} \times \mathbb{N}$ by

$d(n, m) = | 1/n - 1/m |$ if n is odd and m is even;
$d(n, m) = | 1/n - 1/m |$ if n and m are odd with $n > m$;
$d(n, m) = | 1/n - 1/m |$ if n and m are even with $n < m$;
$d(n, n) = 0$ for all $n \in \mathbb{N}$;

and

$d(n, m) = 1$ otherwise.

It is easily seen that d is a quasi-metric on $\mathbb{N}$ such that $\mathcal{T}(d)$ and $\mathcal{T}(d^{-1})$ are the discrete topology on $\mathbb{N}$. Moreover d^s is the discrete metric on $\mathbb{N}$, so $\mathcal{U}_d$ is a fitting quasi-uniformity (see Ex. 1).

Put $U = \{(n,m) \in \mathbb{N} \times \mathbb{N} : d(n,m) < 1\}$. Let $\mathcal{F}$ and $\mathcal{G}$ be the filters on $\mathbb{N}$ generated by $\{\{2k : k \geq n\} : n \in \mathbb{N}\}$ and $\{\{2k-1 : k \geq$ $\geq n\} : n \in \mathbb{N}\}$, respectively. Clearly $(\mathcal{G}, \mathcal{F}) \to 0$ with respect to $\mathcal{U}_d$. Furthermore, for each $\varepsilon > 0$ there are $j, k \in \mathbb{N}$ such that $B_d(2j, \varepsilon) \in \mathcal{F}$ and $B_{d^{-1}}(2k-1, \varepsilon) \in \mathcal{G}$. But $d(2j, 2k-1) = 1$, so $(2j, 2k-1) \notin U$. We have shown that $\mathcal{U}_d$ is not a quiet quasi-uniformity on $\mathbb{N}$.

It was proved in [18] that any bicompletion of a quiet quasi-uniform space is quiet. We next extend this result to fitting quasi-uniform spaces.

Proposition 1. *Let $(Y,\mathcal{U})$ be a quasi-uniform space and let X be a $\mathcal{T}(\mathcal{U}^s)$-dense subset of Y. Then $(Y,\mathcal{U})$ is fitting if and only if $(X, \mathcal{U} \mid$ $\mid X \times X)$ is fitting.*

Proof. Since the necessity is obvious (see Remark 1 (b)), we only prove the sufficiency. Let $U_0 \in \mathcal{U}$. Choose $U_1 \in \mathcal{U}$ such that $U_1^3 \subseteq U_0$. Then, there is $V \in \mathcal{U}$ such that $V \subseteq U_1$ and $V \cap (X \times X)$ is fitting for $U_1 \cap$ $\cap (X \times X)$. Let $W \in \mathcal{U}$ such that $W^3 \subseteq V$. We shall show that W is fitting for U_0.

Indeed, let $\mathcal{F}$ and $\mathcal{G}$ be filters on Y such that $(\mathcal{G}, \mathcal{F}) \to 0$ and let $x, y \in Y$ be such that $W^s(x) \in \mathcal{F}$ and $W^s(y) \in \mathcal{G}$. Since X is a $\mathcal{T}(\mathcal{U}^s)$-dense subset of Y, $\mathcal{F}_1$ and $\mathcal{G}_1$ are filter bases on X, where

$$\mathcal{F}_1 = \{U^s(F) \cap X : F \in \mathcal{F},\ U \in \mathcal{U}\} \quad \text{and}$$
$$\mathcal{G}_1 = \{U^s(G) \cap X : G \in \mathcal{G},\ U \in \mathcal{U}\}.$$

Furthermore $(\mathcal{G}_1, \mathcal{F}_1) \to 0$ because $(\mathcal{G}, \mathcal{F}) \to 0$. Now let $a \in W^s(x) \cap$ $\cap X$ and $b \in W^s(y) \cap X$. Then $W^s(W^s(x)) \cap X \subseteq V^s(a) \cap X$. Since $W^s(x) \in \mathcal{F}$ we deduce that $V^s(a) \cap X \in \mathcal{F}_1$. Similarly, we obtain that $V^s(b) \cap X \in \mathcal{G}_1$. So, by our assumption, $(a,b) \in U_1$. Thus $(x,y) \in$ $\in U_1^3 \subseteq U_0$. We conclude that $(Y,\mathcal{U})$ is fitting. $\Diamond$

From Prop. 1 we immediately deduce the following result.

Theorem 1. *Let $(Y, \mathcal{V})$ be a bicompletion of a fitting quasi-uniform space $(X, \mathcal{U})$. Then $(Y, \mathcal{V})$ is a fitting quasi-uniform space.*

Corollary 1. *The bicompletion of any fitting T_0 quasi-uniform space is a fitting T_0 quasi-uniform space.*

3. Fitting totally bounded quasi-uniformities

In this section we shall extend Fletcher and Lindgren's theorem cited in Section 1 to fitting quasi-uniform spaces.

Let us recall that a quasi-uniform space $(X,\mathcal{U})$ is totally bounded provided that the uniform space $(X,\mathcal{U}^s)$ is totally bounded (see for instance [13]).

Theorem 2. *Let $(X,\mathcal{U})$ be a fitting totally bounded quasi-uniform space. Then $(X,\mathcal{U})$ is a uniform space.*

Proof. We first show that $\mathcal{U}^{-1} \subseteq \mathcal{U}$. Assume the contrary. Then, there exists $U \in \mathcal{U}$ such that $V \backslash U^{-1} \neq \emptyset$ for all $V \in \mathcal{U}$. Let $(x_V, y_V) \in$ $\in V \backslash U^{-1}$ whenever $V \in \mathcal{U}$. Since $(X,\mathcal{U})$ is totally bounded we can construct, without loss of generality, two subnets $(a_\alpha)_{\alpha\in\Lambda}$ and $(b_\alpha)_{\alpha\in\Lambda}$ of the nets $(x_V)_{V\in\mathcal{U}}$ and $(y_V)_{V\in\mathcal{U}}$ respectively, such that both $(a_\alpha)_{\alpha\in\Lambda}$ and $(b_\alpha)_{\alpha\in\Lambda}$ are Cauchy nets in the uniform space $(X,\mathcal{U}^s)$. Moreover, for each $\alpha \in \Lambda$ we may put $a_\alpha = x_{V_\alpha}$ and $b_\alpha = y_{V_\alpha}$, where the corresponding map from Λ to $\mathcal{U}$ witness that $(a_\alpha)_{\alpha\in\Lambda}$ is a subnet of $(x_V)_{V\in\mathcal{U}}$ and $(b_\alpha)_{\alpha\in\Lambda}$ is a subnet of $(y_V)_{V\in\mathcal{U}}$. Since $(X,\mathcal{U})$ is totally bounded there is a bicompletion $(Y,\mathcal{V})$ of $(X,\mathcal{U})$ such that $(Y,\mathcal{V}^s)$ is a compact uniform space ([24] p. 80). Thus $(a_\alpha)_{\alpha\in\Lambda}$ converges to a point $a \in$ $\in Y$ with respect to $\mathcal{T}(\mathcal{V}^s)$ and $(b_\alpha)_{\alpha\in\Lambda}$ converges to a point $b \in Y$ with respect to $\mathcal{T}(\mathcal{V}^s)$. (Recall that both $(a_\alpha)_{\alpha\in\Lambda}$ and $(b_\alpha)_{\alpha\in\Lambda}$ are Cauchy nets in $(X,\mathcal{U}^s)$ and hence in $(Y,\mathcal{V}^s)$.) It immediately follows that $(a,b) \in \cap\{W : W \in \mathcal{V}\}$. On the other hand $(Y,\mathcal{V})$ is fitting by Th. 1, so $\mathcal{T}(\mathcal{V})$ is an R_0 topology, and, thus, $(b,a) \in \cap\{W : W \in \mathcal{V}\}$ ([13] Prop. 1.9). Therefore the net $((y_{V_\alpha}, x_{V_\alpha}))_{\alpha\in\Lambda}$ is eventually in U, which contradicts our assumption that $(x_V, y_V) \notin U^{-1}$ for all $V \in \mathcal{U}$. We conclude that $\mathcal{U}^{-1} \subseteq \mathcal{U}$. Similarly, we prove that $\mathcal{U} \subseteq U^{-1}$. Hence $\mathcal{U}$ is a uniformity on X and the proof is complete. $\Diamond$

Corollary 2 ([11], [15]). *Every quiet totally bounded quasi-uniformity is a uniformity.*

Since a topological space is T_1 if and only if it is T_0 and R_0, and each fitting quasi-uniform space is R_0, it seems interesting to characterize those T_1 quasi-uniform spaces $(X,\mathcal{U})$ whose bicompletion $(\widetilde{X},\widetilde{\mathcal{U}})$ is T_1. Our next result provides such a characterization. In order to help the reader we recall the construction of the bicompletion of a T_0 quasi-uniform space $(X,\mathcal{U})$, as is given in [13] Th. 3.33.

Let $\widetilde{X}$ be the set of all minimal $\mathcal{U}^s$ -Cauchy filters on X. (Recall that a $\mathcal{U}^s$-Cauchy filter is minimal provided that it contains no $\mathcal{U}^s$-

Cauchy filters other than itself.) For each $U \in \mathcal{U}$ let

$$\widetilde{U} = \{(\mathcal{F}, \mathcal{G}) \in \widetilde{X} \times \widetilde{X} : \text{ there exist } F \in \mathcal{F} \text{ and } G \in \mathcal{G} \text{ with } F \times G \subseteq U\}.$$

Then $\{\widetilde{U} : U \in \mathcal{U}\}$ is a base for a quasi-uniformity $\widetilde{\mathcal{U}}$ on $\widetilde{X}$ such that $(\widetilde{X}, \widetilde{\mathcal{U}})$ is a bicomplete T_0 quasi-uniform space and $(X, \mathcal{U})$ is quasi-uniformly embedded as a $\mathcal{T}(\widetilde{\mathcal{U}})$-dense subspace of $(\widetilde{X}, \widetilde{\mathcal{U}})$ by the map $i : X \to \widetilde{X}$ such that for each $x \in X$, $i(x)$ is the $\mathcal{T}(U^s)$-neighborhood filter of x. Furthermore, any T_0 bicompletion of $(X, \mathcal{U})$ is quasi-unimorphic to $(\widetilde{X}, \widetilde{\mathcal{U}})$ (see [13] Th. 3.34) and we may identify X with $i(X)$.

Proposition 2. *A T_1 quasi-uniform space $(X, \mathcal{U})$ has a T_1 quasi-uniform bicompletion if and only if whenever $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{U}^s$-Cauchy filters such that $(\mathcal{F}, \mathcal{G}) \to 0$, then $(\mathcal{G}, \mathcal{F}) \to 0$.*

Proof. We first suppose that the bicompletion $(\widetilde{X}, \widetilde{\mathcal{U}})$ of the T_1 quasi-uniform space $(X, \mathcal{U})$ is T_1. Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{U}^s$ -Cauchy filters on X such that $(\mathcal{F}, \mathcal{G}) \to 0$. Then $i(\mathcal{F})$ and $i(\mathcal{G})$ are $\widetilde{\mathcal{U}}^s$ -Cauchy filter bases on $\widetilde{X}$, so there exist a and b in $\widetilde{X}$ such that $i(\mathcal{F})$ and $i(\mathcal{G})$ are $\mathcal{T}(\widetilde{\mathcal{U}}^s)$-convergent to a and b, respectively. Since $(i(\mathcal{F}), i(\mathcal{G})) \to 0$ and $(\widetilde{X}, \mathcal{T}(\widetilde{\mathcal{U}}))$ is T_1, it follows that $a = b$. Hence $(i(\mathcal{G}), i(\mathcal{F})) \to 0$, and, thus, $(\mathcal{G}, \mathcal{F}) \to 0$.

Conversely, suppose that $\mathcal{F}$ and $\mathcal{G}$ are minimal $\mathcal{U}^s$-Cauchy filters on X such that $(\mathcal{F}, \mathcal{G}) \in \widetilde{U}$ for all $U \in \mathcal{U}$. Then, for each $U \in \mathcal{U}$ there exist $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \times G \subseteq U$. Thus $(\mathcal{F}, \mathcal{G}) \to 0$ and, by assumption, $(\mathcal{G}, \mathcal{F}) \to 0$. This implies that $(\mathcal{G}, \mathcal{F}) \in \widetilde{U}$ for all $U \in \mathcal{U}$. Since $(\widetilde{X}, \mathcal{T}(\widetilde{\mathcal{U}}))$ is T_0, we obtain $\mathcal{F} = G$. Consequently $(\widetilde{X}, \mathcal{T}(\widetilde{\mathcal{U}}))$ is a T_1 topological space. $\Diamond$

4. Other completeness properties on fitting quasi-uniform spaces

Some authors have investigated several kinds of completeness on quiet quasi-uniform spaces ([5], [12], [6], [19], [7], [22], etc.). In this direction, many interesting results were obtained in the more general context of uniformly regular quasi-uniform spaces, which were introduced by Császár in [4]. (Let us recall that a quasi-uniform space $(X, \mathcal{U})$ is uniformly regular provided that for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $\mathrm{cl}_{\mathcal{T}(U)} V(x) \subseteq U(x)$ for all $x \in X$.)

In this section we study several kinds of completeness for fitting and uniformly R_0 quasi-uniform spaces.

A quasi-uniform space $(X,\mathcal{U})$ is *uniformly* R_0 (uniformly weakly regular in [1]) provided that for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $\text{cl}_{\mathcal{T}(U)} V^s(x) \subseteq U(x)$ for all $x \in X$.

Obviously, every uniformly R_0 quasi-uniform space is R_0 and every uniformly regular quasi-uniform space is uniformly R_0.

Proposition 3. *Let $(X,\mathcal{U})$ be a fitting quasi-uniform space. Then both $(X,\mathcal{U})$ and $(X,\mathcal{U}^{-1})$ are uniformly R_0.*

Proof. Let $U \in \mathcal{U}$. By assumption, there is $V \in \mathcal{U}$ which is fitting for U. Fix $x \in X$. We shall show that $\text{cl}_{\mathcal{T}(U)} V^s(x) \subseteq U(x)$. Indeed, given $z \in \text{cl}_{\mathcal{T}(U)} V^s(x)$, denote by $\mathcal{F}$ the filter on X for which $\{V^s(x) \cap W(z) : : W \in \mathcal{U}\}$ is a base and denote by $\mathcal{G}$ the filter on X generated by $\{z\}$. Clearly $(\mathcal{G},\mathcal{F}) \to 0$. Since $V^s(z) \in \mathcal{G}$ and $V^s(x) \in \mathcal{F}$, it follows that $(x,z) \in U$. We conclude that $(X,\mathcal{U})$ is uniformly R_0. By symmetry we show that $(X,\mathcal{U}^{-1})$ is uniformly R_0. ◊

Next we recall some pertinent concepts.

Let $(X,\mathcal{U})$ be a quasi-uniform space and let $\mathcal{F}$ be a filter on X. Then $\mathcal{F}$ is said to be *left K-Cauchy* ([22]) if for each $U \in \mathcal{U}$ there is $F \in \mathcal{F}$ such that $U(x) \in \mathcal{F}$ for all $x \in F$. $\mathcal{F}$ is called *right K-Cauchy* ([22]) if it is left K-Cauchy on $(X,\mathcal{U}^{-1})$.

A quasi-uniform space $(X,\mathcal{U})$ is said to be *left (right) K-complete* ([22]) if each left (right) K-Cauchy filter on $(X,\mathcal{U})$ is $\mathcal{T}(U)$-convergent. $(X,\mathcal{U})$ is *Smyth completable* if and only if each left K-Cauchy filter on $(X,\mathcal{U})$ is a $\mathcal{U}^s$-Cauchy filter ([16]), and it is *Smyth complete* if and only if each left K-Cauchy filter on $(X,\mathcal{U})$ is $\mathcal{T}(U^s)$-convergent ([16]). According to [5], $(X,\mathcal{U})$ is said to be *half-complete* if each $\mathcal{U}^s$-Cauchy filter is $\mathcal{T}(U)$-convergent.

A Smyth complete quasi-uniform space is both bicomplete and left K-complete, and a bicomplete or left K-complete quasi-uniform space is half-complete. However, it is well known that the converse implications do not hold in general. Furthermore, right K-completeness implies half-completeness, while left K-completeness and right K -completeness are independent concepts ([22]).

It is interesting to recall that Smyth completeness and left K -completeness constitute useful tools to explain properties of some interesting examples of quasi-uniform and quasi-metric spaces which arise in several fields of Theoretical Computer Science (see [26], [27], [23], [25], etc.), while bicompleteness, right K-completeness and half-completeness are appropriate notions of completeness in the study of function and multifunction spaces from a quasi-uniform point of view (see [20], [21], [2], etc.).

Lemma 1. *Let $(X,\mathcal{U})$ be a uniformly R_0 quasi-uniform space. If a $\mathcal{U}^s$-Cauchy filter is $\mathcal{T}(U)$-convergent to a point $x_0 \in X$, then it is $\mathcal{T}(U^s)$-convergent to x_0.*

Proof. Let $\mathcal{F}$ be a $\mathcal{U}^s$-Cauchy filter on X such that $\mathcal{F}$ is $\mathcal{T}(U)$-convergent to $x_0 \in X$. By assumption, given $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $\mathrm{cl}_{\mathcal{T}(U)} V^s(x) \subseteq U(x)$ for all $x \in X$. Furthermore, there is $F \in \mathcal{F}$ such that $V^s(x) \in \mathcal{F}$ for all $x \in F$. So $W(x_0) \cap V^s(x) \neq \emptyset$ for all $W \in \mathcal{U}$ and all $x \in F$. Hence $x_0 \in \mathrm{cl}_{\mathcal{T}(U)} V^s(x) \subseteq U(x)$ for all $x \in F$. Therefore $F \subseteq U^{-1}(x_0)$ and, thus, $U^{-1}(x_0) \in \mathcal{F}$ for all $U \in \mathcal{U}$. Consequently $\mathcal{F}$ is $\mathcal{T}(U^{-1})$-convergent to x_0. We conclude that $\mathcal{F}$ is $\mathcal{T}(U^s)$-convergent to x_0. ◊

Proposition 4 ([1]). *Every uniformly R_0 half-complete quasi-uniform space is bicomplete.*

Proof. Let $\mathcal{F}$ be a $\mathcal{U}^s$-Cauchy filter on a uniformly R_0 half-complete quasi-uniform space $(X,\mathcal{U})$. Since $(X,\mathcal{U})$ is half-complete $\mathcal{F}$ is $\mathcal{T}(U)$-convergent to a point $x_0 \in X$. By Lemma 1, $\mathcal{F}$ is $\mathcal{T}(U^s)$-convergent to x_0. We conclude that $(X,\mathcal{U})$ is bicomplete. ◊

Proposition 5. *For a uniformly R_0 Smyth completable quasi-uniform space $(X,\mathcal{U})$ the following are equivalent:*

(1) *$(X,\mathcal{U})$ is Smyth complete;*
(2) *$(X,\mathcal{U})$ is bicomplete;*
(3) *$(X,\mathcal{U})$ is half-complete;*
(4) *$(X,\mathcal{U})$ is left K-complete.*

Proof. (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (4), (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (3) are obvious. We show that (3) $\Rightarrow$ (1) : By Prop. 4, $(X,\mathcal{U})$ is bicomplete. Since by assumption $(X,\mathcal{U})$ is Smyth completable, every left K-Cauchy filter on $(X,\mathcal{U})$ is a $\mathcal{U}^s$-Cauchy filter. Hence, every left K-Cauchy filter on $(X,\mathcal{U})$ is $\mathcal{T}(\mathcal{U}^s)$-convergent. So $(X,\mathcal{U})$ is Smyth complete. ◊

Corollary 3. *Let $(X,\mathcal{U})$ be a uniformly R_0 Smyth completable left K-complete quasi-uniform space. Then $(X,\mathcal{U}^{-1})$ is right K-complete.*

Proof. By Prop. 5, $(X,\mathcal{U})$ is Smyth complete. Let $\mathcal{F}$ be a right K-Cauchy filter on $(X,\mathcal{U}^{-1})$. Then $\mathcal{F}$ is left K-Cauchy on $(X,\mathcal{U})$. So, it is $\mathcal{T}(U^s)$-convergent and, in particular, $\mathcal{T}(U^{-1})$-convergent. We conclude that $(X,\mathcal{U}^{-1})$ is right K-complete. ◊

A quasi-uniform space $(X,\mathcal{U})$ is said to be *D-complete* ([9], [12]) if every D-Cauchy filter on $(X,\mathcal{U})$ is $\mathcal{T}(U)$-convergent and it is said to be *strongly D-complete* ([14]) if whenever $\mathcal{G}$ and $\mathcal{F}$ are filters on X such that $(\mathcal{G},\mathcal{F}) \to 0$, then $\mathcal{G}$ has a $\mathcal{T}(U)$-cluster point. It is known that each strongly D-complete quasi-uniform space is D-complete but the converse implication does not hold in general ([14]).

It was proved in [8] that each co-stable quiet half-complete quasi-uniform space is strongly D-complete. The space of Ex. 2 shows that this result does not hold when "quiet" is replaced by "fitting". (Let us recall ([8]) that a quasi-uniform space $(X,\mathcal{U})$ is *co-stable* provided that for each pair $\mathcal{G},\mathcal{F}$ of filters on X such that $(\mathcal{G},\mathcal{F}) \to 0$, then $\mathcal{G}$ is *stable*, where $\mathcal{G}$ is said to be stable ([4]) if for each $U \in \mathcal{U}$, $\cap_{G\in\mathcal{G}} U(G) \in \mathcal{G}$.)

In fact, it is easy to see that the quasi-uniform space $(\mathbb{N},\mathcal{U}_d)$ of Ex. 2 is co-stable and half-complete. Clearly, it is not D-complete. Note also that both $\mathcal{U}_d$ and $(\mathcal{U}_d)^{-1}$ are uniformly regular because $\mathcal{T}(d)$ and $\mathcal{T}(d^{-1})$ are the discrete topology on $\mathbb{N}$.

The interesting question of obtaining conditions under which a uniformly regular quasi-uniform space is quiet has been investigated by several authors ([6], [12], [19], etc.). In particular, it was shown in [12] that every uniformly regular strongly D-complete quasi-uniform space is quiet. Here we show that strong D-completeness is also an appropriate property for a uniformly R_0 quasi-uniform space to be fitting.

Proposition 6. *Every uniformly R_0 strongly D-complete quasi-uniform space is fitting.*

Proof. Let $(X,\mathcal{U})$ be a uniformly R_0 strongly D-complete quasi-uniform space. Let $U \in \mathcal{U}$ and $W \in \mathcal{U}$ such that $W^2 \subseteq U$. There is $V \in \mathcal{U}$ such that $V^2 \subseteq W$ and $\mathrm{cl}_{\mathcal{T}(U)} V^s(x) \subseteq W(x)$ for all $x \in X$. We shall prove that V is fitting for U. Indeed, suppose $(\mathcal{G},\mathcal{F}) \to 0$ and let $x, y \in X$ such that $V^s(x) \in \mathcal{F}$ and $V^s(y) \in \mathcal{G}$. Since $(X,\mathcal{U})$ is strongly D-complete, the filter $\mathcal{G}$ has a $\mathcal{T}(U)$-cluster point $x_0 \in X$. Thus $\mathcal{F}$ is $\mathcal{T}(U)$-convergent to x_0. Hence $x_0 \in (\mathrm{cl}_{\mathcal{T}(U)} V^s(y)) \cap (\mathrm{cl}_{\mathcal{T}(U)} V^s(x))$. Consequently $y \in V^2(x_0)$ and $x_0 \in W(x)$. Therefore $y \in W^2(x) \subseteq$ $\subseteq U(x)$. We conclude that V is fitting for U and, thus, $(X,\mathcal{U})$ is a fitting quasi-uniform space. ◊

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