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INCREASE OF $\mathbf{x_{n+1} = x_n - \varphi(x_n)}$ ITERATION'S ORDER

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Abstract: A simple inductive method is presented by which an arbitrary rapidly convergent iterative sequence can be constructed for approximation of the single simple zero of a given C^∞-function. Among others, these include sequences obtained by Newton's, Halley's and other well known iterative algorithms. The Maple computer-algebraic system is used for establishing new, higher-order convergent iterations, as well as, for the numeric illustration of their order.

1. Preliminaries

By speed of the convergence $0 < \varepsilon_n \to 0$ $(n \to \infty)$ we mean the least upper bound of those $k \geq 0$ satisfying the

$$(1) \qquad \limsup_n \frac{\varepsilon_{n+1}}{\varepsilon_n^k} < +\infty$$

condition. In this case the $(\varepsilon_n)_{n=0}^\infty$ sequence is said to be *at least k^{th}-order* *convergent* (if, in addition, $\liminf_n \frac{\varepsilon_{n+1}}{\varepsilon_n^k} > 0$, the *order of convergence is exactly* k).

The ideal frame for iterative search of a simple root of a given C^∞-function $\varphi : [a,b] \to \mathbb{R}$ is to have *a* single zero $c \in (a,b)$, at which the following conditions are satisfied:

$$\varphi(c) = 0,\ \varphi'(c) = 1,\ \varphi''(c) = \cdots = \varphi^{(k)}(c) = 0. \tag{2}$$

The set of these functions will be denoted by $Z^{(k)}(c)$. If $\varphi \in \mathcal{Z}^{(k)}(c)$ and $\psi(x) := x - \varphi(x)$ then

$$\psi(c) = c, \psi'(c) = \psi''(c) = \cdots = \psi^{(k)}(c) = 0 \tag{3}$$

thus, by the k^{th}-order Taylor-formula in c, written with Lagrangian rest, we have for every $x \in [a,b]$

$$|\psi(x) - c| \le M|x - c|^{k+1}, \tag{4}$$

where

$$M := \max\{|\varphi^{(k+1)}(t)|/(k+1)! \mid t \in [a,b]\}.$$

Consequently, if $0 < r < \min\{1, b-c, c-a, M^{-1/k}\}$ and $I := [c-r, c+r]$, then $\psi(I) \subset I \subset [a,b]$, so for every $x_0 \in I : x_{n+1} := x_n - \varphi(x_n) = \psi(x_n) \in I$ and

$$\limsup_n \frac{|x_{n+1} - c|}{|x_n - c|^{k+1}} \le M,$$

i.e. $(x_n)_{n=0}^\infty$ is at least $(k+1)^{th}$-order convergent to c.

By the first and second recursive-theorem (see (2.5) and (2.6)), a large class of $\Phi_k : \mathcal{Z}^{(k)}(c) \to \mathcal{Z}^{(k+1)}(c)$ $(k \in \mathbb{N}^*)$ mappings can be constructed. For a fixed sequence $(\Phi_k)_{k=0}^\infty$ taking in account that for a smooth $f : [a,b] \to \mathbb{R}$, with $f(c) = 0 \ne f'(c)$ for some $c \in (a,b)$, the function $\varphi_1 := f/f' \in \mathcal{Z}^{(1)}(c)$, and consequently, the sequence $\varphi_{k+1} := \Phi_k(\varphi_k)$ satisfies the following

Theorem. *If* $f : [a,b] \to \mathbb{R}$ *is a* C^∞*-function for which* $f(a)f(b) < 0$ *and* $f'(x) \ne 0$ $(\forall x \in [a,b])$ *then one may recursively construct a* $(\varphi_k)_{k=1}^\infty$ *sequence of* C^∞*-functions, for which* $\varphi_1 = f/f'$ *and for every* $k \in \mathbb{N}^*$ *and some appropriently choosen initial value* $x_0^{(k)}$, *the sequence* $(x_n^{(k)})_{n=0}^\infty$ *defined by* $x_{n+1}^{(k)} := x_n^{(k)} - \varphi_k(x_n^{(k)})$ *is at least* $(k+1)^{th}$*-order convergent to the (single) zero of* f.

References. J.F. Traub in the 8^{th} paragraph of his [4] monography in an ambitious program generates such a $(\varphi_k)_{k=1}^\infty$ sequence. The adventage of our approach is, that it does not require any pre-construction. W. Gander in [2] – discussing the Halley method – grasps the momentum of transition from φ_1 to φ_2.

The possible chaotic behaviour of the discrete dynamic systems generated by φ_k algorithm ($k \in \mathbb{N}^*$) and the estimation of the attraction basin of the zero point, will be discussed elsewhere. For the complexity of these issues in the case of Newton and Halley algoritms, see [3] and [1].

2. Iteration calculus

In the following we shall discuss $\mathcal{Z}^{(k)}(c)$ and some related function classes. Following Traub, the iterative calculus terminology is used, since iterative algorithms are behind these. We consider the following classes of functions (where $c \in \mathbb{R}$ is an arbitrary fixed point, dom (φ) is the domain of φ and int is the interior of a set):

$$\mathcal{C}^\infty(c) := \{\varphi \mid c \in \text{ int dom } (\varphi) \wedge \varphi \text{ is } C^\infty\text{-function on dom } (\varphi)\},$$

$$\mathcal{Z}^{(k)}(c) := \{\varphi \in \mathcal{C}^\infty(c) \mid \varphi(c) = 0, \varphi'(c) = 1, \varphi''(c) = \cdots = \varphi^{(k)}(c) = 0\} \quad (k \geq 1),$$

$$\mathcal{Z}_*^{(k)}(c) := \{\varphi \in \mathcal{C}^\infty(c) \mid \varphi(c) = 0, \varphi'(c) \neq 0, \varphi''(c) = \cdots = \varphi^{(k)}(c) = 0\} \quad (k \geq 1),$$

$$\mathcal{F}^{(k)}(c) := \{\psi \in \mathcal{C}^\infty(c) \mid \psi(c) =, \psi'(c) = \cdots = \psi^{(k)}(c) = 0\} \quad (k \geq 1),$$

$$\mathcal{N}^{(k)}(c) := \{\nu \in \mathcal{C}^\infty(c) \mid \nu(c) = \nu'(c) = \cdots = \nu^{(k)}(c) = 0\} \quad (k \geq 0).$$

The functions' operations are defined pointwise on the intersection of their domains. It is clear that $\varphi \to \psi := id_{\text{dom}(\varphi)} - \varphi$ is a one-to-one correspondence between $\mathcal{Z}^{(k)}(c)$ and $\mathcal{F}^{(k)}(c)$. The folllowing two statements are self-evident:

(2.1) (The multiplicity of the zero point): *If* $\nu_1 \in \mathcal{N}^{(k)}(c)$ *and* $\nu_2 \in \mathcal{N}^{(\ell)}(c)$, *then* $\nu_1 \cdot \nu_2 \in \mathcal{N}^{(k+\ell+1)}(c)$.

(2.2) (Newton's normalization): *If* $\varphi \in \mathcal{Z}_*^{(k)}(c)$, *then* $\varphi/\varphi' \in \mathcal{Z}^{(k)}(c)$.

Remarks. 1. The necessity of transition from $\mathcal{Z}_*^{(k)}(c)$ to $\mathcal{Z}^{(k)}(c)$ is imposed by the fact that for $\varphi \in \mathcal{Z}_*^{(k)}(c)$ the point c may be a repulsive fixed point of $\psi(x) = x - \varphi(x)$, and in this case the iterative sequence does not converge to c, while for $\varphi \in \mathcal{Z}^{(k)}(c)$, $\psi'(c) = 0$, so the fixed point is superattractive.

2. For an arbitrary $\varphi \in \mathcal{Z}_*^{(1)} : x_{n+1} = x_n - \dfrac{\varphi(x_n)}{\varphi'(x_n)}$ is the classical Newton-algorithm.

3. For arbitrary $k \in \mathbb{N}^*$, verification of $\varphi \in \mathcal{Z}_{(c)}^{(k)}$ is done by

$$(\varphi_1 \varphi')^{(n)}(c) = \varphi_1^{(n)}(c)\varphi'(c) = \varphi^{(n)}(c) \quad (0 \le n \le k).$$

The basic idea of the Lemma on changes of variable below, is the following: We want to construct iterative algoritms with $\varphi_k \in \mathcal{Z}^{(k)}(c)$, for the approximation of the unknown root c of $f : [a,b] \to \mathbb{R}$. By the hypothesis $f'(x) \neq 0$ ($\forall x \in [a,b]$) of the Theorem of 1§, f is a $\mathcal{C}^\infty$-diffeomorphism between approprietly choosen neighbourhoods of c and $0 = f(c)$, and so each element of $\mathcal{C}^\infty(c)$ may be uniquely represented as a composition of f with an element of $\mathcal{C}^\infty(0)$. In this way, instead of c, we will work in 0.

(2.3) (Lemma on change of variable): *If* $\varphi \in \mathcal{Z}_*^{(1)}(c)$ *and* $\nu \in \mathcal{N}_{(0)}^{(k)}$ ($k \in \mathbb{N}$), *then* $\nu \circ \varphi \in \mathcal{N}^{(k)}(c)$ *and the mapping* $\nu \to \nu \circ \varphi : \mathcal{N}^{(k)}(0) \to \to \mathcal{N}^{(k)}(c)$ *is onto.*

Proof. By induction on n, it is easy to realise that for appropiately choosen P_m ($1 \le m \le n$) polinomials

$$(\nu \circ \varphi)^{(n)} = \nu^{(n)} \circ \varphi \cdot (\varphi')^n + \sum_{m=1}^{n} \nu^{(n-m)} \circ \varphi \cdot P_m(\varphi, \varphi', \dots, \varphi^{(n)}).$$

Since for $0 \le n \le k : \nu^{(n)}(0) = 0$, it follows that $(\nu \circ \varphi)^{(n)}(c) = 0$. Because φ^{-1} is also a $\mathcal{C}^\infty$-diffeomorphism, the above argument is applicable to φ^{-1} and means that for every $\nu \in \mathcal{N}^{(k)}(c) : \nu \circ \varphi^{-1} \in \mathcal{N}^{(k)}(c)$ and consequently $\nu = (\nu \circ \varphi^{-1}) \circ \varphi$ i.e. the mapping of the statement is onto. ◊

The key-momentum in constructing a $\mathcal{Z}^{(k)}(c) \to \mathcal{Z}^{(k+1)}(c)$ map is based on the following statement: For a given $\varphi \in \mathcal{Z}^{(k)}(c)$ we characterize those $\psi \in \mathcal{C}^{(\infty)}(c)$ for which $\varphi(1+\psi) \in \mathcal{Z}^{(k+1)}(c)$. From the equalities

$$(\varphi(1+\psi))^{(n)} = \begin{cases} 0, & n = 0 \\ 1 + \psi(c), & n = 1 \\ n\psi^{(n-1)}(c), & 1 < n \le k \\ (k+1)\psi^{(k)}(c) + \varphi^{(k+1)}(c)(1+\psi(c)), & n = k+1 \end{cases}$$

immediately follows that:

(2.4) *If* $\varphi \in \mathcal{Z}^{(k)}(c)$ *and* $\psi \in \mathcal{C}^{(\infty)}(c)$, *the following two statements are equivalent:*

(i) $\varphi(1+\psi) \in \mathcal{Z}^{(k+1)}(c)$;

(ii) *(a)* $\psi \in \mathcal{N}^{(k-1)}(c)$ *and*

(b) $\varphi^{(k+1)}(c) + (k+1)\psi^{(k)}(c) = 0$.

Using (2.3) and (2.4), for $k \ge 1$, we prove:

(2.5) (The first recursive theorem): (i) *If* $\varphi \in \mathcal{Z}^{(k)}(c)$, $\varphi^{(k+1)}(c) \neq 0$, $\nu \in \mathcal{N}^{(k-1)}(0)$ *and* $\nu^{(k)}(0) = -\operatorname{sgn} \varphi^{(k+1)}(c)/(k+1)$, *then*

$$\varphi\left(1+\nu\circ\left(\varphi\cdot|\varphi^{(k+1)}|^{\frac{1}{k}}\right)\right)\in\mathcal{Z}^{(k+1)}(c).$$

(ii) *If* $\varphi\in\mathcal{Z}_*^{(k)}(c)$, $\varphi^{(k+1)}(c)\neq 0$, $\nu\in\mathcal{N}^{(k-1)}(0)$ *and* $\nu^{(k)}(0)=$ $=\dfrac{k}{k+1}\operatorname{sgn}\dfrac{\varphi^{(k+1)}(c)}{(\varphi'(c))^{k+1}}$, *then*

$$\frac{\varphi}{\varphi'}\left(1+\nu\circ\left(\varphi\cdot\left|\frac{\varphi^{(k+1)}}{(\varphi')^{k+1}}\right|^{\frac{1}{k}}\right)\right)\in\mathcal{Z}^{(k+1)}(c).$$

Proof. (i) If $\alpha\in\mathcal{C}^{(\infty)}(c)$ and $\alpha(c)\neq 0$, then $(\alpha\varphi)'(c)\neq 0$, so in (2.3) we may choose the change of variable $\alpha\cdot\varphi\in\mathcal{Z}_*^{(1)}(c)$, and based on this, the function $\psi\in\mathcal{N}^{(k-1)}(c)$ from (2.4) can be written with a $\nu\in\mathcal{N}^{(k-1)}(0)$ as $\psi=\nu\circ(\alpha\varphi)$. Next, by (6) we can state that $\psi^{(k)}(c)=\nu^{(k)}(0)\alpha^k(c)$, so (7) becomes $\varphi^{(k+1)}(c)+(k+1)\nu^{(k)}(0)\alpha^k(c)=0$, and then by $\nu^{(k)}(0)$ from the statement we obtain the function $\alpha=|\varphi^{(k+1)}|^{\frac{1}{k}}$ for which $\varphi(1+\psi)=$ $=\varphi(1+\nu\circ(\alpha\varphi))=\varphi(1+\nu\circ(\varphi\cdot|\varphi^{(k+1)}|^{\frac{1}{k}}))$.

(ii) If $\varphi\in\mathcal{Z}_*^{(k)}(c)$ then $\dfrac{\varphi}{\varphi'}\in\mathcal{Z}^{(k)}(c)$ (cf. (2.2)). For φ_1, the element from $\mathcal{Z}^{(k+1)}(c)$ will be looked for as being of the form $\varphi_1(1+\nu\circ(\alpha\cdot\varphi))$. Since $\varphi_1'(c)=-k\varphi^{(k+1)}(c)/\varphi'(c)$, condition (7) becomes $(k+1)\nu^{(k)}(0)[\varphi'(c)]^{k+1}\alpha^k(c)=k\varphi^{(k+1)}(c)$ and thus the solution is obtained by choosing $\alpha:=|\varphi^{(k+1)}/(\varphi')^{k+1}|^{1/k}$. ◊

For $k\geq 2$ a more simple iteration is generated by

(2.6) (The second recursive theorem):

(i) *If* $\varphi\in\mathcal{Z}^{(k)}(c)$, $k\geq 2$, $\nu\in\mathcal{N}^{(k-2)}(0)$ *and* $\nu^{(k-1)}(0)=-\dfrac{1}{k(k+1)}$, *then*

$$\varphi(1+\varphi^{(k)}\cdot\nu\circ\varphi)\in\mathcal{Z}^{(k+1)}(c).$$

(ii) *If* $\varphi\in\mathcal{Z}_*^{(k)}(c)$, $k\geq 2$, $\nu\in\mathcal{N}^{(k-2)}(0)$ *and* $\nu^{(k-1)}(0)=\dfrac{1}{k+1}$, *then*

$$\frac{\varphi}{\varphi'}\left(1+\frac{\varphi^{(k)}}{(\varphi')^k}\cdot\nu\circ\varphi\right)\in\mathcal{Z}^{(k+1)}(c).$$

Proof. (i) Elements $\mu\in\mathcal{N}^{(0)}(c)$ and $\mu\in\mathcal{N}^{(k-2)}(o)$ are determined so that $\psi:=\mu\cdot\nu\circ\varphi$ fulfills the conditions of (2.4) (ii). From (2.1) follows that $\psi\in\mathcal{N}^{(k-1)}(c)$. Since

$$\psi^{(k)}(c)=\sum_{\ell=0}^{k}\binom{k}{\ell}\mu^{(\ell)}(c)(\nu\circ\varphi)^{(k-\ell)}(c)\overset{(6)}{\to}=k\mu'(c)\nu^{(k-1)}(0)=$$
$$=-\frac{1}{k+1}\mu'(c),$$

for $\mu{:=}\varphi^{(k)}$, (7) is also satisfied and thus $\varphi(1+\mu\cdot\nu\circ\varphi)=\varphi(1+\varphi^{(k)}\cdot\nu\circ\varphi)\in \in \mathcal{Z}^{(k+1)}(c)$.

(ii) In this case the elements of $\mathcal{Z}^{(k+1)}(c)$ will also be looked for, as being of the form $\frac{\varphi}{\varphi'}(1+\mu\cdot\nu\circ\varphi)$. Since

$$\psi^{(k)}(c)=k\mu'(c)\nu^{(k-1)}(0)(\varphi'(c))^{k-1}=\frac{k}{k+1}\mu'(c)(\varphi'(c))^{k-1},$$
$$\left(\frac{\varphi}{\varphi'}\right)^{(k+1)}(c)=-k\varphi^{(k+1)}(c)/\varphi'(c),$$

the condition (7) can be written as $\varphi^{(k+1)}(c)-\mu'(c)(\varphi'(c))^k=0$, this latter, based on $\mu(c)=0$, can be rewritten as $\varphi^{(k+1)}(c)-(\mu\cdot\varphi'^k)(c)=0$, which is satisfied by $\mu=\varphi^{(k)}/(\varphi')^k$. $\Diamond$

If in (2.5), repectively (2.6), we put $\nu(t)=\alpha t^k$ resp. $\nu(t)=\beta t^{k-1}$ (α resp. β are determined according to $\nu^{(k)}(0)$ resp. $\nu^{(k-1)}(0)$), then occur the following two recursive formulas.

(2.7) (i) *For* $\varphi_k\in\mathcal{Z}^{(k)}(c)$

(a) *if* $k\geq 1$ *and* $\varphi^{(k+1)}(c)\neq 0$, *then*

$$\varphi_{k+1}:=\varphi_k\left(1-\frac{1}{(k+1)!}\varphi_k^k\varphi_k^{(k+1)}\right)\in\mathcal{Z}^{(k+1)}(c);$$

(b) *if* $k\geq 2$, *then*

$$\varphi_{k+1}:=\varphi_k\left(1-\frac{1}{(k+1)!}\varphi_k^{(k-1)}\varphi_k^{(k)}\right)\in\mathcal{Z}^{(k+1)}(c).$$

(ii) *For* $\varphi_k\in\mathcal{Z}_*^{(k)}(c)$

(a) *if* $k\geq 1$ *and* $\varphi^{(k+1)}(c)\neq 0$, *then*

$$\varphi_{k+1}:=\frac{\varphi_k}{\varphi_k'}\left(1-\frac{1}{(k-1)!(k+1)}\,\frac{\varphi_k^k\varphi_k^{(k+1)}}{(\varphi')^{k+1}}\right)\in\mathcal{Z}^{(k+1)}(c);$$

(b) *if* $k\geq 2$, *then*

$$\varphi_{k+1}:=\left(1+\frac{1}{(k-1)!(k+1)}\,\frac{\varphi_k^{k-1}\varphi_k^{(k)}}{(\varphi_k')^k}\right)\in\mathcal{Z}^{(k+1)}(c).$$

3. Maple applications

The recursive formulas of previous paragraph for $k = 1$ easily can be computed by hand. For example if in (2.5) we choose $\chi := 1+\nu$, then χ must satisfy the following conditions

$$\chi(0) = 1 \text{ and } \chi'(0) = \begin{cases} -\frac{1}{2}\,\mathrm{sgn}\,\varphi''(c) & \text{in the case (i)} \\ \frac{1}{2}\,\mathrm{sgn}\,\varphi''(c) & \text{in the case (ii)} \end{cases}.$$

Here are three solutions and the corresponding third order iterations:

(a) if $\chi(t) := 2/\big(2 + t\,\mathrm{sgn}\,\varphi''(c)\big)$ and $\varphi \in Z^{(1)}_{(c)}$, then

$$2\varphi/(2+\varphi\varphi'') \in Z^{(2)}(c).$$

(b) if $\chi(t) := 1/\sqrt{1+t\,\mathrm{sgn}\,\varphi''(c)}$ and $\varphi \in Z^{(1)}_*(c)$, then

$$\mathrm{sgn}\,(\varphi')\frac{\varphi}{\sqrt{\varphi'^2-\varphi\varphi''}} \in Z^{(2)}(c).$$

(c) if $\chi(t) := 2/\big(2 - t\,\mathrm{sgn}\,\varphi''(c)\big)$ and $\varphi \in Z^{(1)}_*(c)$, then

$$\frac{2\varphi\varphi''}{2\varphi'^2-\varphi\varphi''} \in Z^{(2)}(c) \quad \text{(the classical Halley-iteration).}$$

To obtain higher order recursive formulas became more difficult task because of the large amount of computations. Therefore we use the Maple computer algebraic system to deduce more complicated formulas and numerically illustrate the accuracy of convergence order. If we choose $k = 2$ in (2.6) (i), $\nu(t) = -\frac{t}{6}$ and $\varphi(t) = \frac{2f\cdot f'}{2\cdot(f')^2 - f\cdot f''} \in Z^2(c)$ (i.e. φ is the Halley-iteration of f) then we obtain by Maple the following iteration function

$$\begin{gathered}\psi = \varphi\cdot(1+\varphi''\cdot(\nu\circ\varphi)) = \frac{2}{3}\cdot f\cdot f'\{-48(f')^8 + 96(f')^6 f\cdot f'' - \\ -84(f')^4f^2(f'')^2 + 48(f')^2f^3(f'')^3 - 3f^4(f'')^4 + +8f^2(f')^5f''' - \\ -24f^3(f')^3f''f''' - 2f^4f'f'''(f'')^2 + 4f^4(f')^2(f''')^2 + \\ +4f^3(f')^4f'''' - 2f^4(f')^2\cdot f''\cdot f''''\}\cdot[-2(f')^2 + f\cdot f'']^{-5}\end{gathered}$$

which belongs to $Z^3(c)$. Applying this to approximate the unique zero $c = \sqrt[3]{2}$ of $f(x) = x^3 - 2$, we calculate the sequence $x_{n+1} = x_n - \psi(x_n)$ (with the above ψ), the deviations $\varepsilon_n = |x_{n+1} - x_n|$ and the ratios

$$\frac{\varepsilon_{n+1}}{\varepsilon_n^3}, \quad \frac{\varepsilon_{n+1}}{\varepsilon_n^4}, \quad \frac{\varepsilon_{n+1}}{\varepsilon_n^5} \qquad (n = 1, 2, 3, 4, 5).$$

The numerical precession of Maple we set 800 digits, in order to get nonzero ε_n differences. Here is a part of the Maple program illustrating the accuracy order of this sequences:

```
> 'ITERATION':=unapply (x-pszi(x)):
  ORDER:=4;
  Digits:=800:x1:=1:
  ORDER1:=NULL:ORDER2:=NULL:ORDER3:=NULL:
  x2:=evalf('ITERATION'(x1)):h1:=abs(x2-x1):
  'approx':=evalf(x2,10):
  for i from 1 to 5 do
     x1:=x2:x2:=evalf('ITERATION'(x1)):
     h2:=abs(x2-x1):
     'approx':='approx',evalf(x2,10):
     ORDER1:=ORDER1,evalf(h2/h1^(ORDER-1),10):
     ORDER2:=ORDER2,evalf(h2/h1^ORDER,10):
     ORDER3:=ORDER3,evalf(h2/h1^(ORDER+1),10):
     h1:=h2:
  od:
> ORDER1;ORDER2;ORDER3;
> 'approximation'='approx';
```

The results of this Maple program: on the one hand the sequences $\varepsilon_{n+1}/\varepsilon_n^3$, $\varepsilon_{n+1}/\varepsilon_n^4$, and $\varepsilon_{n+1}/\varepsilon_n^5$ $(n = 1, 2, 3, 4, 5)$
.1002796317, .0008979861435, $.2608503162\,10^{-11}$, $.1851933811\,10^{-45}$, $.4705023909\,10^{-182}$,
.3831579959, .4995194381, .5000000002, .4999999998, .5000000001,
1.464006670, 277.8658342, $.9584040526\,10^{11}$, $.1349940254\,10^{46}$, $.5313469281\,10^{182}$
and on the other hand the x_n approximating sequence $(n = 1, 2, 3, 4, 5)$
approx. = (1.261718750, 1.259921050, 1.259921050, 1.259921050, 1.259921050, 1.259921050)
These sequences convince that for this new iteration function the order of convergence is exactly 4.

Interesting to compare this result with the classical Halley-iteration. If in the programme we substitute `ITERATION:=unapply(x-Halley(x))` and `ORDER:=3` then for the ratios $\varepsilon_{n+1}/\varepsilon_n^2$, $\varepsilon_{n+1}/\varepsilon_n^3$, $\varepsilon_{n+1}/\varepsilon_n^4$ we obtain the following sequences
.1587301587, .004216403843, $.1742783454\,10^{-6}$, $.1260398211\,10^{-19}$, $.4767615870\,10^{-59}$,

.6349206349, .4250135074, .4199738909, .4199736834, .4199736831, 2.539682540, 42.84136154, $.1012048104\,10^{7}$, $.1399382300\,10^{20}$, $.3699498854\,10^{59}$
and the approximation sequence of the zero is
approx. = (1.250000000, 1.259920635, 1.259921050, 1.259921050, 1.259921050, 1.259921050)
These data reinforce the well known fact that the order of Halley-iteration is exactly 3.

References

[1] ALEFELD, G.: On the convergence of Halley's method, *Amer. Math. Monthly* **88** (1981), 530–536.

[2] GANDER, W.: On Halley's iteration method, *Amer. Math. Monthly* **92** (1985), 131–134.

[3] HURLEY, M. and MARTIN, C.: Newton's algorithm and chaotic dynamical systems, *SIAM I. Math. Anal.* **15** (1984), 238–252.

[4] TRAUB, J.F.: Iterative methods for the solution of equations, New Jersey, Prentice-Hall (1964).