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FURTHER EXTENSION OF STOLARSKY'S INEQUALITY WITH GENERAL WEIGHTS

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Abstract: We consider an inequality

$$\frac{\int_a^b w_1(x)g(x)dx}{\int_a^b w_1(x)dx} \cdot \frac{\int_a^b w_2(x)g(x)dx}{\int_a^b w_2(x)dx} \le g(a)\frac{\int_a^b w_3(x)g(x)dx}{\int_a^b w_3(x)dx},$$

where w_i, $i = 1, 2, 3$ are nonnegative and integrable functions on $[a, b]$ and g is a nonnegative function on $[a, b]$ and we present a number of assumptions on g and w_i when that inequality is valid.

1. Introduction

L. Maligranda, J. Pečarić, L.E. Persson in their paper "Stolarsky's Inequality with General Weights", [5], discussed the so-called Stolarsky's inequality by which new inequalities for gamma function can be pointed out. They remarked that if in Th. 1 there the function g is a nonincreasing function then the inequality (that extend Stolarsky's inequality) holds even if the assumption $W_1W_2 = W_3$ is replaced by the assumption $W_1W_2 \leq W_3$, [5, Remark 2]. In this paper we continue the discussion about that condition. In Chapter 2 of paper we present an improvement of Th. 1 from [5] using some helpful lemmas, one of which is due to Hardy, [4], and the others can be proved by elementary transformations (see [1], [2]). In Chapter 3 we present some applications to beta and incomplete gamma functions. In Chapter 4 we present Jensen's type inequality which in some special case give us a generalization of Gauss-Pólya's inequality.

In this paper if an inequality has a number (n) then its reverse version (the reversed inequality) is denoted by (Rn).

2. Main results

Let us suppose that $w_i, i = 1,2,3$, are nonnegative and integrable functions on $[a, b]$ and W_i is defined by

$$W_i(x) = \frac{\int_a^x w_i(t)dt}{\int_a^b w_i(t)dt} \qquad i = 1,2,3.$$

Also, let g be a function of bounded variation and

$$Q(g, w_i) = \frac{\int_a^b w_i(x)g(x)dx}{\int_a^b w_i(x)dx} \qquad i = 1,2,3.$$

In [5] the following theorem is proven:

Theorem MPP. *If g is a function of bounded variation on $[a, b] = [0,1]$ such that $0 \leq g(1) \leq g(x) \leq g(0)$ for all $x \in [0,1]$ and if*

$$W_1(x)W_2(x) = W_3(x) \quad \textit{for all} \quad x \in [0,1] \tag{1}$$

then

$$g(0)Q(g, w_3) \geq Q(g, w_1)Q(g, w_2). \tag{2}$$

It is easy to check that the theorem still holds if the interval is [a,b], and also, if g is a function of bounded variation on $[a, b]$ such that $0 \leq$ $\leq g(a) \leq g(x) \leq g(b)$ for all $x \in [a, b]$ and if (1) holds then (R2) holds. The proof of the reverse inequality is very similar to the proof which is represented in [5], only instead of discrete Chebyshev's inequality the following inequality is used: If $p_1p_2 \leq 0$, $a_1 \geq a_2$ and $b_1 \geq b_2$, then

$$(p_1 + p_2)(p_1a_1b_1 + p_2a_2b_2) \leq (p_1a_1 + p_2a_2)(p_1b_1 + p_2b_2). \tag{3}$$

When g is a monotone function condition (1) can be replaced by a weaker assumptions. Namely, the following theorem is valid.

Theorem 1. *Let g be a nonnegative function on $[a,b]$.*

a) *If g is differentiable and $g'(x) \le 0$ for all $x \in [a,b]$, g is convex on $[a,b]$ and*

$$\int_a^x W_3(t)dt \ge \int_a^x W_1(t)W_2(t)dt \quad \text{for all} \quad x \in [a,b] \tag{4}$$

then

$$Q(g,w_1)Q(g,w_2) \le g(a)Q(g,w_3) \tag{5}$$

holds.

If $g'(x) \ge 0$ for all $x \in [a,b]$ and g is concave on $[a,b]$ and if (4) *holds then* (R5) *is valid.*

b) *If g is a nonnegative nonincreasing function on $[a,b]$ and*

$$\begin{aligned} W_3'(x) &\ge (W_1W_2)'(x), \quad \text{for} \quad x \in [a,(a+b)/2], \\ W_3(b-x) - W_3(a+x) &\ge (W_1W_2)(b-x) - (W_1W_2)(a+x) \end{aligned} \tag{6}$$

holds for $x \in [0,(b-a)/2]$, then (5) *holds.*

For the proof of Th. 1 we need the following lemmas.

Lemma 1. a) *If S is a nonnegative and nondecreasing function on $[a,b]$ and*

$$\int_x^b H(t)dt \le 0 \quad \text{for all} \quad x \in [a,b],$$

then

$$\int_a^b H(t)S(t)dt \le 0.$$

b) *If S is a nonnegative and nonincreasing function on $[a,b]$ and*

$$\int_a^x H(t)dt \le 0 \quad \text{for all} \quad x \in [a,b],$$

then

$$\int_a^b H(t)S(t)dt \le 0.$$

Lemma 2. *Let S be a nonnegative left balanced function on $[a,b]$ (left balanced means that $S(a+x) \ge S(b-x)$ for all $x \in [0,\frac{b-a}{2}]$,* [3]*) and let S be nonincreasing on $[\frac{a+b}{2},b]$. If*

$$H(x) \le 0 \quad \text{for all} \quad x \in [a,\frac{a+b}{2}]$$

and

$$\int_{a+x}^{b-x} H(t)dt \le 0 \quad \text{for} \quad x \in [0,\frac{b-a}{2}],$$

then

$$\int_a^b H(t)S(t)dt \le 0.$$

The statement in Lemma 1a) appears in [4, p. 298], and other cases can be proven using integration by parts, [1], [2].

Proof. a) When $g' \leq 0$ then putting $S = -g'$ and $H = W_1W_2 - W_3$ and applying Lemma 1b) we have

$$\int_a^b W_1(t)W_2(t)g'(t)dt \geq \int_a^b W_3(t)g'(t)dt.$$

Putting in the discrete Chebyshev inequality [6, p.240] $p_1 = g(b)$, $p_2 = g(a) - g(b)$, $a_1 = b_1 = 1$ and

$$a_2 = \frac{1}{g(b)-g(a)} \int_a^b W_1(x)dg(x), \quad b_2 = \frac{1}{g(b)-g(a)} \int_a^b W_2(x)dg(x)$$

after simple calculation we have: $p_1 > 0$, $p_2 > 0$, $a_1 \leq a_2$ and $b_1 \leq b_2$ and so,

$$\Big(g(b) - \int_a^b W_1(x)dg(x)\Big)\Big(g(b) - \int_a^b W_2(x)dg(x)\Big) \leq$$
$$\leq g(a)\Big(g(b) - \frac{1}{g(b)-g(a)} \int_a^b W_1(x)dg(x) \int_a^b W_2(x)dg(x)\Big)$$

Now, using that inequality and the integral Chebyshev inequality, [6, p.239], we obtain

$$\begin{aligned} Q(g,w_1)Q(g,w_2) &= \frac{\int_a^b w_1(x)g(x)dx}{\int_a^b w_1(x)dx} \cdot \frac{\int_a^b w_2(x)g(x)dx}{\int_a^b w_2(x)dx} = \\ &= \Big(g(b) - \int_a^b W_1(x)dg(x)\Big)\Big(g(b) - \int_a^b W_2(x)dg(x)\Big) = \\ &\leq g(a)\Big(g(b) - \frac{1}{g(b)-g(a)} \int_a^b W_1(x)dg(x) \int_a^b W_2(x)dg(x)\Big) \leq \\ &\leq g(a)\Big(g(b) - \int_a^b W_1(x)W_2(x)dg(x)\Big) \end{aligned} \tag{7}$$

Therefore,

$$Q(g,w_1)Q(g,w_2) \leq g(a) \int_a^b W_3(x)g'(x)dx = g(a)Q(g,w_3).$$

When $g' \geq 0$ we replace $S = g'$ and the same method is used.

Let us prove the case b). Setting $S = g$ and $H = (W_1W_2)' - W_3'$ and applying Lemma 2 we get

$$\int_a^b (W_1W_2)'(x)g(x)dx \leq \int_a^b W_3'(x)g(x)dx.$$

Now, using (7) we have

$$Q(g,w_1)Q(g,w_2) \le g(a)\left(g(b) - \int_a^b W_1(x)W_2(x)dg(x)\right) =$$
$$= g(a)\int_a^b (W_1W_2)'(x)g(x)dx \le$$
$$\le g(a)\int_a^b W_3'(x)g(x)dx = g(a)Q(g,w_3).\Diamond$$

In a similar way we can prove

Theorem 2. *Let g be a nonnegative function on $[a,b]$. In cases a)–f) we will suppose that g is a diferentiable function.*

a) *If $g'(x) \le 0$ for all $x \in [a,b]$, g is concave on $[a,b]$ and*

$$(8) \qquad \int_x^b W_3(t)dt \ge \int_x^b W_1(t)W_2(t)dt \quad \textit{for all} \quad x \in [a,b]$$

then (5) *holds.*

If $g'(x) \ge 0$ for $x \in [a,b]$, g is convex on $[a,b]$ and (8) *is valid then* (R5) *holds.*

b) *If g' is a nonpositive symmetrical function on $[a,b]$ and is nondecreasing on $[\frac{a+b}{2}, b]$ and if*

$$(9) \qquad \int_{a+x}^{b-x} W_3(t)dt \ge \int_{a+x}^{b-x} W_1(t)W_2(t)dt \quad \textit{for all} \quad x \in [0, \frac{b-a}{2}]$$

then (5) *holds.*

If g' is a nonnegative symmetrical function on $[a,b]$ and is nonincreasing on $[\frac{a+b}{2}, b]$ and if (9) *holds then* (R5) *holds.*

c) *If g is concave on $[a, \frac{a+b}{2}]$, g' is nonpositive left balanced on $[a,b]$ and if*

$$(10) \qquad W_3(x) \ge W_1(x)W_2(x) \quad \textit{for all} \quad x \in [\frac{a+b}{2}, b]$$

and if (9) *holds, then* (5) *holds.*

If g' is a nonnegative right balanced function on $[a,b]$, g is convex on $[a,(a+b)/2]$ and if (10) *and* (9) *hold, then* (R5) *holds.*

d) *If g' is a nonpositive right balanced function on $[a,b]$, g is concave on $[a,(a+b)/2]$,* (R10) *and* (9) *hold, then* (5) *holds.*

If g' is a nonnegative left balanced function on $[a,b]$, g is convex on $[a,(a+b)/2]$, (R10) *and* (9) *hold, then* (R5) *holds.*

e) *If g is convex on $[(a+b)/2, b]$, g' is nonpositive right balanced on $[a,b]$ and if*

(11) $$W_3(x) \geq W_1(x)W_2(x) \quad \textit{for all} \quad x \in [a, \frac{a+b}{2}]$$

and if (9) *holds, then* (5) *holds.*

If g' is a nonnegative left balanced function on $[a,b]$, g is concave on $[(a+b)/2, b]$ and if (11) *and* (9) *hold, then* (R5) *holds.*

f) *If g' is a nonpositive left balanced function on $[a,b]$, g is convex on $[(a+b)/2, b]$,* (R11) *and* (9) *are valid, then* (5) *holds.*

If g' is a nonnegative right balanced function on $[a,b]$, g is concave on $[(a+b)/2, b]$, (R11) *and* (9) *are valid, then* (R5) *holds.*

g) *If g is a nonnegative nondecreasing function on $[a,b]$,*

$$W_3'(x) \leq (W_1W_2)'(x), \quad \textit{for} \quad x \in [\frac{a+b}{2}, b]$$

and if (R6) *holds then* (R5) *holds.*

The proof is based on several lemmas which are similar to lemmas 1 and 2 and are given in [2].

3. Applications on beta and incomplete gamma functions

Theorem 3. *Let a_1, a_2, a_3 and y be positive real numbers such that $a_1 + a_2 \geq a_3$ and $y \geq 2$. Then*

$$a_1a_2B(a_1,y)B(a_2,y) \leq a_3B(a_3,y),$$

where $B(x,y)$ is Beta function defined as $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$.

Proof. This theorem is a consequence of case a) from Th. 1. Namely, let us suppose that

$$w_i(t) = t^{a_i - 1}, \qquad t \in [0,1], \quad i = 1,2,3.$$

Then $W_i(t) = t^{a_i}$, $i = 1,2,3$ and for a_1, a_2 and a_3 such that $a_1 + a_2 \geq a_3$ inequality (4) holds. If we take $g(x) = (1-x)^{y-1}$, $y > 2$ then g' is nonpositive and g is convex on $[0,1]$ and we have $Q(g, w_i) = a_iB(a_i, y)$. Applying Th. 1a we obtain the above mentioned inequality for beta function. ◊

Remark 1. An inequality for gamma function is hidden in Th. 3. Namely, since $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ we have that

$$\frac{a_1a_2}{a_3}\frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(a_3)} \leq \frac{\Gamma(a_1+y)\Gamma(a_2+y)}{\Gamma(y)\Gamma(a_3+y)}$$

for $a_1 + a_2 \geq a_3 > 0$ and $y \geq 2$.

Remark 2. From Th. 1a we can deduce an inequality for incomplete gamma function $\gamma(x) = \int_0^1 t^{x-1}e^{-t}dt$ using the same functions w_i, $i = 1, 2, 3$, as in Th. 3, and putting $g(x) = e^{-x}$. In fact, we have that for $a_1 + a_2 \geq a_3 > 0$

$$a_1 a_2 \gamma(a_1)\gamma(a_2) \leq a_3 \gamma(a_3)$$

holds.

4. On the Jensen's type inequality

In the following theorem we get inequality of Jensen's type.

Theorem 4. *Let f be a concave function and g be a nonnegative nonincreasing function on $[a, b]$. If*

$$W_3 \leq \alpha W_1 f\left(\frac{W_2}{W_1} f^{-1}(\frac{1}{\alpha})\right) \quad on \quad [a, b], \tag{12}$$

then

$$Q(g, w_3) \leq \alpha Q(g, w_1) f\left(\frac{Q(g, w_2)}{Q(g, w_1)} f^{-1}(\frac{1}{\alpha})\right)$$

for every α such that there is a $\beta \in [a, b]$ that satisfies $\frac{1}{\alpha} = f(\beta)$.

Proof. Using integration by parts, condition (12) and Jensen's inequality for concave function we have

$$Q(g, w_3) = g(b) + \int_a^b W_3(x) d\bar{g}(x) \leq$$

$$\leq g(b) + \int_a^b \alpha W_1(x) f\left(\frac{W_2(x)}{W_1(x)} f^{-1}(\frac{1}{\alpha})\right) d\bar{g}(x) \leq$$

$$\leq g(b) + \alpha \int_a^b W_1(x) d\bar{g}(x) f\left(\frac{\int_a^b W_2(x) f^{-1}(\frac{1}{\alpha}) d\bar{g}(x)}{\int_a^b W_1(x) d\bar{g}(x)}\right) =$$

$$= \alpha g(b) f(f^{-1}(\frac{1}{\alpha})) + \alpha \int_a^b W_1(x) d\bar{g}(x) f\left(\frac{\int_a^b W_2(x) f^{-1}(\frac{1}{\alpha}) d\bar{g}(x)}{\int_a^b W_1(x) d\bar{g}(x)}\right) \leq$$

$$\leq \alpha \left(g(b) + \int_a^b W_1(x) d\bar{g}(x)\right) f\left(\frac{g(b) + \int_a^b W_2(x) d\bar{g}(x)}{g(b) + \int_a^b W_1(x) d\bar{g}(x)} f^{-1}(\frac{1}{\alpha})\right) =$$

$$= \alpha Q(g, w_1) f\left(\frac{Q(g, w_2)}{Q(g, w_1)} f^{-1}(\frac{1}{\alpha})\right),$$

where $\bar{g} = -g$. ◊

Remark 3. For the special case that $f(x) = x^{\frac{1}{p}}$, $p > 1$ we get from Th. 4 that when

$$W_3 \leq W_1^{1-\frac{1}{p}} W_2^{\frac{1}{p}}$$

holds, the inequality of Hölder's type

$$Q(g, w_3) \leq Q(g, w_1)^{1-\frac{1}{p}} Q(g, w_2)^{\frac{1}{p}}$$

is satisfied. This type of the inequality is discused in [2]. In that paper it is shown that for suitable choice of functions w_i, $i = 1, 2, 3$, we get a generalization of the so–called Gauss-Pólya inequality, [2],[8].

Remark 4. If we replace $f(x)$ by $(1 + x^{\frac{1}{p}})^p$, $p > 1$ and denote $\alpha^{\frac{1}{p}} = p_1$, $p_2 = 1 - p_1$ we get from

$$W_3 \leq \left(p_1 W_1^{\frac{1}{p}} + p_2 W_2^{\frac{1}{p}}\right)^p$$

the inequality of Minkowski's type

$$Q(g, w_3) \leq (p_1 Q(g, w_1)^{\frac{1}{p}} + p_2 Q(g, w_2)^{\frac{1}{p}})^p$$

which is also considered in [2].

Remark 5. Th. 4 is an analogue of Th. MPP [5], therefore so we can state results of Jensen's type similar to Ths. 1 and 2.

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