Mathematica Pannonica
10/2 (1999), 235–248

DIGITAL EXPANSIONS IN REAL ALGEBRAIC QUADRATIC FIELDS

Gábor **Farkas**

Department of Computer Algebra, Eötvös L. University, H–1117 Budapest, Pázmány Péter sétány I/D, Hungary

Dedicated to Professor Ferenc Schipp on his 60th birthday

Received: November 1998

MSC 1991: 11 R 11

Keywords: Number system, real quadratic extension field, digital expansion.

Abstract: I. Kátai proved that if I is the set of integers in some imaginary quadratic extension field the $\alpha \in I$ is a base of a number system with an appropriate digit set iff $\alpha \neq 0$, $|\alpha| \neq 1$, $|1-\alpha| \neq 1$. In this paper we attempt to arrive at a similar result in real quadratic extension fields of rational numbers.

1. Introduction

For an algebraic number β, let $\mathbb{Q}(\beta)$ be the number field that we can get by adjoining β to $\mathbb{Q}$, and let I be the set of integers in $\mathbb{Q}(\beta)$. Let $\alpha \in I$ and E_α ($\subseteq I$) be a complete residue system mod α containing 0, i.e. such a collection of $f_0 = 0, f_1, \dots, f_{t-1} \in I$ for which for every $\gamma \in I$ there exists a unique $f \in E_\alpha$ such that $\gamma = \alpha\gamma_1 + f$ with a suitable $\gamma_1 \in I$. Let $J : I \to I$ be defined by $J(\gamma) = \gamma_1$. Let us name this *transition* $\gamma \to \gamma_1$. J^k denotes the $k-fold$ iterate of J. We say that (α, E_α) is a *number system* in I if each $\gamma \in I$ can be written as a finite sum

(1.1) $$\gamma = e_0 + e_1\alpha + \cdots + e_k\alpha^k, \quad (e_i \in E_\alpha,\ i = 0, 1, \dots, k.)$$

The uniqueness of the representation follows from the fact that E_α is a complete residue system mod α.

The following questions are natural. (I) For a given α find such a digit system E_α, if any, for which (α, E_α) is a number system. (II) For a given α and digit set $E_c = \{0, 1, \dots, |N(\alpha)| - 1\}$ decide whether E_c is a complete residue system mod α and (α, E_α) is a number system or not.

(II) was solved for quadratic extension fields in [2], [3], [4] and [6]. In these it was shown that only very special numbers α can be served as bases with such special digit set.

With respect to problem (I) G. Steidl observed in [5] that in the case of Gaussian integers a good strategy for choosing an appropriate digit set is to take one for which $f \in E_\alpha \max |f|$ is close to the minimum. Later I. Kátai [1] proved that if I is the set of integers in some imaginary quadratic extension field then $\alpha \in I$ is a base of a number system with an appropriate digit set, iff $\alpha \neq 0, |\alpha| \neq 1, |1 - \alpha| \neq 1$. The concept of number systems can also be generalized as the group of lattices of the k-dimensional Euclidean space. In [8] an effective algorithm was presented to decide whether the system (M, A) is a number system or not for a given invertible expanding linear operator M and a given digit set A.

The case of real quadratic extension fields seems to be complicated due to the fact that the modules of the eigenvalues of the mapping corresponding to $\gamma \to \alpha\gamma$ are different. We shall concentrate our attention on the real quadratic extension of rational numbers denoted by $\mathbb{Q}(\sqrt{D})$. Let $D > 1$ be a square-free integer, and I be the set of integers in $\mathbb{Q}(\sqrt{D})$. For some $\beta \in \mathbb{Q}(\sqrt{D})$ let $\overline{\beta}$ be the algebraic conjugate of β. One can observe easily that if $\alpha \in I$ is a base of a number system in $\mathbb{Q}(\sqrt{D})$ with a suitable digit set then (a) $\alpha \neq 0$, (b) $\alpha \neq$ unit, (c) 1-$\alpha \neq$ unit, (d) $|\alpha| > 1, |\overline{\alpha}| > 1$. The assertions (a) and (b) are obvious. Assume that $1 - \alpha = \varepsilon$=unit and (α, E_α) is a number system. Let $f \in E_\alpha, f \neq 0, \gamma = f\overline{\varepsilon}\delta$, where $\delta = \varepsilon\overline{\varepsilon}$. Then $\gamma = f + \alpha\gamma$ and $\gamma \neq 0$, consequently γ cannot be expanded as (1.1). Assume that $|\alpha| < 1$and (α, E_α) is a number system. Then the set of γ having the finite representation (1.1) is bounded, while the whole set I is not bounded, which is a contradiction. Let us observe finally that (α, E_α) is a number system if and only if $(\overline{\alpha}, \overline{E}_\alpha)$ is a number system, where $\overline{E}_\alpha$ consists of the set of the algebraic conjugates of the elements of E_α. This implies that $|\overline{\alpha}| > 1$ is also necessary.

In [7] was proved that if $|\alpha| \geq 2, |\overline{\alpha}| \geq 2$, then there exists an E_α for which (α, E_α) is a number system. The digit set was explicitly computed.

Furthermore, let us assume that $|\alpha|, |\overline{\alpha}| > 1$. Then for each $\gamma \in I$ the path $\gamma, J(\gamma), J^2(\gamma), \dots$ is ultimately periodic. We say that $\pi \in I$ is a periodic element, if there exists a positive integer k such that $J^k(\pi) = \pi$. Let P be the set of periodic elements, and $G(P)$ be the directed graph getting by directing an edge from π to $J(\pi)$ for every $\pi \in P$. It is clear that $G(P)$ is a disjoint union of directed circles. Furthermore, (α, E_α) is a number system if and only if $P = \{0\}$. Another interesting problem is to find such a coefficient system E_α for which $G(P)$ has a simple structure.

2. Formulation of our results

Let $D > 1$ be a square-free integer, $\mathbb{Q}(\sqrt{D})$ be the extension field generated by $\sqrt{D}$, I be the set of the integers of $\mathbb{Q}(\sqrt{D})$. We classify the possible D-s according to

$$\textit{Case (A)}: \ D \not\equiv 1 \ (\mathrm{mod} 4), \qquad \textit{Case (B)}: \ D \equiv 1 \ (\mathrm{mod} 4).$$

It is known that$\{1, \sqrt{D}\}$ in Case (A), $\{1, \frac{\sqrt{D}+1}{2}\}$ in Case (B) are integral bases in I. Let $\varepsilon = \pm 1$, $\delta = \pm 1$ and for some $\alpha \in \mathbb{Q}(\sqrt{D})$ let $d = N(\alpha) = \alpha\bar{\alpha}$.

2.1. Choice of the digit set $\mathbf{E}_\alpha^{(\varepsilon,\delta)}$

In Case (A) let $\alpha = a + b\sqrt{D}$. Then let $E_\alpha^{(\varepsilon,\delta)}$ be the sets of those $f = k+l\sqrt{D}$, $k, l \in \mathbb{Z}$ for which $f\bar{\alpha} = (k+l\sqrt{D})(a-b\sqrt{D}) = (ka-blD)+ +(la-kb)\sqrt{D}=r+s\sqrt{D}$ satisfiy the following conditions: if $(\varepsilon, \delta) = (1, 1)$, then $r, s \in (-|d|/2, |d|/2]$, if $(\varepsilon, \delta) = (-1, -1)$, then $r, s \in [-|d|/2, |d|/2)$, if $(\varepsilon, \delta) = (-1, 1)$, then $r \in [-|d|/2, |d|/2)$, $s \in [-|d|/2, |d|/2)$, if $(\varepsilon, \delta) = (1, -1)$, then $r \in (-|d|/2, |d|/2]$, $s \in [-|d|/2, |d|/2)$.

In Case (B) we proceed in the same way except that here $\alpha = a+b\omega$, where $\omega = \frac{1+\sqrt{D}}{2}$. Then $\bar{\alpha} = a + b\bar{\omega} = a + \frac{b}{2} - \frac{b}{2}\sqrt{D} = a + b - b\omega$, thus, $f\bar{\alpha} = (a+b)k + bl\frac{1-D}{4} + (la - kb)\,\omega = r + s\omega$. It is known from number theory that $E_\alpha^{(\varepsilon,\delta)}$ is a complete residue system mod α. Since during the proof we never specify which value (ε, δ) will equal, therefore our proof will hold true for each $E_\alpha^{(\varepsilon,\delta)}$.

2.2. Exceptions

Let T be a subset of I defined here: $T = \pm\{1+2\sqrt{2}, 3+\sqrt{2}, 3+3\sqrt{2}, 4+4\sqrt{2}, 6+3\sqrt{2}, 2+2\sqrt{3}, 1+\sqrt{6}, 1+\sqrt{7}, 1+\sqrt{8}, 1+2\frac{\sqrt{13}+1}{2}, 2+3\frac{\sqrt{13}+1}{2}, 3+\frac{\sqrt{13}+1}{2}, 4+2\frac{\sqrt{13}+1}{2}, 4+4\frac{\sqrt{13}+1}{2}, 5+5\frac{\sqrt{13}+1}{2}, 6+6\frac{\sqrt{13}+1}{2}, 7+4\frac{\sqrt{13}+1}{2}, 8+5\frac{\sqrt{13}+1}{2}, 3+\frac{\sqrt{17}+1}{2}, 3+3\frac{\sqrt{17}+1}{2}, 5+2\frac{\sqrt{17}+1}{2}, 2+2\frac{\sqrt{21}+1}{2}, 4+\frac{\sqrt{29}+1}{2}, 1+\frac{\sqrt{33}+1}{2}, 1+\frac{\sqrt{37}+1}{2}, 1+\frac{\sqrt{41}+1}{2}, 1+\frac{\sqrt{45}+1}{2}, 1+4\frac{\sqrt{5}+1}{2}, 3+2\frac{\sqrt{5}+1}{2}\}$.

The argument which will be used in the next sections does not work for the elements of T. But it is easy to compute the coefficient systems and determine the set P for these numbers. We can observe that the assertion of the Th. will be valid in each case. The computed results are presented in the Appendix. Now we can state our theorem.

2.3. The theorem and the sketch of the proof

Theorem. *Let α be an arbitrary integer in $\mathbb{Q}(\sqrt{D})$ such that $\alpha, \alpha-1$ is not a unit, and $1 < \min(|\alpha|, |\overline{\alpha}|) < 2$, where $D > 1$ is a square-free integer. Let E_α be any of $E_\alpha^{(\varepsilon,\delta)}$, $(\varepsilon = \pm 1, \delta = \pm 1)$ a coefficient system defined above. Then $G(P)$ is the disjoint union of loops, if either $|\alpha| >$ $> 2 > |\overline{\alpha}|, \overline{\alpha} > 0$ or $|\overline{\alpha}| > 2 > |\alpha|, \alpha > 0$, and beside the loop $0 \to 0$ it contains only circles of order two of type $\pi \to (-\pi) \to \pi$, if either $|\alpha| > 2 > |\overline{\alpha}|, \overline{\alpha} < 0$ or $|\overline{\alpha}| > 2 > |\alpha|, \alpha < 0$ holds.*

Our proof is based on the investigation of the possible transitions. For the sake of the clarity of the proof in Subsect. 2.4 we discuss some general statements separately, which are often referred to throughout the paper. In Sect. 3 we show under what conditions the transitions are possible in an arbitrary circle of the periodic elements. In Sect. 4 we prove that the modules of the irrational parts of π and $J(\pi)$ are the same for each $\pi \in P$. In Sect. 5 we prove the same assertion for the rational parts of the modules of π and $J(\pi)$. This basically completes the proof of the teorem. In case D=5 the proof is slightly different, however the differences are presented at the appropriate places. Further we assume that the conditions of Theorem are valid in the whole paper and $\alpha \notin T$.

2.4. Preparing the proof

Remark 1. Observe that if d is an odd number, then we get the same digit set for arbitrary value of (ε, δ). In addition in Case (A) we can say that $E_{-\alpha}^{(\varepsilon,\delta)} = E_\alpha^{(-\varepsilon,-\delta)}$, because $f(-\bar{\alpha}) = -r - s\sqrt{D}$, and $\overline{E_\alpha^{(\varepsilon,\delta)}} = E_{\overline{\alpha}}^{(\varepsilon,-\delta)}$, because $\overline{f}\alpha = r - s\sqrt{D}$.

We get the same results in Case (B). Thus, if we have proved our assertions in case $|\alpha| > |\overline{\alpha}|$, we get the Th. in case $|\alpha| < |\overline{\alpha}|$ simply by reversing the roles of $|\alpha|, |\overline{\alpha}|$ and those of $E_\alpha^{(\varepsilon,\delta)}, E_{\overline{\alpha}}^{(\varepsilon,-\delta)}$. Further we assume that $|\alpha| > 2$ and $1 < |\overline{\alpha}| < 2$.

Remark 2. In Case (A) $|\pi| \leq (\frac{1}{2} + \frac{1}{2}\sqrt{D})\frac{|\alpha|}{|\alpha|-1}$ holds, where π is an arbitrary element of P. Then we can compute easily that $|\pi| < \sqrt{D}$, because $\frac{|\alpha|}{|\alpha|-1}$ decreases if $|\alpha|$ increases.

Proof. Assume that the absolute value of π is the maximum in P. If the cardinality of P is 1, then $\pi = 0$, in the opposite case there exists an π_2, for which $\pi_2 = \pi\alpha + f$, where $f \in A$. Thus, $\pi_2\overline{\alpha} = \pi d + f\overline{\alpha}$. We have two cases: If $|\pi_2\overline{\alpha}| = |\pi d| - |f\overline{\alpha}|$, then we get from the definitions $|\pi\overline{\alpha}| > |\pi d| - |f\overline{\alpha}| \geq |d|\big(|\pi| - (\frac{1}{2}+\frac{1}{2}\sqrt{D})\big)$ from this $\frac{|\pi|}{|\alpha|} > |\pi| - (\frac{1}{2}+\frac{1}{2}\sqrt{D})$. We get the proof, if we rearrange this inequality. If $|\pi_2\overline{\alpha}| = |f\overline{\alpha}| - |\pi d|$ from this $0 < (\frac{1}{2}+\frac{1}{2}\sqrt{D}) - |\pi|$, and this implies that $(\frac{1}{2}+\frac{1}{2}\sqrt{D}) > |\pi|$. ◊

In Case (B) we get with same calculation: $|\pi| \leq (\frac{1}{2} + \frac{1}{2}\omega)\frac{|\alpha|}{|\alpha|-1}$, therefore, $|\pi| < \omega$ if $\alpha \neq 2 + \frac{\sqrt{5}+1}{2}$. But for this number, $\alpha - 1$ would be a unit, therefore, we need not consider case $\alpha = 2 + \frac{\sqrt{5}+1}{2}$.

Remark 3. Since $|\alpha| > 2$ and $1 < |\overline{\alpha}| < 2$, therefore $a \neq 0$ and $b \neq 0$.

Remark 4. $\operatorname{sgn}(a) = \operatorname{sgn}(b)$.

Proof. Assume that $\operatorname{sgn}(a) \neq \operatorname{sgn}(b)$. Then in Case (A) $|\overline{\alpha}| = |a| + |b|\sqrt{D} < 2$ is impossible. In Case (B) $|\overline{\alpha}| = |a| + |b|(\frac{\sqrt{D}-1}{2}) < 2$ implies that $|a| = 1, D = 5$ and $|b| = 1$. But then $|\alpha| = \frac{\sqrt{D}+1}{2} - |a| < 2$, which is a contradiction. ◊

Remark 5. $\overline{\alpha} > 0$ implies that $\operatorname{sgn}(a) = \operatorname{sgn}(b) = \operatorname{sgn}(\alpha) = \operatorname{sgn}(d)$. $\overline{\alpha} < 0$ implies that $\operatorname{sgn}(a) = \operatorname{sgn}(b) = \operatorname{sgn}(\alpha) \neq \operatorname{sgn}(d)$.

Remark 6. In Case (A) $|a| + |b|\sqrt{D} < |d| < 2|a| + 2|b|\sqrt{D}$. In Case (B) $|a| + |b|\omega < |d| < 2|a| + 2|b|\omega$. In both cases $|d| > 3$ follows from the conditions of the Th.

Remark 7. Let $\pi \in P$, $\pi \neq 0$, $\pi = p + q\sqrt{D}$ or $\pi = p + q\omega$ according to cases (A),(B) respectively. Then $\operatorname{sgn}(p) = \operatorname{sgn}(\overline{\pi})$ and $\operatorname{sgn}(p) \neq \operatorname{sgn}(q)$ if $q \neq 0$.

Proof. Let us assume that $\operatorname{sgn}(p) = \operatorname{sgn}(q)$. Then $|\pi| = |p| + |q|\sqrt{D} \geq \sqrt{D}$ in Case (A). This is a contradiction, therefore $\operatorname{sgn}(p) \neq \operatorname{sgn}(q)$. Since $\operatorname{sgn}(p) = \operatorname{sgn}(-q)$, then $\operatorname{sgn}(p) = \operatorname{sgn}(p - q\sqrt{D}) = \operatorname{sgn}(\overline{\pi})$. In Case (B) we can proceed in a similar way. ◊

Remark 8. If $f \in E_\alpha$ then $|\overline{f}| < \sqrt{D} + 1$ in Case (A) and $|\overline{f}| < \omega$ in (B).

Proof. Case (A): $|\overline{f}\alpha| = |r - s\sqrt{D}| \leq \frac{\sqrt{D}+1}{2}|d|$, therefore we obtain $|\overline{f}| < \sqrt{D} + 1$. Case (B): $|\overline{f}\alpha| = |r + s\frac{1-\sqrt{D}}{2}| \leq \frac{\frac{\sqrt{D}-1}{2}+1}{2}|d|$, we get $|\overline{f}| \leq \frac{\omega}{2}|\overline{\alpha}|$, thus, $|\overline{f}| < \omega$.◊

Remark 9. If there exists such a $\pi_2 \to \pi$ transition, where $|\pi| > |\pi_2|$ and $q \neq 0$, then $|\pi| - |\pi_2| < \frac{\sqrt{D}+1}{2}$ in Case (A), and $|\pi| - |\pi_2| < \frac{\omega}{2}$ in (B).

Proof. Remark 7 and the proof of Remark 8 imply that $|\pi_2| = |\pi\overline{\alpha}| - |\overline{f}|$, therefore in Case (A) $\frac{\sqrt{D}+1}{2}|\overline{\alpha}| \geq |\overline{f}| = |\pi\overline{\alpha}| - |\pi_2|$ holds, and from this $\frac{\sqrt{D}+1}{2} \geq |\pi| - \frac{|\pi_2|}{2}$ thus, $\frac{\sqrt{D}+1}{2} > |\pi| - |\pi_2|$. In Case (B) $\frac{\frac{\sqrt{D}-1}{2}+1}{2}|\overline{\alpha}| \geq |\overline{f}| = |\pi\overline{\alpha}| - |\pi_2|$, from which we get $\frac{\frac{\sqrt{D}+1}{2}+1}{2} = \frac{\omega}{2} \geq |\pi| - \frac{|\pi_2|}{2}$ thus, $\frac{\omega}{2} > |\pi| - |\pi_2|$.◊

Remark 10. Let π_i be an irrational and π_j be a rational element in P. Then $|\overline{\pi}_i| > |\overline{\pi}_j|$.

Proof. In Case (A) let $\pi_j = p_j$, $\pi_i = p_i + q_i\sqrt{D}, q_i = 0$. Since $|\pi_j| = |\overline{\pi}_j| = |p_j| < \sqrt{D}$, and $|\overline{\pi}_i| = |p_i| + |q_i|\sqrt{D} \geq \sqrt{D}$, the assertion is true. In Case (B) let $\pi_j = p_j$ and $\pi_i = p_i + q_i\omega$ in . We have $|\pi_j| = |\overline{\pi}_j| = |p_j| < \omega$. $|p_i| > 0$, since $p_i = 0$ would imply that $|\pi_i| = |q_i\omega| \geq \omega$, which cannot hold, therefore, $|\overline{\pi}_i| \geq \omega - 1 + |p_i|$. Hence, we obtain immediately that $|\overline{\pi}_i| > |\overline{\pi}_j| > 0$.◊

Remark 11. Each circle that contains exclusively rational elements, has the shape $\pi \to \pi$ *or* $\pi \to -\pi \to \pi$.

Proof. (Indirect) Assume that there exists a circle each element of which is rational and let the absolute value of $\pi (\neq 0)$ be maximum in this circle. Let π_2 be defined by $\pi_2 \to \pi$. If $|\pi_2| = |\pi|$, then $\pi_2 = \pm\pi$, and we are ready. So we may assume that $|\pi_2| < |\pi|$. Then $\pi = p \neq 0$, $\pi_2 = p_2 \neq 0$ where p, $p_2 \in \mathbb{Z}$. Let $\pi_2 = \pi\alpha + f$, for some $f \in A$. We can write in Case (A): $p_2 = pa + bp\sqrt{D} + k + l\sqrt{D}$,we get from this: $k = p_2 - pa$ and $l = -bp$, thus, $r = ak - bDl = ap_2 - a^2p + b^2Dp = ap_2 - dp$. Assume that $\operatorname{sgn}(ap_2) \neq \operatorname{sgn}(dp)$. Then $|r| = |ap_2| + |dp| > |d|$. This implies that $p = p_2 = 0$, which is a contradiction, therefore, $\operatorname{sgn}(ap_2) = \operatorname{sgn}(dp)$. Remark 6 implies that $|d| > |a|$ and $|p| \geq |p_2|$, so we get that $|r| = |dp| - |ap_2| \geq |p_2||d| - |a||p_2| = |p_2|(|d| - |a|)$. The Case (B) can be proved in the same way. Remark 6 implies that in both (A) and (B) cases $|r| > |d|$. This is a contradiction, therefore we proved that $|\pi_2| = |\pi|$. It means that each element has same absolute value in the circle thus, we completed the proof of Remark 11.◊

Remark 12. The assertion of our theorem is true for the circle defined in Remark 11.

Proof. We get from $\pi_2 = \pi\alpha + f$ that

(1) $f = \pi(1 - \alpha) = n(1 - \alpha), n \in \mathbb{N}$ in case $\pi_2 = \pi$,

(2) $f = -\pi(\alpha + 1) = n(\alpha + 1), n \in \mathbb{N}$ in case $\pi_2 = -\pi$.

Let us consider the value of r in every case. Case (A): from (1) it follows that $r = n(a - d)$, and from (2) follows that $r = -n(a + d)$. Since $\operatorname{sgn}(a) = \operatorname{sgn}(\alpha)$ and $d = \alpha\overline{\alpha}$, therefore $\overline{\alpha} > 0$ in case (1) and $\overline{\alpha} < 0$ in case (2) otherwise $r > \frac{|d|}{2}$ would be valid. In Case (B) from (1) follows that $r = n(a + b - d)$, and from (2) follows that $r = -n(a + b + d)$. Since $\operatorname{sgn}(a) = \operatorname{sgn}(b) = \operatorname{sgn}(\alpha)$ and $d = \alpha\overline{\alpha}$, therefore $\overline{\alpha} > 0$ in case (1) and $\overline{\alpha} < 0$ in case (2).◊

In what follows, let us assume that each considered circle contains at least one irrational element. Now we state an important assertion.

Lemma 1. *Let π_1, π_2 be nonzero elements of P.*

(1) *Assume that $D \not\equiv 1 \bmod 4$, and $\pi_i = p_i + q_i\sqrt{D}$, $(i = 1, 2)$. Then $|q_1| > |q_2|$ implies that $|\overline{\pi}_1| > |\overline{\pi}_2|$.*

(2) *Assume that $D \equiv 1 \bmod 4$, $D \neq 5$, and $\pi_i = p_i + q_i\omega$, $(i = 1, 2)$. Then $|q_1| > |q_2|$ implies that $|\overline{\pi}_1| > |\overline{\pi}_2|$.*

Proof. Case (A): Let $|q_1| = |q_2| + n$, $n \in \mathbb{N}$. Assume in contrary that $|\overline{\pi}_1| \leq |\overline{\pi}_2|$. Since $|q_1| \neq |q_2|$, therefore $|\overline{\pi}_1| < |\overline{\pi}_2|$. Furthermore, $0 < |\overline{\pi}_2| - |\overline{\pi}_1| = (|p_2| - |p_1|) + (|q_2| - |q_1|)\sqrt{D} = |p_2| - |p_1| - n\sqrt{D}$, i.e. $|p_2| - |p_1| > n\sqrt{D}$. To prove that this is impossible, we have to investigate four cases: If $|p_1| > |q_1|\sqrt{D}$ & $|p_2| > |q_2|\sqrt{D}$ holds, then $|p_2| > |p_1| + n\sqrt{D} > |q_1|\sqrt{D} + n\sqrt{D} = |q_2|\sqrt{D} + 2n\sqrt{D}$, therefore, $|\pi_2| = |p_2| - |q_2|\sqrt{D} > 2n\sqrt{D}$. If $|p_1| > |q_1|\sqrt{D}$ & $|p_2| < |q_2|\sqrt{D}$ holds, then $|\pi_1| + |\pi_2| = |p_1| - |p_2| + n\sqrt{D} > 2n\sqrt{D}$. If $|p_1| < |q_1|\sqrt{D}$ & $|p_2| > |q_2|\sqrt{D}$ holds, then $|p_1| < |p_2|$ & $|p_1| > |p_2|$. If $|p_1| < |q_1|\sqrt{D}$ & $|p_2| < |q_2|\sqrt{D}$ holds, then $|p_2| < |p_1| - n\sqrt{D}$ and $|q_2|\sqrt{D} = |q_1|\sqrt{D} + n\sqrt{D}$ so we can write $|\pi_2| = |q_2|\sqrt{D} - |p_2| > |q_1|\sqrt{D} + n\sqrt{D} - |p_1| + n\sqrt{D} = |\pi_1| + 2n\sqrt{D}$. Observe that each of the four cases leads to a contradiction, therefore we have finished the proof of the Lemma 1 in Case (A).

In Case (B) we get similar results with similar computing, except for case $|p_1| > |q_1|\omega$ & $|p_2| < |q_2|\omega$. Since $|\overline{\pi}_1| - |\overline{\pi}_2| > 0$, therefore $|\overline{\pi}_1| - |\overline{\pi}_2| = |p_1| - |p_2| - n\omega + n$, and from this $|p_1| - |p_2| > n\omega - n$. We get that $|\pi_1| + |\pi_2| = |p_1| - |q_1|\omega - |p_2| + |q_2|\omega = |p_1| - |p_2| + n\omega$. Observe that if $n > 1$, then $|\pi_1| + |\pi_2| > 2\omega$, which is a contradiction, therefore we have to assume that $n = 1$. Then $|\pi_1| + |\pi_2| = |p_1| - |p_2| + \omega$. Since $|p_1| - |p_2| > \omega - 1$, therefore $|p_1| - |p_2| = [\omega]$. Thus, $|\pi_1| + |\pi_2| = |p_1| - |p_2| + \omega$ and by Remark 2 $(\omega + 1)\frac{|\alpha|}{|\alpha| - 1} > [\omega] + \omega$. We can compute easily that this is impossible if $D > 5$.◊

3. Investigation of the possible transitions

We shall investigate such periodic elements π, π_2 for which

(∗) $\qquad \pi_2 \to \pi$ and $|\overline{\pi}| > |\overline{\pi}_2|$.

In Case (A) Remark 2 and Remark 9 imply that

(3.1) $\quad ||\pi| - |\pi_2|| < \sqrt{D}, \qquad |\pi| + |\pi_2| < 2\sqrt{D}, \qquad |\overline{\pi}| - |\overline{\pi}_2| < \frac{\sqrt{D}+1}{2}$.

We can deduce some consequences of (3.1). If $|p| = |p_2|$ & $|q| = |q_2|$ then we get $|\overline{\pi}| = |\overline{\pi}_2|$, which contradicts (∗). Now assume that $|p| = |p_2|$ & $|q| \neq |q_2|$. If $|p| > |q|\sqrt{D}$ & $|p_2| > |q_2|\sqrt{D}$ or $|p| < |q|\sqrt{D}$ & $|p_2| < |q_2|\sqrt{D}$ is true, than $||\pi| - |\pi_2|| \geq \sqrt{D}$, which contradicts (3.1). If $|p| > |q|\sqrt{D}$ & $|p_2| < |q_2|\sqrt{D}$ and from this $|\overline{\pi}| - |\overline{\pi}_2| = |p| - |p_2| +$

$+(|q|-|q_2|)\sqrt{D}<0$, because $|q_2|>|q|$. This contradicts (*). If $|p|<|q|\sqrt{D}$ & $|p_2|>|q_2|\sqrt{D}$ is valid, then $|\overline{\pi}|-|\overline{\pi}_2|=(|q|-|q_2|)\sqrt{D}\geq\sqrt{D}$, which contradicts (3.1). In the end assume that $|p|\neq|p_2|$ & $|q|\neq|q_2|$, then we have to consider four cases. If $|p|>|p_2|$ & $|q|>|q_2|$ holds, then $|\overline{\pi}|-|\overline{\pi}_2|=|p|-|p_2|+(|q|-|q_2|)\sqrt{D}\geq\sqrt{D}+1$, which contradicts (3.1). If $|p|>|p_2|$ & $|q|<|q_2|$ holds, and if $|p|>|q|\sqrt{D}$ & $|p_2|>|q_2|\sqrt{D}$ or $|p|<|q|\sqrt{D}$ & $|p_2|<|q_2|\sqrt{D}$ is true, than $||\pi|-|\pi_2||=||p|-|p_2||+|(|q_2|-|q|)\sqrt{D}|\geq\sqrt{D}+1$, which contradicts (3.1). On the other hand if $|p|>|q|\sqrt{D}$ & $|p_2|<|q_2|\sqrt{D}$ is valid, then $|\overline{\pi}|-|\overline{\pi}_2|>0$, which implies that $|p|-|p_2|>|(|q_2|-|q|)|\sqrt{D}$, thus, $|p|-|p_2|>\sqrt{D}$ so $|\pi|+|\pi_2|=|p|-|p_2|+(|q_2|-|q|)\sqrt{D}>2\sqrt{D}$, which contradicts (3.1). If $|p|<|q|\sqrt{D}$ & $|p_2|>|q_2|\sqrt{D}$ then $|p|>|p_2|$ & $|p|<|p_2|$. This is impossible. If $|p|<|p_2|$ & $|q|>|q_2|$ holds, we get that if $|p|>|q|\sqrt{D}$ & $|p_2|>|q_2|\sqrt{D}$ then $|\pi_2|-|\pi|=|p_2|-|p|+(|q|-|q_2|)\sqrt{D}\geq\sqrt{D}+1$, which contradicts (3.1). If $|p|>|q|\sqrt{D}$ & $|p_2|<|q_2|\sqrt{D}$ then $|p|>|p_2|$ & $|p|<|p_2|$. This is impossible. If $|p|<|q|\sqrt{D}$ & $|p_2|>|q_2|\sqrt{D}$ then $|\pi|+|\pi_2|=|p_2|-|p|+(|q|-|q_2|)\sqrt{D}$, thus, $|q|=|q_2|+1$, otherwise, $|\pi|+|\pi_2|>2\sqrt{D}$ would be the consequence leading to a contradiction. If $|p|<|q|\sqrt{D}$ & $|p_2|<|q_2|\sqrt{D}$ then $|\pi|-|\pi_2|=|p_2|-|p|+(|q|-|q_2|)\sqrt{D}\geq 1+\sqrt{D}$, which contradicts (3.1). If $|p|<|p_2|$ & $|q|<|q_2|$ holds, then $|\overline{\pi}|-|\overline{\pi}_2|=|p|-|p_2|+(|q|-|q_2|)\sqrt{D}<0$, which contradicts (*).

In Case (B) we get a similar result for $D>5$. On the other hand, for $D=5$ if $|p|<|q|\omega$ & $|p_2|>|q_2|\omega$&$|p|<|p_2|$ & $|q|>|q_2|$ is valid, then $|\overline{\pi}|-|\overline{\pi}_2|=-m+\omega-1$, and from this $\omega>m+1\geq 2$, which is impossible. If we assume that $|q|=|q_2|$ and $|p|\neq|p_2|$, then $|\overline{\pi}|-|\overline{\pi}_2|=|p|-|p_2|\geq 1$, which is impossible because $|\overline{\pi}|-|\overline{\pi}_2|<\frac{\omega}{2}$. Unfortunately, we do not get a contradiction in either case $|p|>|p_2|$ & $|q|<|q_2|$ & $|p|>|q|\omega$ & $|p_2|<|q_2|\omega$, or case $|p|=|p_2|$ & $|q|=|q_2|+1$ & $|p|<|q|\omega$ & $|p_2|>|q_2|\omega$.

We can summarise our result as follows.

Lemma 2. *Assume that π,π_2 belong to P, $\pi_2\to\pi$ and $|\overline{\pi}|>|\overline{\pi}_2|$. Then the following assertion holds.*

(1) *In case $D\not\equiv 1$ (mod4) : if $\pi=p+q\sqrt{D},\pi_2=p_2+q_2\sqrt{D}$, then either $|p|\neq|p_2|$ & $|q|=|q_2|$, or $|q|=|q_2|+1$ & $0<|p|-|p_2|<\sqrt{D}$ & $|p|<|q|\sqrt{D}$ & $|p_2|>|q_2|\sqrt{D}$.*

(2) *In case $D\equiv 1$ (mod4) & $D>5$: if $\pi=p+q\omega,\pi_2=p_2+q_2\omega$ then either $|p|\neq|p_2|$ & $|q|=|q_2|$, or $|q|=|q_2|+1$ & $0<|p|-|p_2|<\omega-1$ & $|p|<|q|\omega$ & $|p_2|>|q_2|\omega$.*

(3) *In case* $D \equiv 1 \pmod 4$ & $D = 5$: *if* $\pi = p + q\omega, \pi_2 = p_2 + q_2\omega$ *then either* $|p| > |p_2|$ & $|q| < |q_2|$ & $|p| > |q|\omega$ & $|p_2| < |q_2|\omega$, *or* $|p| = |p_2|$ & $|q| = |q_2| + 1$ & $|p| < |q|\omega$ & $|p_2| > |q_2|\omega$.

4. Transitions, where $|q| \neq |q_2|$

Let us consider an arbitrary circle of periodical elements. We assume that there exists a transition $\pi_2 \to \pi$, where $|\overline{\pi}| > |\overline{\pi}_2|, |q| \neq |q_2|$ and $\pi, \pi_2 \neq 0$.

In Case (A) according to the assertions of Lemma 2 $\pi = p + q\sqrt{D}, \pi_2 = p_2 + q_2\sqrt{D}$ and $|q| = |q_2| + 1$ & $0 < |p| - |p_2| < \sqrt{D}$ & $|p| < |q|\sqrt{D}$ & $|p_2| > |q_2|\sqrt{D}$ is true. Then the next two relations are valid:

$$|\pi| + |\pi_2| = \sqrt{D} + m, \tag{4.1}$$

$$|\overline{\pi}| - |\overline{\pi}_2| = \sqrt{D} - m, \text{ where } m = |p_2| - |p|. \tag{4.2}$$

Then Remark 11 implies that $0 < m < \sqrt{D}$. Since $\pi_2 \to \pi$, there exists an $f \in E_\alpha$, for which $\pi_2 = \pi\alpha + f$. We will try to give an upper bound for $|\pi|$ and $|\overline{\pi}| - |\overline{\pi}_2|$. Then we have to investigate two cases. If $|f\overline{\alpha}| = |r| + |s|\sqrt{D}$, then $|\overline{f}\alpha| = ||r| - |s|\sqrt{D}|$ and so $|\overline{f}| = |\overline{\pi}\overline{\alpha}| - |\overline{\pi}_2|$. Multiplying by α, we get $|\overline{f}\alpha| = ||r| - |s|\sqrt{D}| = |\overline{\pi}d| - |\overline{\pi}_2\alpha|$ and then $\frac{||r|-|s|\sqrt{D}|}{|d|} > |\overline{\pi}| - |\overline{\pi}_2|$. Furthermore, if $|\pi_2| = |\pi\alpha| + |f|$ is valid, then $|\pi_2| \geq |\pi\alpha|$. But (4.2) implies that $|\pi|, |\pi_2| > m$ and from this we get that $|\pi_2| > |\alpha| \geq \sqrt{D} + 1$. This is impossible. If $|\pi_2| = |\pi\alpha| - |f|$ is valid, we get that $|f\overline{\alpha}| = |\pi d| - |\pi_2\overline{\alpha}|$, and we get from this $\frac{|r|+|s|\sqrt{D}+|\pi_2||\overline{\alpha}|}{|d|} \geq |\pi|$. Finally, if $|\pi_2| = |f| - |\pi\alpha|$, then $\frac{|r|+|s|\sqrt{D}-|\pi_2\overline{\alpha}|}{|d|} \geq |\pi|$. Thus, $|\pi| \leq \frac{|r|+|s|\sqrt{D}+|\pi_2||\overline{\alpha}|}{|d|}$. So we have got that:

$$|\pi| + |\overline{\pi}| - |\overline{\pi}_2| < \frac{|r|+|s|\sqrt{D}+|\pi_2||\overline{\alpha}|}{|d|} + \frac{||r|-|s|\sqrt{D}|}{|d|}. \tag{4.3}$$

If $|f\overline{\alpha}| = ||r| - |s|\sqrt{D}|$, then $|\overline{f}\alpha| = |r| + |s|\sqrt{D}$ and we get with a similar computation that:

$$|\pi| + |\overline{\pi}| - |\overline{\pi}_2| < \frac{||r|-|s|\sqrt{D}|+|\pi_2||\overline{\alpha}|}{|d|} + \frac{|r|+|s|\sqrt{D}}{|d|}. \tag{4.4}$$

Observe that (4.3) and (4.4) yield the same results: If $|r| > |s|\sqrt{D}$, then $|\pi| + |\overline{\pi}| - |\overline{\pi}_2| < \frac{2|r|+|\pi_2||\overline{\alpha}|}{|d|} \leq 1 + \frac{|\pi_2|}{|\alpha|}$. If $|r| < |s|\sqrt{D}$, then $|\pi| + |\overline{\pi}| - |\overline{\pi}_2| < \frac{2|s|\sqrt{D}+|\pi_2||\overline{\alpha}|}{|d|} \leq \sqrt{D} + \frac{|\pi_2|}{|\alpha|}$. Since $\sqrt{D} + \frac{|\pi_2|}{|\alpha|} > 1 + \frac{|\pi_2|}{|\alpha|}$ it is enough for us to consider the latter case. Then, from (4.2) it follows that $|\pi| + \sqrt{D} - m < \sqrt{D} + \frac{|\pi_2|}{|\alpha|}$ and

$$|\pi| - m < \frac{|\pi_2|}{|\alpha|}. \tag{4.5}$$

Assume that $|r| < |s|\sqrt{D}$ is valid. Then we get from (4.1) and (4.5) that $|\pi| - m \geq \sqrt{D} - |\pi_2|$. If $\sqrt{D} - |\pi_2| \geq \frac{|\pi_2|}{|\alpha|}$ then $|\pi| - m \geq \frac{|\pi_2|}{|\alpha|}$. This contradicts (4.5). Thus, we have to find a K_D for each D, such that $|\pi_2| \leq K_D$ and $\sqrt{D} - K_D \geq \frac{K_D}{|\alpha|}$ would be valid. If it will be successfull, then we get that the considered transition does not exist!

Thus, let $K_D = \sqrt{D}\frac{|\alpha|}{|\alpha|+1}$. Then $\sqrt{D} - K_D = \frac{K_D}{|\alpha|}$. From Remark 2 it follows that K_D is a suitable upper bound, if $K_D \geq \frac{\sqrt{D}+1}{2}\frac{|\alpha|}{|\alpha|-1}$. This is true, if $\sqrt{D}\frac{|\alpha|}{|\alpha|+1} \geq \frac{\sqrt{D}+1}{2}\frac{|\alpha|}{|\alpha|-1}$. We can check with simple computing that this is always true except the cases where $\alpha \in T_D$. We got that $\pi_2 \to \pi$ does not exist in Case (A).

In Case (B) let $\pi = p + q\omega, \pi_2 = p_2 + q_2\omega$. If $D > 5$, then we can proceed in the same way, except that here $K_D = \left(\omega - \frac{1}{2}\right)\frac{|\alpha|}{|\alpha|+1}$. Next, we assume that $D = 5$, then $|p| > |p_2|$ & $|q| < |q_2|$ & $|p| > |q|\omega$ & $|p_2| < < |q_2|\omega$, where $\omega = \frac{\sqrt{5}+1}{2}$. We can compute easily from Remark 2 that for arbitrary $\pi \in P : |\pi| < \omega - 0.1$ if $\alpha \notin T$, therefore, we get that $|\pi| + |\pi_2| = |p| - |p_2| + (|q_2| - |q|)\,\omega = \omega + 1$ from which $|\pi|, |\pi_2| > 1.1$ follows, because $0 < |p| - |p_2| < \omega$. In addition we get that $|\overline{\pi}| - |\overline{\pi}_2| = = |p| - |p_2| + (|q| - |q_2|)\,(\omega - 1) = 1 - (\omega - 1) = 2 - \omega$. We have to consider four cases.

If $|\overline{f}\alpha| = |r| - |s|\omega - 1)$ let us assume that $|s| \geq \frac{|d|}{4}$ then $2 - -\omega = |\overline{\pi}| - |\overline{\pi}_2| < \frac{|r|-|s|(\omega-1)}{|d|} \leq \frac{\frac{|d|}{2} - \frac{|d|}{4}(\omega-1)}{|d|} = \frac{3-\omega}{4} < 2 - \omega$, which is a contradiction, therefore $|s| < \frac{|d|}{4}$. But then $|\pi| \leq \frac{|r|+|s|\omega}{|d|} + \frac{|\pi_2|}{|\alpha|} \leq \leq \frac{\frac{|d|}{2} + \frac{|d|}{4}\omega}{|d|} + \frac{|\pi_2|}{|\alpha|} = \frac{2+\omega}{4} + \frac{|\pi_2|}{|\alpha|}$. Since $|\alpha| > 7.8$ and $|\pi_2| \leq \omega - 0.1$ we get that $|\pi| < \frac{2+\omega}{4} + \frac{|\pi_2|}{|\alpha|} < 1.1$.

If $|\overline{f}\alpha| = |s|\,(\omega - 1) - |r|$ then $2 - \omega = |\overline{\pi}| - |\overline{\pi}_2| < \frac{|s|(\omega-1)-|r|}{|d|} \leq \leq \frac{|s|(\omega-1)}{|d|} \leq \frac{\frac{|d|}{2}(\omega-1)}{|d|} = \frac{\omega-1}{2} < 2 - \omega$, which is impossible.

If $|f\overline{\alpha}| = |r| - |s|\,(\omega - 1)$ then $|\pi| \leq \frac{|r|}{|d|} + \frac{|\pi_2|}{|\alpha|} \leq \frac{1}{2} + \frac{|\pi_2|}{|\alpha|} < 1.1$.

If $|f\overline{\alpha}| = |s|\,(\omega - 1) - |r|$ then $|\pi| \leq \frac{|s|\omega}{|d|} + \frac{|\pi_2|}{|\alpha|} \leq \frac{\omega}{2} + \frac{|\pi_2|}{|\alpha|} < 1.1$.

We arrive at contradictions in all cases, because $|\pi| > 1.1$, therefore, we can say that transition $\pi_2 \to \pi$ does not exist. We can summarize our results as follows:

Lemma 3. *For arbitrary $\pi_2 \to \pi$ transition, if $|\overline{\pi}| > |\overline{\pi}_2|, |q| \neq |q_2|$ and $\pi, \pi_2 \neq 0$ are valid, then* $D = 5, |p| = |p_2|$ & $|q| = |q_2| + 1$ & $|p| < < |q|\omega$ & $|p_2| > |q_2|\omega$, *where $\pi = p + q\omega, \pi_2 = p_2 + q_2\omega$.*

5. The completion of the proof of the Theorem

Let us consider an arbitrary circle, and assume that the following conditions are valid. (i) π is the element whose conjugate has the maximal absolute value in the circle. (ii) Assume that there exist π_1, π_2 in the circle such that $\pi_2 \to \pi \to \pi_1$ & $|\pi| \neq |\pi_1|, |\pi_2|$.

First, assume that $D \neq 5$. We know that $|q| \geq |q_1|, |q_2|$ because of Lemma 1 and (i). Lemma 3 implies that $|q_2| = |q|$. Now, assume that $|q_1| < |q|$. Then there must be a transition $\pi_x \to \pi_y$ in the circle such that $|q_x| < |q_y|$ and then $|\overline{\pi}_x| < |\overline{\pi}_y|$ follows from Lemma 1. But according to the Lemma 3 such kind of transition does not exist, therefore, we can observe that $|q| = |q_1| = |q_2|$. Thus, in Case (A), we can describe these two transitions with two equations for some $f, f_1 \in E_\alpha$:

(5.1) $\pi_2 = \pi\alpha + f$, i.e. $p_2 + q_2\sqrt{D} = \left(p + q\sqrt{D}\right)\left(a + b\sqrt{D}\right) + k + l\sqrt{D}$,

(5.2) $\pi = \pi_1\alpha + f_1$, i.e. $p + q\sqrt{D} = \left(p_1 + q_1\sqrt{D}\right)\left(a + b\sqrt{D}\right) + k_1 + l_1\sqrt{D}$.

Depending on the sign of $\overline{\alpha}$ and p, we can write the following four equation systems, where $\pi = p + q\sqrt{D}$ and $m, z > 0$ integers:

(5.3) If $\overline{\alpha} > 0$ & $p < 0$ then $\pi_2 = p + m + q\sqrt{D}$ and $\pi_1 = p + z + q\sqrt{D}$.

(5.4) If $\overline{\alpha} > 0$ & $p > 0$ then $\pi_2 = p - m + q\sqrt{D}$ and $\pi_1 = p - z + q\sqrt{D}$.

(5.5) If $\overline{\alpha} < 0$ & $p < 0$ then $\pi_2 = -p - m - q\sqrt{D}$ and $\pi_1 = -p - z - q\sqrt{D}$.

(5.6) If $\overline{\alpha} < 0$ & $p > 0$ then $\pi_2 = -p + m - q\sqrt{D}$ and $\pi_1 = -p + z - q\sqrt{D}$.

From (5.1), (5.2) and (5.3) we get $p + m + q\sqrt{D} = (p + q\sqrt{D})(a + b\sqrt{D}) + f$ and $p + q\sqrt{D} = (p + z + q\sqrt{D})(a + b\sqrt{D}) + f_1$. From it follows $k - k_1 = m + az$ and $l - l_1 = zb$. Using $r = ak - bDl = a(k_1 + m + az) - bD(l_1 + zb)$ we obtain that $r = r_1 + dz + am$ in Case (A).

We will receive some results in the other cases with the same computing. In Case (A) we have from (5.4) $r = r_1 - dz - am$, from (5.5) $r = -r_1 + dz - am$ and from (5.6) $r = -r_1 - dz + am$. We can see that all these four cases imply that $|r| > \frac{|d|}{2}$, which is a contradiction. In Case (B) we get the same result. Thus, we got that there may exist only such transitions where $|\pi| = |\pi_1| = |\pi_2|$.

Next, we assume that $D = 5$. We have already proved in Sect. 4 that $|q| < |q_2|$ will never be true, therefore, we have to consider only the next case: $D = 5, |p| = |p_2|$ & $|q| = |q_2| + 1$ & $|p| < |q|\omega$ & $|p_2| > |q_2|\omega$. For some $f, f_1 \in E_\alpha$:

(5.7) $\pi_2 = \pi\alpha + f$, in detail: $p_2 + q_2\omega = (p + q\omega)(a + b\omega) + k + l\omega$.

(5.8) $\pi = \pi_1\alpha + f_1$, in detail: $p + q\omega = (p_1 + q_1\omega)(a + b\omega) + k_1 + l_1\omega$.

Determine k and l, where $f = k + l\omega$. Assume that $p < 0$, and $\overline{\alpha} > 0$. Then we get from (5.7) that $p_2 + q_2\omega = p + q\omega - \omega = (p + q\omega)(a + b\omega) +$

$+k+l\omega$. We get that $k=p\,(1-a)-\frac{bq}{4}D+\frac{qb}{4}$ and $l=q\,(1-a)-b\,(q+p)-1$. What can we say about the transition $\pi\to\pi_1$?

Assertion 1. $|\overline{\pi}|-|\overline{\pi}_1|<\omega$.

Proof. We know that $|\overline{\pi}|\geq\omega$ and Remark 8 imply that $|\overline{f}|<\omega$. We get from this that either $|\overline{\pi}|=|\overline{\pi}_1\overline{\alpha}|-|\overline{f}_1|$ or $|\overline{\pi}|=|\overline{\pi}_1\overline{\alpha}|+|\overline{f}_1|$. Since $|\overline{\pi}_2|=|\overline{\pi}\overline{\alpha}|-|\overline{f}|$, therefore, $|\overline{\pi}\overline{\alpha}|-|\overline{f}|<|\overline{\pi}_1\overline{\alpha}|+|\overline{f}_1|$. We get from this that $|\overline{\pi}|-|\overline{\pi}_1|<\frac{|\overline{f}|+|\overline{f}_1|}{|\overline{\alpha}|}\leq\frac{2|r|+2|s|(\omega-1)}{|d|}\leq\omega.\lozenge$

We saw in Sect. 4 that if $|q|<|q_1|$, then the transition $\pi\to\pi_1$ does not exist. Thus we can see that $|q|\geq|q_1|$. Therefore, there are two cases:

Assertion 2. *There does not exist such a transition* $\pi\to\pi_1$ *where* $|q|>|q_1|$.

Proof. Assume that $|q|=|q_1|+n$, where $n>0$ is an integer. Observe that $0<|\overline{\pi}|-|\overline{\pi}_1|=|p|-|p_1|+(|q|-|q_1|)(\omega-1)=|p|-|p_1|+n(\omega-1)<\omega$ and this implies that $|p_1|\geq|p|$, and in this case either $|\pi_1|=|q|\omega-\ -n\omega-|p_1|$ and from this $|\pi_1|+|\pi_2|=|p|-|p_1|+\omega\,(1-n)\leq 0$, or $|\pi_1|=|p_1|-|q|\omega+n\omega$ and from this $|\pi_1|+|\pi|=|p_1|-|p|+n\omega\geq n\omega$ thus, $n=1$ is valid. Thus, we have got that $|\pi_1|=|\pi_2|$.

Now we shall calculate the value of k_1 and l_1 from (5.8) that $p+q\omega=\ =(p+q\omega-\omega)\,(a+b\omega)+k_1+l_1\omega$. We obtain that $k_1=p\,(1-a)-\frac{bq}{4}D+\ +\frac{qb}{4}+\frac{b}{4}D-\frac{b}{4}$ and $l_1=q\,(1-a)+a-b\,(q+p)+b$. Now we can compute s from $k-k_1=-\frac{b}{4}D+\frac{b}{4}=\frac{1-D}{4}$ and $l-l_1=-a-b-1$. Thus, $s=al-bk=\ =a\,(l_1-a-b-1)-b\left(k_1+\frac{1-D}{4}\right)=al_1-a^2-ab-a-bk_1-b^2\frac{1-D}{4}$. Observe that $d=a^2+ab+b^2\frac{1-D}{4}$. This implies that $s=s_1-d-a$. Since $\overline{\alpha}>0$, therefore, $|s|>\frac{|d|}{2}$. This is a contradiction. We arrive at the same result even if we make this deducing in all cases depending on sign of $\overline{\alpha}$ and $p.\lozenge$

Assertion 3. *If* $|q|=|q_1|$ *then* $|\pi|=|\pi_1|$.

Proof. In this case $|p|=|p_1|+1$, since Ass. 1 implies that $0<|\overline{\pi}|-\ -|\overline{\pi}_1|=|p|-|p_1|<\omega$. We can calculate the value of k_1 and l_1 in the same way that we used in the proof of Ass. 2. Then, from $p+q\omega=\ =(p+1+q\omega)(a+b\omega)+k_1+l_1\omega$. We get that $k_1=p\,(1-a)-a-\frac{bq}{4}D+\frac{qb}{4}$ and $l_1=q\,(1-a)-b\,(q+p)-b$. Now we can compute r. Since $k-k_1=a$ and $l-l_1=b-1$ then $r=(a+b)\,(k_1+a)+b\frac{1-D}{4}\,(l_1+b-1)=\ =(a+b)\,k_1+a^2+ab+b\frac{1-D}{4}l_1+b^2\frac{1-D}{4}+b$. $d=a^2+ab+b^2\frac{1-D}{4}$ implies that $r=r_1+d+b$. Since $\overline{\alpha}>0$, therefore $|r|>\frac{|d|}{2}$. This is a contradiction. We reach the same result even if we make this deducing in all cases depending on sign of $\overline{\alpha}$ and $p.\lozenge$

We have got a contradiction in all the cases of this subsection, thus this way so we have proved the following assertion.

Lemma 4. *Each node of an arbitrary circle has the same absolute value.*
Corollary. $\overline{\alpha} > 0$ *implies that each circle is a loop,* $\overline{\alpha} < 0$ *implies that each circle either consists of the loop* 0, *or contains a number and its negative.*
Proof. Assume that $\overline{\alpha} > 0$. Then from $\overline{\pi}_2 = \overline{\pi}\overline{\alpha} + \overline{f}$ we get that $\overline{f} = \overline{\pi}_2 - \overline{\pi}\overline{\alpha}$. Observe that if $\mathrm{sgn}\,(\overline{\pi}) \neq \mathrm{sgn}\,(\overline{\pi}_2)$, then $|\overline{f}| = |\overline{\pi}_2| + |\overline{\pi}\overline{\alpha}| > \omega+1$, which is impossible. Thus, we have got that $\mathrm{sgn}\,(\overline{\pi}) = \mathrm{sgn}\,(\overline{\pi}_2) = \mathrm{sgn}\,(p) = \mathrm{sgn}\,(p_2)$, which implies that $\mathrm{sgn}\,(\pi) = \mathrm{sgn}\,(\pi_2)$ and according to the Lemma 4 $\pi = \pi_2$. In this case, naturally, $\pi = \pi_1$ is valid too. With similar computation, we can get that $\overline{\alpha} < 0$, which implies that $\pi = -\pi_2$. Observe that in this case $\pi = -\pi_1$ will be valid, and then $\pi_1 = \pi_2$. Thus, we have finished the proof of our Theorem.◊

6. Appendix

Here are presented the elements of T, the suitable digit sets. If $P \neq \{0\}$, then P and the transitions belonging to it are also provided. Let $\left(a + b\frac{\sqrt{D}+1}{2}\right)$ and $\left(a + b\sqrt{D}\right)$denoted by (a, b).

D = 2. **(1)** $\alpha=\pm(1,2)$, $\pm E_\alpha=\pm\{(1,1),\ (2,1),\ 1\} \cup \{0\}$. **(2)** $\alpha=\pm(3,1)$, $\pm E_\alpha = \pm\{(1,1),\ (0,1),\ 1\} \cup \{0\}$. **(3)** $\alpha = \pm(3,3)$, $\pm E_\alpha = \pm\{(1,1),\ (2,1),\ (3,2),\ 1\} \cup \{0\}$, $\pm P = \{0,\ \pm(1,-1)\}$. The transitions: $(1,-1) = (-1,1)\alpha + (-2,-2)$, $(-1,1) = (1,-1)\alpha+(2,2)$ **(4)** $\alpha = \pm(4,4)$, $\pm E_\alpha = \pm\{(1,1),\ (2,1),\ 1,\ (3,2)\} \cup \{0,\ (-3,-1),\ (-4,-2),\ (-2,-2),\ (-5,-3),\ (-4,-3),\ (-6,-4),\ (0,-1)\}$. **(5)** $\alpha = \pm(6,3)$, $\pm E_\alpha = \pm\{(1,1),\ (2,1),\ 1,\ (3,2),\ (4,3),\ (0,1),\ (2,2)\} \cup \{0,\ (3,3),\ (1,2),\ (5,4)\}$.

D=3. **(6)** $\alpha=\pm(2,2)$, $\pm E_\alpha=\{(2,1),1\}\cup\{0,(-3,-1),(-1,-1),(-4,-2)\}$.

D = 6. **(7)** $\alpha=\pm(1,1)$, $\pm E_\alpha=\pm\{1,\ 2\} \cup \{0\}$, $\pm P=\{0,\ \pm(2,-1)\}$. The transitions: $(2,-1) = (-2,1)\alpha - 2$, $(-2,1) = (2,-1)\alpha + 2$.

D = 7. **(8)** $\alpha = \pm(1,1)$, $\pm E_\alpha = \pm\{1,\ 2\} \cup \{0,\ (-4,-1)\}$.

D = 8. **(9)** $\alpha = \pm(1,1)$, $\pm E_\alpha = \pm\{1,\ 2,\ 3\} \cup \{0\}$.

D = 13. **(10)** $\alpha = (1,2)$, $\pm E_\alpha = \pm\{(1,0),\ (2,1),\ (1,1),\ (3,2)\} \cup \{0\}$. **(11)** $\alpha = (2,3)$, $\pm E_\alpha = \pm\{(1,0),\ (2,1),\ (1,1),\ (3,2),\ (2,2),\ (4,3),\ (3,1),\ (4,2)\}\cup\{0\}$. **(12)** $\alpha = (3,1)$, $\pm E_\alpha = \pm\{(1,0),\ (0,1),\ (1,1),\ (2,2)\}\cup\{0\}$. **(13)** $\alpha = (4,2)$, $\pm E_\alpha = \pm\{(2,2),\ (1,1)\} \cup \{0,\ (3,3),\ (0,-1),\ (1,0),\ (2,1),\ (3,2),\ (4,3),\ (5,4)\}$, $\pm P = \{0,\ (-1,0),\ (-2,1),\ (1,-1)\}$. The transitions: $\pm(-1,0) = \pm(-1,0)\alpha \pm (3,2)$, $\pm(-2,1) = \pm(-2,1)\alpha \pm (0,-1)$, $\pm(1,-1) = \pm(1,-1)\alpha \pm (3,3)$. **(14)** $\alpha = (4,4)$, $\pm E_\alpha = \pm\{(2,1),\ (1,1),\ (4,3),\ (3,2)\} \cup \{0,\ (-8,-6),\ (-2,-2),\ (1,0),\ (-5,-3),\ (-6,-4),\ (-7,-5),\ (-5,-4)\}$, $\pm P = \{0,\ \pm(-2,1)\}$. The transitions: $(-2,1) = (2,-1)\alpha + (2,1)$, $(2,-1) = (-2,1)\alpha + (-2,-1)$. **(15)** $\alpha = (5,5)$, $\pm E_\alpha = \pm\{(1,0),\ (2,1),\ (1,1),\ (3,2),\ (2,2),\ (4,3),\ (8,6),\ (4,2),\ (5,4),\ (7,5),\ (5,3),\ (6,4)\} \cup \{0\}$. **(16)** $\alpha = (6,6)$, $\pm E_\alpha = \pm\{(1,0),\ (2,1),$

$(1,1)$, $(3,2)$, $(2,2)$, $(4,3)$, $(8,6)$, $(4,2)$, $(5,4)$, $(7,5)$, $(5,3)$, $(6,4)\}\cup\{0$, $(-7,-4)$, $(-8,-5)$, $(-10,-7)$, $(-11,-8)$, $(3,1)$, $(0,-1)$, $(-9,-6)$, $(-3,-3)$, $(-6,-5)$, $(-9,-7)$, $(-12,-9)$, $\}$. **(17)** $\alpha=(7,4)$, $\pm E_\alpha=\pm\ \{(1,0)$, $(2,1)$, $(1,1)$, $(3,2)$, $(2,2)$, $(4,3)$, $(4,4)$, $(5,4)$, $(7,6)$, $(3,3)$, $(6,5)$, $(2,3)$, $(1,2)$, $(0,1)\}\cup\{0\}$. **(18)** $\alpha=(8,5)$, $\pm E_\alpha=\pm\{(1,0)$, $(2,1)$, $(1,1)$, $(3,2)$, $(2,2)$, $(4,3)$, $(4,4)$, $(5,4)$, $(7,6)$, $(3,3)$, $(6,5)$, $(8,6)$, $(9,7)$, $(10,8)\}\cup\{0\}$.

D = 17. **(19)** $\alpha=(3,1)$, $\pm E_\alpha=\pm\{(1,1)\}\cup\{0$, $(2,2)$, $(0,-1)$, $(1,0)$, $(2,1)$, $(3,2)\}$, $\pm P=\{0$, $(-1,0)$, $(-2,1)$, $(1,-1)\}$. The transitions: $\pm(-1,0)=\pm(-1,0)\alpha\pm(2,1)$, $\pm(-2,1)=\pm(-2,1)\alpha\pm(0,-1)$, $\pm(1,-1)=\pm(1,-1)\alpha\pm(2,2)$. **(20)** $\alpha=(3,3)$, $\pm E_\alpha=\pm\{(1,0)$, $(2,1)$, $(1,1)$, $(3,2)$, $(5,3)$, $(4,2)$, $(3,1)\}\cup\{0$, $(-7,-4)$, $(-5,-2)$, $(-6,-3)\}$. **(21)** $\alpha=(5,2)$, $\pm E_\alpha=\pm\{(1,0)$, $(2,1)$, $(1,1)$, $(3,2)$, $(2,2)$, $(4,3)$, $(3,3)$, $(1,2)$, $(0,1)\}\cup\{0\}$.

D = 22. **(22)** $\alpha=(2,2)$, $\pm E_\alpha=\pm\{(1,0)$, $(2,1)$, $(4,2)$, $(3,1)\}\cup\{0$, $(-1,-1)$, $(-5,-2)$, $(-6,-3)\}$.

D = 29. **(23)** $\alpha=(4,1)$, $\pm E_\alpha=\pm\{(1,0)$, $(2,1)$, $(1,1)$, $(3,2)$, $(2,2)$, $(4,2)\}\cup\{0\}$.

D = 33. **(24)** $\alpha=(1,1)$, $\pm E_\alpha=\pm\{(1,0)$, $(2,1)\}\cup\{0$, $(-4,-1)\}$.

D = 37. **(25)** $\alpha=(1,1)$, $\pm E_\alpha=\pm\{(1,0)$, $(3,1)$, $(4,1)\}\cup\{0\}$.

D = 41. **(26)** $\alpha=(1,1)$, $\pm E_\alpha=\pm\{(1,0)$, $(4,1)\}\cup\{0$, $(2,0)$, $(-5,-1)$, $(-3,-1)\}$.

D = 45. **(27)** $\alpha=(1,1)$, $\pm E_\alpha=\pm\{(1,0)$, $(4,1)$, $(5,1)$, $(2,0)\}\cup\{0\}$.

D = 5. **(28)** $\alpha=(1,4)$, $\pm E_\alpha=\pm\{(1,0),(0,1),(1,2),(2,0),(1,1)\}\cup\{0\}$. **(29)** $\alpha=(3,2)$, $\pm E_\alpha=\pm\{(1,0)$, $(0,1)$, $(1,2)$, $(2,0)$, $(1,1)\}\cup\{0\}$.

References

[1] KÁTAI, I.: Number systems in imaginary quadratic fields, *Annales Univ. Sci. Budapest., Sect. Comp.* **14** (1994), 159–164.

[2] KÁTAI, I. and SZABÓ, J.: Canonical number systems for complex integers, *Acta Sci. Math.* **37** (1975), 255–260.

[3] KÁTAI, I. and KOVÁCS, B.: Kanonische Zahlensysteme bei reelen quadratischen algebraischen Zahlen, *Acta Sci. Math.* **42** (1980), 99–107.

[4] KÁTAI, I. and KOVÁCS, B.: Canonical number systems in imaginary quadratic fields, *Acta Math. Hung.* **37** (1981), 159–164.

[5] STEIDL G.: On symmetric representation of Gaussian integers, *BIT* **29** (1989), 563–571.

[6] GILBERT, W.: Redix representations of quadratic fields, *J. Math. Anal. and Appl.* **83** (1981), 264–274.

[7] FARKAS, G.: Number systems in real quadratic fields, *Annales Univ. Sci. Budapest., Sect. Comp.* **18** (1999) (to appear).

[8] KOVÁCS, A.: On computation of attractors for invertible expanding linear operators in $\mathbb{Z}^k$, Two pages abstract in: Leaflets in Mathematics, Jannus Pannonius Univ. (Pécs).