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FOURIER COEFFICIENTS AND ABSOLUTE CONVERGENCE ON COMPACT TOTALLY DISCONNECTED GROUPS

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Abstract: In this paper we introduce the concept of modulus of continuity of complex functions defined on compact totally disconnected groups. These groups are not necessarily Abelian groups. We use the modulus of continuity in the estimations of the Fourier coefficients of integrable functions and functions with bounded fluctuation. Moreover, we investigate the absolute convergence of functions that are constant on the conjugacy classes of totally disconnected groups.

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In [6] and [7] the authors studied the compact totally disconnected (CTD) groups and the product system ψ as the generalization of the systems Vilenkin and Walsh. In fact, the structure of these groups is similar. We remark that if G_k is the discrete cyclic group of order $m_k, k \in \in \mathbb{N}$ then the CTD group G coincides with the Vilenkin group [13]. The difference is that if the CTD group is not Abelian then the ψ system is not bounded and can take on the value 0. For this reason, we can rarely use the methods with which we can treat the Vilenkin groups. For a more general case see e.g. [2, 10].

In Section 2 of this paper, according to the Vilenkin group, we introduce the concept of modulus of continuity for complex functions defined on compact totally disconnected groups. This concept is due to Fine [4] and Morgenthaler [9] for the Walsh group. Functions of bounded fluctuation were introduced by Onneweer and Waterman [11].

In Section 3 we use the modulus of continuity in the estimations of the Fourier coefficients of integrable functions and functions of bounded fluctuation. Before formulating our statements in this section, we should remark that if the CTD group is not Abelian group then the system ψ is not bounded. This fact is important because of the norm of the operators $T_n : L^1(G) \to \mathbb{C}$, $T_n f := \int_G f \overline{\psi_n}\, d\mu$ is $\|\psi_n\|_\infty$. For this reason if the CTD group is not Abelian group then there is an $f \in L^1(G)$ such that $\widehat{f}(n) \not\to 0$. We should not be surprised that $\|\psi_n\|_\infty$ appears in the estimation.

In [3] Benke proved that the Lipschitz class to which a function belongs can be identified by the best approximation characteristics of the function by trigonometric polynomials, and that functions which are easily approximated by trigonometric polynomials have absolutely convergent Fourier series. In Section 4, according to the above work we have some conditions under which a function has absolutely convergent Fourier series based on the system of characters of G in case that the function is constant on the conjugacy classes of G.

1. Preliminaries

The notations that we use in this paper are similar to those in the books of Hewitt-Ross [8] and Schipp-Wade-Simon [13]. Let σ be an equivalence class of continuous irreducible unitary representations of a compact group G. Denote Σ the set of all such σ. Σ is called the *dual object* of G. Denote by d_σ the dimension of a representation $U^{(\sigma)}$, $\sigma \in \Sigma$ and let $u_{i,j}^{(\sigma)}(x) := \langle U_x^{(\sigma)} \xi_i, \xi_j \rangle$ $i, j \in \{1, ..., d_\sigma\}$ be the coordinate functions for $U^{(\sigma)}$, where $\xi_1, ..., \xi_{d_\sigma}$ is an orthonormal base in the representation space of $U^{(\sigma)}$. According to the Weyl-Peter's theorem, the system of functions $\sqrt{d_\sigma} u_{i,j}^{(\sigma)}$, $\sigma \in \Sigma$, $i, j \in \{1, ..., d_\sigma\}$ is an orthonormal base for $L^2(G)$. If G is a finite group, then also Σ is finite. If $\Sigma := \{\sigma_1, ..., \sigma_s\}$, then $|G| = d_{\sigma_1}^2 + ... + d_{\sigma_s}^2$.

We now restrict our attention to infinite compact totally disconnected groups. For the sake of simplicity we shall call a compact totally disconnected group a CTD group. It is known that these groups have a countable neighborhood base $G = H_0 \supset H_1 \supset ...$ at the identity e consisting of open and closed normal subgroups which satisfy that for every $n \in \mathbb{N}$, the factor structure H_n/H_{n+1} is finite. Moreover, G is a complete direct product of these factor structures.

Gát and Toledo [7] gave the infinite compact totally disconnected groups in the following form: Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{C}$ denote the set of complex numbers. Denote by $m := (m_k : k \in \mathbb{N})$ a sequence of positive integers such that $m_k \geq 2$, $k \in \mathbb{N}$ and G_k a finite group with order m_k and with number of conjugacy classes p_k, $k \in \mathbb{N}$. We will use the same notation $(+)$ for the group operation of G_k, $k \in \mathbb{N}$. Denote the identity of these groups by e. Suppose that each group has discrete topology and right and left Haar measure μ_k with $\mu_k(G_k) = 1$. Thus each group has similar measure which maps every singleton of G_k to $\frac{1}{m_k}$, $k \in \mathbb{N}$. Let G be the compact group formed by the complete direct product of G_k with the product of the topologies, operations and measures (μ). Thus each $x \in G$ consist of sequences $x := (x_0, x_1, ...)$, where $x_k \in G_k$, $k \in \mathbb{N}$. Define by G^0 the set of sequences of G terminating in e's (i.e. the set of "finite" sequences), $I_0(x) := G$,

$$I_n(x) := \{y \in G : y_k = x_k, \text{ for } 0 \leq k < n\} \quad (x \in G, n \in \mathbb{N})$$

$I_n := I_n(e)$. We say that every set $I_n(x)$ is an *interval*. The intervals I_n form a countable neighborhood base at the identity of the product topology on G. Denote by $\mathcal{A}_n$ the σ algebra generated by the sets $I_n(x)$ $(x \in G)$ and by E_n the conditional expectation operator with respect to $\mathcal{A}_n$ $(n \in \mathbb{N})$.

If $M_0 := P_0 := 1$ and $M_{k+1} := m_k M_k$, $P_{k+1} := p_k P_k$ $k \in \mathbb{N}$, then every $n \in \mathbb{N}$ can be uniquely expressed as

$$n = \sum_{k=0}^{\infty} n_k M_k, \quad 0 \leq n_k < m_k, \; n_k \in \mathbb{N}.$$

This allows one to say that the $(n_0, n_1, \dots)$ sequence is the expansion of

n with respect to m. In this case let $n^* = (n_0, n_1, \dots) \in G$. In Section 4 we use the expansion of n with respect to the sequence $(p_0, p_1, \dots)$.

Now we denote the dual object of G_k by Σ_k. Let $\{\varphi_k^s : 0 \leq s < m_k\}$ be the set of all normalized coordinate functions of the group G_k and suppose that $\varphi_k^0 \equiv 1$. Thus for every $0 \leq s < m_k$ there exist $\sigma \in \Sigma_k$, $i, j \in \{1, ..., d_\sigma\}$ such that

$$\varphi_k^s(x) = \sqrt{d_\sigma} u_{i,j}^{(\sigma)}(x) \qquad (x \in G_k)$$

(d_σ is the dimension of σ). Let ψ be the *product system* of φ_k^s, namely

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \qquad (x \in G),$$

where n is of the form $n = \sum_{k=0}^{\infty} n_k M_k$ and $x = (x_0, x_1, ...)$. Similarly denote by $\chi_k^0 = 1$, χ_k^1, ... ,$\chi_k^{p_k - 1}$ the characters of the group G_k and let d_k^j be the dimension of the representation corresponding to the character χ_k^j. Then we obtain the characters of G in the form

$$\chi_n = \prod_{k=0}^{\infty} \chi_k^{n_k} \qquad (n = \sum_{k=0}^{\infty} n_k P_k;\ k \in \mathbb{N}).$$

The system $\psi = (\psi_n, n \in \mathbb{N})$ is orthonormal and complete in $L^1(G)$. In harmonic analysis it usual to use the characters of representations in approximation. In this case we restrict the space $L^p(G)$ for the functions that are constant on every conjugacy classes of G. We denote this new space by $\mathcal{L}^p(G)$. The system of characters $\chi = (\chi_n, n \in \mathbb{N})$ of a non Abelian group is not complete in $L^1(G)$, but it is orthonormal and complete in $\mathcal{L}^1(G)$.

For $f \in L^1(G)$ we define the *Fourier coefficients* by

$$\widehat{f}(k) := \int_G f \overline{\psi}_k d\mu \quad (k \in \mathbb{N}).$$

In Section 4 we denote by $\mathcal{A}$ the set of functions which have absolutely convergent Fourier series based on the system of characters of G. The Lipschitz class of order α will be denoted by $\mathrm{Lip}(\alpha)$ and is a closed subspace of the continuous functions endowed with the norm

$$\|f\|_{\mathrm{Lip}(\alpha)} := \sup_k \left[\sup_{x \in I_k} \|f(x\cdot) - f(\cdot)\|_\infty M_k^\alpha \right] + \|f\|_\infty.$$

2. The modulus of continuity

We introduce the concept of modulus of continuity using the analogy between the structure of CTD groups and Vilenkin groups. Let $f \in L^p(G)$, $1 \le p \le \infty$ and I an interval. Then $I = I_n(x)$ for some $x \in G, n \in \mathbb{N}$. Denote by

$$\omega^{(p)}(f, I) := \sup_{h \in I_n} \left(\frac{1}{\mu(I)} \int_I |\tau_h f - f|^p \, d\mu \right)^{\frac{1}{p}}, \quad \omega(f, I) := \omega^{(1)}(f, I)$$

the local modulus of continuity of f on I and let

$$\omega_n^{(p)}(f) := \sup_{h \in I_n} \|\tau_h f - f\|_p, \quad (n \in \mathbb{N}), \qquad \omega_n(f) := \omega_n^{(1)}(f)$$

be the n-th *modulus of continuity* of f on L^p, where $\tau_h f(x) := f(x+h)$ is the right translation operator. We remark that if we use the left translation operator, we obtain identical value for the modulus of continuity, because the measure is both left and right translation invariant and I_n is a normal subgroup of G. Notice that $\omega_n^{(p)}(f) \searrow 0$, $n \to \infty$ and $\omega_n^{(p)}(f)$ increases when the value of p increases.

$\omega_n(f, I)$ is a measure of the oscillation of f on I. Thus we say that a function f is of *p-bounded fluctuation* for some $1 \le p \le \infty$ if

$$\sup_{n \in \mathbb{N}} \left(\sum_{k=0}^{M_n - 1} |\omega(f, I_n(k^*))|^p \right)^{\frac{1}{p}} < \infty.$$

A function is said to be of *bounded fluctuation* if it is of 1-bounded fluctuation. In this case define the *total fluctuation* by:

$$\mathcal{F}\ell(f) := \sup_{n \in \mathbb{N}} \left(\sum_{k=0}^{M_n - 1} |\omega(f, I_n(k^*))| \right)$$

3. Estimates of the Fourier coefficients

Lemma 1. *Let $f \in L^1(G)$, n, $k \in \mathbb{N}$. If $n > M_k$ then there is a $h \in I_k$ such that $|\widehat{\tau_h f}(n) - \widehat{f}(n)| \ge |\widehat{f}(n)|$.*

Proof. Let $s = \max\{j \in \mathbb{N} : n_j \neq 0\}$ and let p be an arbitrary element of G_s. Moreover let $h(p)$ be the element of G with expansion $h(p) = (\underset{0}{e}, \underset{1}{e}, \dots, \underset{s-1}{e}, \underset{s}{p}, \underset{s+1}{e}, \dots)$. Then $h(p) \in I_s \subset I_k$.

If $\varphi_s^{n_s}$ is a normalized coordinate function of the group G_s, then there exist $\sigma \in \Sigma_s$, and $i, j \in \{1, \dots, d_\sigma\}$ such that

$$\varphi_s^{n_s} = \sqrt{d_\sigma} u_{i,j}^{(\sigma)}.$$

Using the fact that the measure μ is both right and left translation invariant and $n < M_{s+1}$ we have

$$\widehat{\tau_{h(p)} f}(n) = \int_G f(x)\overline{\psi_n(x - h(p))}\, d\mu(x) =$$

$$= \int_G f(x)\overline{\varphi_s(x_s - p)} \prod_{l=0}^{s-1} \overline{\varphi_l(x_l)}\, d\mu(x) =$$

$$= \int_G f(x)\overline{\sqrt{d_\sigma} u_{i,j}^{(\sigma)}(x_s - p)} \prod_{l=0}^{s-1} \overline{\varphi_l(x_l)}\, d\mu(x) =$$

$$= \int_G f(x)\sqrt{d_\sigma} \sum_{r=1}^{d_\sigma} \overline{u_{i,r}^{(\sigma)}(x_s)} u_{j,r}^{(\sigma)}(p) \prod_{l=0}^{s-1} \overline{\varphi_l(x_l)}\, d\mu(x) =$$

$$= \sum_{r=1}^{d_\sigma} u_{j,r}^{(\sigma)}(p) \int_G f(x)\overline{\sqrt{d_\sigma} u_{i,r}^{(\sigma)}(x_s)} \prod_{l=0}^{s-1} \overline{\varphi_l(x_l)}\, d\mu(x).$$

The coordinate functions form an orthonormal system:

$$\sum_{p\in G_s} u_{j,r}^{(\sigma)}(p) = m_s \int_{G_s} u_{j,r}^{(\sigma)}(x)\, d\mu_s(x) = 0 \implies \sum_{p\in G_s} \widehat{\tau_{h(p)} f}(n) = 0.$$

On the other hand $\widehat{\tau_{h(e)} f}(n) = \widehat{f}(n)$. Denote by $\langle .,. \rangle$ the inner product (for the complex numbers $a + b\imath, c + d\imath$ $\langle a + b\imath, c + d\imath\rangle = ac + bd$). Thus

$$\widehat{f}(n) + \sum_{\substack{p\in G_s\\ p\neq e}} \widehat{\tau_{h(p)} f}(n) = 0 \implies |\widehat{f}(n)|^2 + \sum_{\substack{p\in G_s\\ p\neq e}} \langle\widehat{\tau_{h(p)} f}(n), \widehat{f}(n)\rangle = 0.$$

Then there exists $p \in G_s$, $p \neq e$ such that

$$\langle\widehat{\tau_{h(p)} f}(n), \widehat{f}(n)\rangle < 0 \qquad \implies \qquad |\widehat{\tau_{h(p)} f}(n) - \widehat{f}(n)| \geq |\widehat{f}(n)|.$$

This completes the proof of Lemma 1.◊

Corollary 1. *Let $f \in L^1(G)$, n, $k \in \mathbb{N}$. If $n > M_k$ then*

$$|\widehat{f}(n)| < \omega_k(f)\|\psi_n\|_\infty.$$

Proof. Let h be an element of G which satisfies the conditions of Lemma 1. By the linearity of $\widehat{\ }$ we see that

$$|\widehat{f}(n)| \leq |\widehat{\tau_h f}(n) - \widehat{f}(n)| = |\widehat{\tau_h f - f}(n)| =$$

$$= \left|\int_G (\tau_h f(x) - f(x))\overline{\psi_n(x)}\, d\mu(x)\right| \leq \|\tau_h f - f\|_1 \|\psi_n\|_\infty \leq \omega_k(f)\|\psi_n\|_\infty$$

which was to be proved.◊

Similarly, we prove the following statement

Corollary 2. *Let $n \in \mathbb{N}$ and $s = \max\{j \in \mathbb{N} : n_j \neq 0\}$. If f is of bounded fluctuation, then*

$$|\widehat{f}(n)| \leq \frac{\mathcal{F}\ell(f)}{M_s}\|\psi_n\|_\infty.$$

Proof. Let h be an element of G that satisfies the conditions of Lemma 1. By the linearity of $\widehat{\ }$ we see that

$$\begin{aligned}
|\widehat{f}(n)| &\leq |\widehat{\tau_h f}(n) - \widehat{f}(n)| = |\widehat{\tau_h f - f}(n)| = \\
&= \left|\int_G (\tau_h f(x) - f(x))\overline{\psi_n(x)}\, d\mu(x)\right| \leq \\
&\leq \int_G |\tau_h f(x) - f(x)|\, d\mu(x)\|\psi_n\|_\infty = \\
&= \sum_{k=0}^{M_s-1} \int_{I_s(k^*)} |\tau_h f(x) - f(x)|\, d\mu(x)\|\psi_n\|_\infty \leq \\
&\leq \frac{1}{M_s}\sum_{k=0}^{M_s-1} |\omega(f, I_s(k^*))|\|\psi_n\|_\infty \leq \frac{\mathcal{F}\ell(f)}{M_s}\|\psi_n\|_\infty,
\end{aligned}$$

since the sets $I_s(k^*)$ $(0 \leq k \leq M_s - 1)$ are disjoints, cover the set G and $\mu(I_s(k^*)) = \frac{1}{M_s}$. This completes the proof of Cor. 2.◊

4. Absolute convergence of functions in $\mathcal{L}^p(G)$

Lemma 2. *Let $f : G_i \to \mathbb{C}$, $j \in \{0, 1, \ldots, p_i - 1\}$, $i \in \mathbb{N}$. Thus there is a $h \in G_i$ such that ($\chi_i^j \not\equiv 1$)*

$$\left|\sum_{x\in G_i} f(x+h)\overline{\chi_i^j}(x) - \sum_{x\in G_i} f(x)\overline{\chi_i^j}(x)\right| \geq \left|\sum_{x\in G_i} f(x)\overline{\chi_i^j}(x)\right|$$

Proof. Let $x \in G_i$. For simplicity we assume that the complex number $A := -\sum_{x\in G_i} f(x)\overline{\chi_i^j}(x)$ is on the first quadrant of the complex plane. If the complex number $B(h) := \sum_{x\in G_i} f(x+h)\overline{\chi_i^j}(x)$ is also on the first quadrant for some $h \in G_i$, then our statement follows for this $h \in G_i$. If $B(h)$ is on the fourth quadrant of the complex plane for some $h \in G_i$, then we replace h by h^{-1}. By using the property $\chi_i^j(h^{-1}) = \overline{\chi_i^j(h)}$ we have that $B(h^{-1})$ is on the first quadrant, so we proved our statement for $h^{-1} \in G_i$. On the other hand,

$$\sum_{h\in G_i}\sum_{x\in G_i} f(x+h)\overline{\chi_i^j}(x) = \sum_{h\in G_i}\sum_{x\in G_i} f(x)\overline{\chi_i^j}(x-h) =$$

$$= \sum_{h\in G_i}\sum_{x\in G_i} f(x)\sum_{r=1}^{d_i^j}\overline{u_{r,r}^\sigma}(x-h) = \sum_{h\in G_i}\sum_{x\in G_i} f(x)\sum_{r=1}^{d_i^j}\sum_{s=1}^{d_i^j}\overline{u_{r,s}^\sigma}(x)u_{s,r}^\sigma(h) =$$

$$= \sum_{r,s=1}^{d_i^j}\left(\sum_{x\in G_i} f(x)\overline{u_{r,s}^\sigma}(x)\right)\sum_{h\in G_i} u_{s,r}^\sigma(h) = 0.$$

Thus we have that there is an $h \in G_i$ such that the corresponding complex number $B(h)$ is on the first or fourth quadrant of the complex plane.

This completes the proof of the lemma.◊

Lemma 3. *Let $f \in \mathcal{L}^1(G)$, $P_n \le k < P_{n+1}$ $(k, n \in \mathbb{N})$. Then there is a $h_n \in G_n$ and $h := h_n e_n = (e, e, \dots, e, h_n, e, \dots)$ such that*

$$|\widehat{\tau_h f}(k) - \widehat{f}(k)| \ge |\widehat{f}(k)|$$

Proof. Let $x \in G$, $k_{(n-1)} := \sum_{i=0}^{n-1} k_i P_i$.

$$\overline{\chi_k}(x-h) = \prod_{i=0}^{n}\chi_i^{k_i}(x-h) = \left(\prod_{i=0}^{n-1}\chi_i^{k_i}(x)\right)\chi_n^{k_n}(x-h) =$$

$$= \chi_{k_{(n-1)}}(x)\chi_n^{k_n}(x-h).$$

Define $g : G_n \to \mathbb{C}$ by

$$g(x_n) := M_n^{-1}\sum_{x_0,\dots,x_{n-1}} (E_{n+1}f)(x)\overline{\chi_{k_{(n-1)}}}(x) \qquad (x_n \in G_n).$$

Thus $\widehat{f}(k) = m_n^{-1}\sum_{x_n\in G_n} g(x_n)\overline{\chi}_n^{k_n}(x)$,

$$\widehat{\tau_h f}(k) = m_n^{-1}\sum_{x_n\in G_n} g(x_n)\overline{\chi}_n^{k_n}(x-h) = m_n^{-1}\sum_{x_n\in G_n} g(x_n+h)\overline{\chi}_n^{k_n}(x).$$

Finally, Lemma 2 completes the proof of Lemma 3.◊

Theorem 1. *Let* $\sup m < \infty$, $f \in \mathcal{L}^2(G)$. *If*

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{M_n-1}|\omega^{(2)}(f, I_n(k^*))|^2\right)^{\frac{1}{2}} < \infty \qquad \textit{then} \qquad f \in \mathcal{A}.$$

Remark. If $f \in \mathcal{L}^2(G)$, m is arbitrary and

$$\sum_{n=0}^{\infty} m_n \left(\sum_{k=0}^{M_n-1} |\omega^{(2)}(f, I_n(k^*))|^2 \right)^{\frac{1}{2}} < \infty \quad \text{then} \quad f \in \mathcal{A}.$$

Proof. Let $P_n \leq k < P_{n+1}$, $a := k_n$. Lemma 3 guaranties that there is an $h = (e, e, \ldots, e, h_n, e, \ldots)$ such that

$$|\widehat{\tau_h f}(k) - \widehat{f}(k)| \geq |\widehat{f}(k)|.$$

By Cauchy's inequality we have

$$\sum_{k=P_n}^{P_{n+1}-1} |\widehat{f}(k)| d_k \leq \left(\sum_{k=P_n}^{P_{n+1}-1} (d_k)^2 \right)^{\frac{1}{2}} \left[\sum_{a=0}^{p_n-1} \sum_{k=aP_n}^{(a+1)P_n-1} |\widehat{f}(k)|^2 \right]^{\frac{1}{2}} \leq$$

$$\leq \left(\sum_{k=P_n}^{P_{n+1}-1} (d_k)^2 \right)^{\frac{1}{2}} \left[\sum_{a=0}^{p_n-1} \sum_{k=aP_n}^{(a+1)P_n-1} |\widehat{\tau_{h(a)} f}(k) - \widehat{f}(k)|^2 \right]^{\frac{1}{2}} \leq$$

$$\leq \sqrt{M_{n+1}} \sqrt{\sum_{a=0}^{p_n-1} \|\tau_{h(a)} f - f\|_2^2},$$

since $d_k = \prod_{i=0}^{n} d_i^{k_i}$, $\sum_{k_i=0}^{p_i-1} (d_i^{k_i})^2 = m_i \quad (i \in \mathbb{N})$. χ is an orthonormal system, therefore we can use the Bessel's inequality. On the other hand,

$$M_{n+1} \|\tau_{h(a)} f - f\|_2^2 = M_{n+1} \sum_{k=0}^{M_n-1} \int_{I_n(k^*)} |f(x+h) - f(x)|^2 \, dx \leq$$

$$\leq m_n \sum_{k=0}^{M_n-1} |\omega^{(2)}(f, I_n(k^*))|^2.$$

Since $p_n \leq m_n$ (and sequence m is bounded) we have

$$\sum_{k=P_n}^{P_{n+1}-1} |\widehat{f}(k)| d_k \leq m_n \left(\sum_{k=0}^{M_n-1} |\omega^{(2)}(f, I_n(k^*))|^2 \right)^{\frac{1}{2}}.$$

Thus

$$\|f\|_A := \sum_{k=0}^{\infty} |\widehat{f}(k)| d_k \leq m_n \sum_{n=0}^{\infty} \left(\sum_{\substack{t_i \in G_i \\ i<n}} |\omega^{(2)}(f, I_n(t))|^2 \right)^{\frac{1}{2}} < \infty.$$

This completes the proof of Th. 1.◊

The following statement is the generalization of a similar statement that appeared in [13].

Theorem 2. *Let $f : G \to \mathbb{C}$ be a continuous function that is constant in the conjugacy classes of G and suppose that there exists $1 \le p \le 2$ such that*

$$\sum_{n=0}^{\infty}\Big(\sum_{\substack{t_i\in G_i\\ i<n}} |\omega(f, I_n(t))|^p\Big)^{1/p} < \infty. \qquad \text{Then} \qquad f \in \mathcal{A}.$$

Proof. Since $f \in \mathcal{L}^2(G)$ we have

$$\omega^{(2)}(f, I_n(t)) = \sup_{h\in I_n}\left[M_n \int_{I_n(t)} |f(x+h) - f(x)|^2\, dx\right]^{\frac{1}{2}} \le$$
$$\le \sup_{\substack{h\in I_n\\ x\in I_n(t)}} |f(x+h) - f(x)| = \omega(f, I_n(t)).$$

Thus

$$\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{M_n-1} |\omega^{(2)}(f, I_n(k^*))|^p\Big)^{1/p} < \infty.$$

Using the inequality $\left(\sum_{i=1}^N |a_i|^2\right)^{\frac{1}{2}} \le \left(\sum_{i=1}^N |a_i|^p\right)^{\frac{1}{p}}$ $(1 \le p \le 2)$ we have

$$\sum_{n=0}^{\infty}\Big(\sum_{\substack{t_i\in G_i\\ i<n}} |\omega^{(2)}(f, I_n(t))|^2\Big)^{1/2} < \infty.$$

That is, the conditions of Th. 1 is fulfilled. This completes the proof of Th. 2.◊

Corollary 3. *Let m be bounded, $f : G \to \mathbb{C}$ be a continuous function that is constant in the conjugacy classes of G and suppose that $\sum_{n=0}^{\infty} \sqrt{M_n}\omega_n(f) < \infty$. Then $f \in \mathcal{A}$.*

Proof. The corollary is a consequence of Th. 2 since $\omega(f, I_n(t)) \le \omega_n(f)$ $(t \in G)$. For this reason

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{M_n-1} |\omega^{(2)}(f, I_n(k^*))|^2\right)^{\frac{1}{2}} < \sum_{n=0}^{\infty} \sqrt{M_n}\omega_n(f) < \infty. \;\Diamond$$

Corollary 4. *Let m be bounded and $f \in \mathrm{Lip}\,(\alpha)$ for some $\alpha > \frac{1}{2}$. Then $f \in \mathcal{A}$.*

Proof. $\omega_n(f) \le cM_n^{-\alpha}$ $(n \in \mathbb{N})$, thus

$$\sum_{n=0}^{\infty} \sqrt{M_n}\omega_n(f) \le c\sum_{n=0}^{\infty} M_n^{\frac{1}{2}-\alpha} < \infty.$$

Thus the conditions of Cor. 3 is satisfied.◊

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