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# RECTANGULAR MODULUS AND GEOMETRIC PROPERTIES OF NORMED SPACES

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**Abstract**: Recently, [18] we have introduced the rectangular (*-rectangular) modulus of a normed space $X$. It is a convex function strongly related to some known constants of $X$. The aim of this paper is to characterize some geometric properties of normed spaces in terms of the rectangular modulus. We prove that a normed space of dimension $\geq 3$ is an inner product space if and only if the right derivative in 0 of the rectangular modulus is zero. The case of two-dimensional spaces is also treated. A characterization of the uniform convexity of $X$ is given in terms of the *-rectangular modulus.

## 1. Introduction and notation

The geometry of a real linear normed space $X$ with $\dim X \geq 2$ may be described, among others, using some moduli attached to $X$ and their properties. For instance, the moduli of convexity [5], and of smoothness [11] are well known and often used in various applications.

Let us denote by $B(x,r)$ the closed ball of $X$, $(\dim X \geq 2)$ with center $x$ and radius $r > 0$ and by $B = B(0,1)$ the closed unit ball of $X$. Let $S(x,r)$, respectively $S = S(0,1)$ be the corresponding spheres of $X$. The symbol $\perp$ will be used for Birkhoff orthogonality in the normed space $(X, \|\cdot\|)$, namely $x \perp y$ iff $\|x\| \leq \|x + \mu y\|$ holds for all $\mu \in \mathbb{R}$.

For $x, y \in X, x \neq y$ denote $L(x,y)$ the straight line passing through $x$ and $y$. Similarly, $[x;y]$ will be the suitable closed segment. Recall that the *modulus of convexity* of $X$ is the function $\delta_X : [0,2] \to \mathbb{R}$ defined by:

$$\delta_X(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in S, \|x-y\| = \varepsilon\right\}, \ \varepsilon \in [0,2],$$

while the *modulus of smoothness* of $X$ is the function $\rho_X : [0,\infty) \to \mathbb{R}$ defined by:

$$\rho_X(\tau) = \sup\left\{\frac{1}{2}(\|x+\tau y\| + \|x-\tau y\| - 2) : x, y \in S\right\}, \tau \geq 0.$$

The following modulus of smoothness, modified with a condition of orthogonality, was defined in [9] as being the function $\overline{\rho}_X : [0,\infty) \to \mathbb{R}$

$$\overline{\rho}_X(\tau) = \sup\left\{\frac{1}{2}(\|x+\tau y\| + \|x-\tau y\| - 2) : x, y \in S, x \perp y\right\}, \tau \geq 0.$$

T. Figiel [9] has proved that $\rho_X$ and $\overline{\rho}_X$ are equivalent, more precisely we have:

$$(1) \qquad \frac{1}{8}\rho_X(\tau) \leq \overline{\rho}_X(\tau) \leq \rho_X(\tau), \forall \tau \geq 0.$$

Now, a normed space is said to be *uniformly convex* if $\delta_X(\varepsilon) > 0, \forall \varepsilon \in (0,2]$ and *uniformly smooth* if $\lim_{\tau \searrow 0} \rho_X(\tau)/\tau = 0$, (or equivalently if $\lim_{\tau \searrow 0} \overline{\rho}_X(\tau)/\tau = 0$). The normed space $X$ is said to be *smooth at* $x_0 \in S$ whenever there exists a unique $f \in X^*, \|f\| = 1$ such that $f(x_0) = 1$. If $X$ is smooth at each point of $S$ then we say that $X$ is *smooth*, [8, p.21]. A normed space $X$ is said to be *strictly convex* whenever $S$ contains no non-trivial line segments, [8, p.23]. A uniformly smooth space is said to have *modulus of smoothness of power type* $p$, with $p > 1$ if there exists a number $C > 0$ such that $\rho_X(\tau) \leq C\tau^p, \forall \tau \geq 0$, [12, p.63].

K. Przeslawski and D. Yost [13], [14] have introduced the modulus of squareness. It appears, in a natural way, in some estimates for the Lipschitz constants of multivalued mappings in Banach spaces. They considered a pair $(x,y)$ of points in $X$ with $\|y\| < 1 < \|x\|$. Then there is a unique $z = z(x,y)$ in the line segment $[x;y]$ with $\|z\| = 1$. As in [14] we put

$$\omega(x,y) = \frac{\|x - z(x,y)\|}{\|x\| - 1}$$

and define the *modulus of squareness* $\xi_X : [0,1) \to \mathbb{R}$ by

$$\xi_X(\beta) = \sup\{\omega(x,y) : \|y\| \leq \beta < 1 < \|x\|\}, \beta \in [0,1).$$

In [15] we have obtained the following alternative formula for $\xi_X$ :

(2)
$$\xi_X(\beta) = \sup\{\|x-y\| : x \in S, y \in X, x \perp y, \min_{\lambda\geq 0}\|(1-\lambda)x+\lambda y\| = \beta\},$$

$\beta \in [0,1)$. Surprisingly, from the behaviour of $\xi_X$ in the neighbourhood of 1 and of 0 respectively, it is possible to characterize uniformly convex and uniformly smooth normed spaces. The relation

(3) $$\lim_{\beta \nearrow 1}(1-\beta)\xi_X(\beta) = 0,$$

characterizes the uniform convexity of $X$, [3,13], while the relation

(4) $$\lim_{\beta \searrow 0}\frac{\xi_X(\beta)-1}{\beta} = 0,$$

characterizes the uniform smoothness of $X$, [4, 16]. On the other hand $\xi_X$ is an increasing function, convex in the neighbourhood of 1, it verifies a Day-Nordlander type inequality and characterizes inner product spaces (i.p.s for short) [4, 17]. Recently, we have introduced the *rectangular modulus* of $X$ [18], as the function $\mu_X : (0,\infty) \to \mathbb{R}$

$$\mu_X(\lambda) = \sup\{\max\{\varphi_{\lambda,x,y}(t), \lambda\varphi_{\frac{1}{\lambda},x,y}(t)\} : t > 0, x, y \in S, x \perp y\}, \lambda > 0,$$

where

$$\varphi_{\lambda,x,y}(t) = \frac{\lambda+t}{\|x+ty\|}, \lambda, t > 0, x, y \in S, x \perp y.$$

The function $\varphi_{\lambda,x,y}$ is a useful ingredient in some characterizations of i.p.s in terms of Birkhoff orthogonality. In the same paper it was also proved that

a) $\mu_X$ is a convex function; if $H$ is an i.p.s then $\mu_H(\lambda) = \sqrt{1+\lambda^2}$;

b) $\mu_X$ verifies a Day-Nordlander inequality i.e.: $\mu_X(\lambda) \geq \mu_H(\lambda) = \sqrt{1+\lambda^2}, \forall\lambda > 0$;

c) $\mu_X(\lambda) = \sqrt{1+\lambda^2}$ for a fixed $\lambda > 0$, then $X$ is an i.p.s.

The **-rectangular modulus* [18] defined by the simpler formula

$$\mu_X^*(\lambda) = \sup\{\varphi_{\lambda,x,y}(t) : t > 0, x, y \in S, x \perp y\}, \lambda > 0,$$

verifies also the properties a), b) and c). Moreover $\mu_X^*(\lambda) \leq \lambda+2, \forall\lambda > 0$.

On the other hand $\mu_X(1) = \mu_X^*(1) = \mu(X)$, where $\mu(X)$ is the rectangular constant of $X$ defined by J.L. Joly [10]. Let $\mu_X(0+)$ be given by $\mu_X(0+) = \lim_{\lambda\searrow 0}\mu_X(\lambda)$. Then $\mu_X(0+) = \mu_X^*(0+) \in [1,2]$ and $\mu_X(0+)$ is the known radial constant of $X$, denoted by $k(X)$, [20], which in turn is equal to other four constants of $X$, denoted by $MPB(X), MPB'(X)$, $\overline{MPB}(X), \beta(X)$, respectively. For more information on this subject see [2], [3], [6], [7], [19], [20].

## 2. Main results

In this paper we obtain some relations between the properties of $\mu_X, (\mu_X^*)$ and the geometry of the normed space $X$. A characterization of i.p.s of dimension $\geq 3$ is deduced from the knowledge of the right derivative of $\mu_X^*$ in the origin. The two-dimensional case is partially treated. A characterization of uniformly convex spaces is obtained from the behaviour of $\mu_X^*$ at infinity.

For $x, y \in X$ let $\tau(x,y)$ be defined by:

$$\tau(x,y) = \lim_{t \searrow 0} \frac{\|x+ty\| - \|x\|}{t} = \inf_{t>0} \frac{\|x+ty\| - \|x\|}{t}.$$

It is clear that $X$ is smooth if and only if $\tau(x,y) = -\tau(x,-y)$, for any pair $(x,y) \in X \times X$ with $x \neq 0$.

**Lemma A** [3]. *A normed space $X$ is smooth if and only if the following condition holds:*

$$\{(x,y) \in S \times S : x \perp y\} = \{(x,y) \in S \times S : \tau(x,y) = 0\}.$$

A uniformly smooth variant of Lemma A is given by

**Lemma 2.1.** *A normed space $X$ is uniformly smooth if and only if the following condition holds:*

$\alpha$) $x, y \in S, x \perp y \Rightarrow \|x+ty\| = 1 + o(x,y,t)$,

*where* $\lim_{t\searrow 0} o(x,y,t)/t = 0$, *uniformly with respect to* $x, y \in S, x \perp y$.

**Proof.** i) If $X$ is uniformly smooth and $x, y \in S, x \perp y$ then by Lemma A

$$\lim_{t\searrow 0} \frac{\|x+ty\| - \|x\|}{t} = \lim_{t\searrow 0} \frac{o(x,y,t)}{t} = \tau(x,y) = 0.$$

By uniform smoothness this limit is uniform with respect to $x, y \in S$, $x \perp y$, and $\alpha$) follows.

ii) Suppose that $\alpha$) holds and that $X$ is not uniformly smooth. Then $\lim_{t\searrow 0} \rho_X(t)/t = \inf_{t>0} \rho_X(t)/t = a > 0$. Using (1) it follows that $\lim_{t\searrow 0} \overline{\rho}_X(t)/t = \inf_{t>0} \overline{\rho}_X(t)/t \geq a/8$. There exists then a sufficiently small $\varepsilon > 0$ such that $\overline{\rho}_X(t)/t > a/16$, for all $t \in (0,\varepsilon)$. For any $t \in (0,\varepsilon)$ choose a pair $(x_t, y_t) \in S \times S, x_t \perp y_t$ such that

$$\frac{1}{2t}(\|x_t + ty_t\| + \|x_t - ty_t\| - 2) > a/32.$$

Let $\overline{y}_t \in \{y_t, -y_t\}$ be such that $\|x_t + t\overline{y}_t\| = \max\{\|x_t+ty_t\|, \|x_t - ty_t\|\}$. One obtains $(\|x_t + t\overline{y}_t\| - 1)/t > a/32$, for all $t \in (0,\varepsilon)$. It follows that

$$\frac{o(x_t, \overline{y}_t, t)}{t} \geq a/32, \forall t \in (0,\varepsilon),$$

contradicting $\alpha$). $\Diamond$

The relation (4) characterizes the uniform smoothness in terms of the squareness modulus. In the sequel we will see that similar formula for *-rectangular modulus of $X$ has a different interpretation.

**Lemma 2.2.** *If the *-rectangular modulus of $X$ verifies the relation*

$$\lim_{\lambda\searrow 0}\frac{\mu_X^*(\lambda)-\mu_X^*(0+)}{\lambda}=0, \tag{5}$$

*then the Birkhoff orthogonality in $X$ is symmetric.*

**Proof.** Let $\lambda\in(0,1)$ and $x,y\in S, x\perp y$ be given. It follows that

$$\varphi_{\lambda,x,y}(t)=\frac{\lambda+t}{\|x+ty\|}=\lambda\cdot\left\|\frac{\lambda x+t\lambda y}{\lambda+t}\right\|^{-1},\quad \forall t>0,$$

and

$$\psi(x,y,\lambda)\overset{\text{def}}{=}\sup_{t>0}\varphi_{\lambda,x,y}(t)=\lambda\cdot\big(\min_{\mu\in[0,1]}\|\mu x+(1-\mu)\lambda y\|\big)^{-1}=$$
$$=\lambda\cdot\|\mu_0x+(1-\mu_0)\lambda y\|^{-1},$$

where $\mu_0=\mu_0(x,y,\lambda)\in[0,1)$ and $\mu_0$ is not necessarily unique. If any $\mu_0(x,y,\lambda)$ is $\neq 0$, then the straight line $L(x,\lambda y)$ is a support line for the sphere $S(0,\|\mu_0x+(1-\mu_0)\lambda y\|)$ and $\psi(x,y,\lambda)>1$. In the opposite case $\psi(x,y,\lambda)=1$. Supposing that $\psi(x,y,\lambda)=1$, for all $x,y\in S,x\perp y$ one obtains that $\mu_X^*(\lambda)=1<\sqrt{1+\lambda^2}$, in contradiction with the property b) of $\mu_X^*$. This means that in order to obtain $\sup_{x,y\in S,x\perp y}\psi(x,y,\lambda)=\mu_X^*(\lambda)$, we can consider only the pairs $x,y\in S,x\perp y$ with any $\mu_0(x,y,\lambda)\in(0,1)$.

A parallel to the straight line $L(x,\lambda y)$ from the origin intersects the parallel to the straight line $L(0,\mu_0x+(1-\mu_0)\lambda y)$ from $y$ in $y_0=y_0(x,y,\lambda)$. The triangle with vertices $0,\mu_0x+(1-\mu_0)\lambda y,\lambda y$ is similar to the triangle with vertices $y,y_0,0$. From this we obtain:

$$\lambda\cdot\|\mu_0x+(1-\mu_0)\lambda y\|^{-1}=\|y\|\cdot\|y-y_0\|^{-1},$$

and

$$\mu_X^*(\lambda)=\sup_{x,y\in S,x\perp y}\psi(x,y,\lambda y)=\big(\inf\{\|y-y_0\|:x,y\in S,x\perp y\}\big)^{-1}.$$

On the other hand we have

$$\begin{aligned}\mu_X^*(0+)&=\sup\{\|tx+y\|^{-1}:t>0,x,y\in S,x\perp y\}=\\&=\big(\inf\{\|y-x_0\|:x,y\in S,x\perp y\}\big)^{-1},\end{aligned}\tag{6}$$

where $x_0=x_0(x,y)\in L(0,x), y-x_0\perp x$. By changing $x$ in $-x$ we can suppose that $0\in[x_0,x]$ (see Fig. 1).

In general, $x_0(x,y)$ is not uniquely determined. Let $z_0=z_0(x,y,\lambda)$ be defined by $\{z_0\}=L(y,x_0)\cap L(0,y_0)$. Since $y-y_0\perp y_0$ we have $\|y-y_0\|\leq\|y-z_0\|$. One obtains

$$\mu_X^*(\lambda) \geq \frac{1}{\inf\{\|y - z_0(x,y,\lambda)\| : x,y \in S, x \perp y\}}. \tag{7}$$

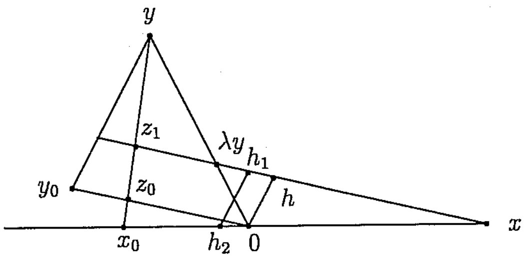


Fig. 1

Let $z_1 = z_1(x,y,\lambda)$ be given by $\{z_1\} = L(y,x_0) \cap L(x,\lambda y)$. Since for two nonzero collinear vectors $u$ and $v$, $\|u\|/\|v\|$ is independent of the norm, we can apply the Menelaus Theorem in a two-dimensional normed space. Consider the triangle with vertices $0, x_0, y$ and the transversal $L(x,\lambda y)$. The relation

$$\frac{\lambda}{1-\lambda} \cdot \frac{\|y - z_1\|}{\|z_1 - x_0\|} \cdot \frac{1 + \|x_0\|}{1} = 1,$$

implies

$$\|y - z_1\| = \frac{1-\lambda}{1+\lambda\|x_0\|} \cdot \|y - x_0\|,$$

and

$$\|y - z_0\| = \frac{1}{1+\lambda\|x_0\|} \cdot \|y - x_0\|.$$

Finally, suppose that (5) holds and that Birkhoff orthogonality in $X$ is not symmetric. Then from [17] $\mu_X^*(0+) > 1$. Let again $\lambda \in (0,1)$ be fixed. By (6) there exists a pair $x', y' \in S, x' \perp y'$ such that

$$\mu_X^*(0+) \geq \frac{1}{\|y' - x_0(x',y')\|} > \mu_X^*(0+) - \lambda^2, \tag{8}$$

and such that $0 \in [x_0(x',y'), x']$. It follows that

$$\frac{1}{\|y' - z_0(x',y',\lambda)\|} = \frac{1 + \lambda\|x_0(x',y')\|}{\|y' - x_0(x',y')\|}. \tag{9}$$

From (7), (8) and (9) we obtain:

$$\frac{\mu_X^*(\lambda) - \mu_X^*(0+)}{\lambda} \geq \frac{\dfrac{1}{\|y' - z_0(x',y',\lambda)\|} - \dfrac{1}{\|y' - x_0(x',y')\|} - \lambda^2}{\lambda} =$$

$$= \frac{\frac{\lambda\|x_0(x',y')\|}{\|y'-x_0(x',y')\|}-\lambda^2}{\lambda} = \frac{\|x_0(x',y')\|}{\|y'-x_0(x',y')\|}-\lambda \geq$$

$$\geq \frac{\|y'\|-\|y'-x_0(x',y')\|}{\|y'-x_0(x',y')\|}-\lambda > \mu_X^*(0+)-\lambda^2-\lambda-1.$$

We have

$$\lim_{\lambda\searrow 0}\frac{\mu_X^*(\lambda)-\mu_X^*(0+)}{\lambda} \geq \mu_X^*(0+)-1>0,$$

in contradiction with (5). ◊

**Theorem 2.3.** *Let $X$ be a real normed space, $\dim X \geq 3$. The following are equivalent:*

1) $\lim_{\lambda\searrow 0}\dfrac{\mu_X(\lambda)-\mu_X(0+)}{\lambda}=0.$

2) $\lim_{\lambda\searrow 0}\dfrac{\mu_X^*(\lambda)-\mu_X^*(0+)}{\lambda}=0.$

3) *$X$ is an inner product space.*

**Proof.** 1) ⇒ 2). Suppose that 1) is valid. This implies that

$$0\leq \lim_{\lambda\searrow 0}\frac{\mu_X^*(\lambda)-\mu_X^*(0+)}{\lambda}\leq \lim_{\lambda\searrow 0}\frac{\mu_X(\lambda)-\mu_X(0+)}{\lambda}=0,$$

and 2) follows.

2) ⇒ 3). By Lemma 2 and 2) we have that the Birkhoff orthogonality is symmetric. Since $\dim X\geq 3$, it follows (see [1, p. 143]) that $X$ is an i.p.s.

3) ⇒ 1). $X$ being an i.p.s we have $\mu_X(\lambda)=\sqrt{1+\lambda^2}, \forall\lambda>0$, and 1) is obvious. ◊

**Theorem 2.4.** *Let $X$ be a two-dimensional real Banach space. If*

$$\lim_{\lambda\searrow 0}\frac{\mu_X^*(\lambda)-\mu_X^*(0+)}{\lambda}=0,$$

*then $X$ is strictly convex.*

**Proof.** Suppose that $X$ is not strictly convex. Then, using the notation from Lemma 2, there exists a pair $x'',y''\in S, x''\perp y''$ such that $\|x_0(x'',y'')\|>0$, and $0\in[x_0(x'',y''),x'']$. The symmetry of orthogonality implies that $\|y''-x_0(x'',y'')\|=1$. As in Lemma 2 we have:

$$\frac{\mu_X^*(\lambda)-\mu_X^*(0+)}{\lambda}\geq \frac{\|x_0(x'',y'')\|}{\|y''-x_0(x'',y'')\|}-\lambda=\|x_0(x'',y'')\|-\lambda.$$

One obtains that

$$\lim_{\lambda \searrow 0} \frac{\mu_X^*(\lambda) - \mu_X^*(0+)}{\lambda} \geq \|x_0(x'', y'')\| > 0,$$

a contradiction. Now, it is well-known that a two-dimensional space with symmetric orthogonality is strictly convex, iff it is smooth (see [1, p.78]). So, $X$ is uniformly smooth, uniformly convex and the Birkhoff orthogonality in $X$ is symmetric.◊

**Therem 2.5.** *Let $X$ be a two-dimensional real Banach space. We suppose that the Birkhoff orthogonality in $X$ is symmetric and that $X$ is smooth with the modulus of smoothness of power type $p, p > (\sqrt{5}+1)/2$, then*

$$\lim_{\lambda \searrow 0} \frac{\mu_X^*(\lambda) - \mu_X^*(0+)}{\lambda} = 0.$$

**Proof.** Let $x, y \in S, x \perp y$ and $\lambda > 0$ be fixed. We have

$$\frac{\|x - \lambda y\| - 1}{2} \leq \frac{\|x + \lambda y\| - 1 + \|x - \lambda y\| - 1}{2} \leq \overline{\rho}_X(\lambda) \leq \rho_X(\lambda) \leq C\lambda^p.$$

Then $\|x - \lambda y\| \leq 1 + C_1\lambda^p, \forall \lambda > 0, \forall x, y \in S, x \perp y$, and by Lemma 1, $o(x, -y, \lambda) \leq C_1\lambda^p, \forall x, y \in S, x \perp y, \forall \lambda > 0$. Moreover the function $o(x, -y, \cdot)$ is increasing in $(0, \infty)$. Using again the notation in Lemma 2, we observe that $h = h(x, y, \lambda) = \mu_0 x + (1 - \mu_0)\lambda y \perp x - \lambda y$, ($h$ is unique) and by the symmetry of orthogonality $h - x \perp h$. We have $\|h - x\| \leq \|x\| = 1$ and $\|h - \lambda y\| \leq \lambda$. Let $h_1$ be the unique vector in the line segment $[x; \lambda y]$ verifying $\|h_1 - x\| = 1$. A parallel from $h_1$ to the straight line $L(0, h)$ intersects $L(0, x)$ in $h_2$. We have

$$\frac{\|x - h\|}{1} = \frac{\|h\|}{\|h_2 - h_1\|},$$

and $h_2 - h_1 \perp x - \lambda y$. Now, by Lemma 1

$$\|h_2 - h_1\| = \frac{\|h\|}{1 + o(x, -y, \lambda) - \|h - \lambda y\|} \leq \frac{\|h\|}{1 + o(x, -y, \lambda) - \lambda}.$$

By orthogonality and Lemma 1 it follows that:

$$\|h - h_1\| \leq \|x - h_2\| - \|x\| = \|x - h_2\| - \|x - h_1\| = 1 +$$

$$+ o\left(\frac{x - \lambda h}{\|x - \lambda h\|}, \frac{h}{\|h\|}, \|h_2 - h_1\|\right) - 1 \leq C_1\|h_2 - h_1\|^p \leq$$

$$\leq C_1 \frac{\|h\|^p}{(1 + o(x, -y, \lambda) - \lambda)^p}.$$

But from $y - y_0 \perp x - \lambda y$ we obtain

$$1 = \|y\| = \|y - y_0 + y_0\| = \|y - y_0\| \cdot \left\| \frac{y - y_0}{\|y - y_0\|} + \frac{y_0}{\|y - y_0\|} \right\| =$$
$$= \|y - y_0\| \cdot \left(1 + o\left(\frac{y - y_0}{\|y - y_0\|}, \frac{x - \lambda y}{\|x - \lambda y\|}, \frac{\|y_0\|}{\|y - y_0\|}\right)\right) \leq$$
$$\leq \|y - y_0\| \cdot \left(1 + C_1 \frac{\|y_0\|^p}{\|y - y_0\|^p}\right).$$

The triangle with vertices $0, h, \lambda y$ is similar to the triangle with vertices $y, y_0, 0$ and this means that

$$\frac{\|y_0\|}{\|y - y_0\|} = \frac{\|h - \lambda y\|}{\|h\|} = \frac{\|h - h_1\| + \|h_1 - \lambda y\|}{\|h\|} \leq \frac{\|h - h_1\|}{\|h\|} +$$
$$+ \frac{\|x - \lambda y\| - 1}{\|h\|} \leq C_1 \frac{\|h\|^{p-1}}{(1 + o(x, -y, \lambda) - \lambda)^p} + \frac{o(x, -y, \lambda)}{\|h\|}.$$

Since $\lambda/\|h\| \leq \mu_X^*(\lambda) \leq \lambda + 2$, for $\lambda > 0$ small enough

$$\frac{\|y_0\|}{\|y - y_0\|} \leq 2C_1\|h\|^{p-1} + C_1 \frac{\lambda^p}{\|h\|} \leq 2C_1\lambda^{p-1} + 3C_1\lambda^{p-1} = 5C_1\lambda^{p-1},$$

which implies that

$$1 - \|y - y_0\| \leq \|y - y_0\| C_1 \frac{\|y_0\|^p}{\|y - y_0\|^p} \leq$$
$$\leq \|y - y_0\| \cdot C_1 \cdot 5^p \cdot C_1^p \lambda^{p^2 - p} = \|y - y_0\| \cdot 5^p \cdot C_1^{p+1} \lambda^{p^2 - p}.$$

The symmetry of orthogonality yields $\mu_X^*(0+) = 1$ and:

$$0 \leq \frac{\mu_X^*(\lambda) - \mu_X^*(0+)}{\lambda} = \frac{\sup\left\{\frac{1}{\|y - y_0(x, y, \lambda)\|} : x, y \in S, x \perp y\right\} - 1}{\lambda} \leq$$
$$\leq \frac{5^p C_1^{p+1} \lambda^{p^2 - p}}{\lambda} = 5^p \cdot C_1^{p+1} \cdot \lambda^{p^2 - p - 1},$$

with $\lambda$ close to 0. If $p > (\sqrt{5} + 1)/2$ then $\lim_{\lambda \searrow 0}(\mu_X^*(\lambda) - \mu_X^*(0+))/\lambda =$ $= 0$. $\Diamond$

**Remark.** Denoting by $H$ an inner product space it is well-known [11] that

$$\rho_X(\tau) \geq \rho_H(\tau) = \sqrt{1 + \tau^2} - 1 = \frac{\tau^2}{2} + o(\tau^2).$$

This implies that $\rho_X$ is of power type at most 2.

**Example.** Let $p \in ((\sqrt{5} + 1)/2, 2)$ be a given number and let $q$ be its conjugate $1/p + 1/q = 1$. In $\mathbb{R}^2$ define the norm:

$$\|(\alpha,\beta)\| = \begin{cases} (|\alpha|^p+|\beta|^p)^{1/p} = \|(\alpha,\beta)\|_p, \text{for } \alpha\beta \geq 0 \\ (|\alpha|^q+|\beta|^q)^{1/q} = \|(\alpha,\beta)\|_q, \text{for } \alpha\beta < 0. \end{cases}$$

Then $(\mathbb{R}^2, \|\cdot\|)$ is a Banach space and the Birkhoff orthogonality is symmetric, (see [1, p.77]). Let $x_1 = (\alpha_1, \beta_1), x_2 = (\alpha_2, \beta_2)$ be two unit vectors with $x_1 \perp x_2$. We have

$$\|x_1 + \lambda x_2\| - 1 \leq \max\{\|x_1+\lambda x_2\|_p - 1, \|x_1+\lambda x_2\|_q - 1\} \leq$$
$$\leq \max\{C_1\lambda^p, C_2\lambda^2\} \leq (C_1+C_2)\lambda^p, \forall\lambda \in [0,1).$$

The space $(\mathbb{R}^2, \|\cdot\|)$ is two-dimensional, uniformly convex and uniformly smooth with modulus of smoothness of pover type $> (\sqrt{5}+1)/2$. From Th. 2.5 it follows that $\lim_{\lambda\searrow 0}(\mu_X^*(\lambda) - \mu_X^*(0+))/\lambda = 0$. However $(\mathbb{R}^2, \|\cdot\|)$ is not a Hilbert space.

**Theorem 2.6.** *The real normed space $X$ is uniformly convex if and only if*

$$\lim_{\lambda\to\infty}(\mu_X^*(\lambda) - \lambda) = 0. \tag{10}$$

**Proof.** Let $x, y \in S, x \perp y$ and $\lambda > 1$ be fixed. Denote by $h_0(\lambda) = \inf\{\|h(x,y,\lambda)\| : x, y \in S, x \perp y\}$ where $h(x,y,\lambda)$ is as in Th. 2.5. In the two-dimensional subspace $X_1$ of $X$, generated by $x$ and $y$ we consider the ball $B(0, h_0(\lambda))$ and a support line $l_x$ to $B(0, h_0(\lambda))$ passing through $x$. Suppose that $\{\lambda_1 y\} = L(0,y) \cap l_x$ is choosen such that $\lambda_1 > 0$. Then $0 < \lambda_1 \leq \lambda$ and from $x \perp y$ it follows:

$$\|x - \lambda_1 y\| \leq \|x - \lambda y\| \leq 1 + \lambda.$$

Using formula (2) for the definition of the squareness modulus we obtain:

$$\xi_X(h_0(\lambda)) \leq 1 + \lambda, \forall\lambda > 0.$$

From $\mu_X^*(\lambda) = \lambda/h_0(\lambda) \geq \sqrt{1+\lambda^2} = \mu_H^*(\lambda), \lambda > 0$, we have that $h_0(\lambda) + 1/(4\lambda^2) < 1$, for all $\lambda > 1$. Pick now $x, y \in S, x \perp y$ such that $\|h(x,y,\lambda)\| \leq h_0(\lambda) + 1/(4\lambda^2)$. For large $\lambda$ one obtains

$$\xi_X\left(h_0(\lambda) + \frac{1}{4\lambda^2}\right) \geq \xi_X(\|h(x,y,\lambda)\|) \geq \|x - \lambda y\| \geq \lambda - 1,$$

implying $h_0(\lambda) \geq \xi_X^{-1}(\lambda - 1) - 1/(4\lambda^2)$. On the other hand $h_0(\lambda) \leq \xi_X^{-1}(\lambda+1)$, and

$$\lambda(1 - \xi_X^{-1}(\lambda+1)) \leq \lambda(1 - h_0(\lambda)) \leq \lambda(1 - \xi_X^{-1}(\lambda-1)) + \frac{1}{4\lambda}.$$

Letting $\beta(\lambda) = \xi_X^{-1}(\lambda+1), \gamma(\lambda) = \xi_X^{-1}(\lambda-1)$, it follows $\beta(\lambda), \gamma(\lambda) \to 1$ for $\lambda \to \infty$ and

$$(1-\beta(\lambda))\xi_X(\beta(\lambda))-1+\beta(\lambda)\le\frac{\lambda}{\mu_X^*(\lambda)}(\mu_X^*(\lambda)-\lambda)\le$$
$$\le(1-\gamma(\lambda))\xi_X(\gamma(\lambda))+1-\gamma(\lambda)+\frac{1}{4\lambda}.$$

Suppose that $X$ is uniformly convex. Using formula (4) we get

$$\lim_{\lambda\to\infty}\frac{\lambda}{\mu_X^*(\lambda)}\cdot(\mu_X^*(\lambda)-\lambda)=0.$$

Now, from [18] we have

$$\lambda/(\lambda+2)\le\lambda/\mu_X^*(\lambda)\le\lambda/\sqrt{1+\lambda^2};\ \lim_{\lambda\to\infty}\lambda/\mu_X^*(\lambda)=1.$$

and $\lim_{\lambda\to\infty}(\mu_X^*(\lambda)-\lambda)=0$. Finally, if (10) holds then

$$0\le\lim_{\lambda\to\infty}[(1-\beta(\lambda))(\xi_X(\beta(\lambda))-1)]\le\lim_{\lambda\to\infty}\frac{\lambda}{\mu_X^*(\lambda)}(\mu_X^*(\lambda)-\lambda)=0,$$

implying $\lim_{\beta\nearrow 1}(1-\beta)\xi_X(\beta)=0$, i.e. $X$ is uniformly convex. ◊

**Corollary 2.7.** *The real normed space $X$ is uniformly smooth if and only if*

$$\lim_{\lambda\to\infty}(\mu_{X^*}^*(\lambda)-\lambda)=0.$$

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