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NEAR-RINGS WITH A SPECIAL CONDITION ON IDEMPOTENTS

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Abstract: Many of the properties enjoyed by idempotents in rings do not carry over into the theory of near-rings. ("Near-ring" in this paper means a left, zero-symmetric near-ring with identity.) This paper examines a subclass of near-rings in which idempotents behave in much the same fashion as in rings. A near-ring N has Property $\mathcal{I}d$ if for each idempotent e and each element x, $(1-e)\,x = x - ex$. Several equivalent formulations are given in terms of right annihilator sets and principal N-subgroups. Examples are given and processes for constructing further examples via direct products, homomorphic images, and sub-direct products are developed. Property $\mathcal{I}d$ is shown to distinguish between various types of near-ring building blocks, for example: near-fields, which have Property $\mathcal{I}d$, and certain centralizer near-rings which do not. These results are applied to 2-primitive and semisimple near-rings. If N has Property $\mathcal{I}d$, then an idempotent e in N is central if either of the following conditions holds: (i) e commutes with all idempotents, (ii) e commutes with all nilpotent elements; if every idempotent satisfies (i) (or(ii)) and N is also von Neumann regular, then N is a sub-direct product of near-fields. A program analogous to von Neumann's for regular rings can be carried out for regular near-rings with Property $\mathcal{I}d$: every principal N-subgroup is a right ideal and a direct summand; finitely generated N-subgroups are principal; and with an additional technical hypothesis, the set of principal N-subgroups form a complemented, modular lattice. This opens up the possibility of connections with continuous geometries and certain algebras of operators.

1. Introduction

Many of the properties enjoyed by idempotents in rings do not carry over into the theory of near-rings. For example, the following three properties are trivial in rings, yet need not hold in near-rings, as Example 3.1 will illustrate:

(i) e is an idempotent if and only if $1-e$ is an idempotent;

(ii) the only element which annihilates both e and $1-e$ is zero;

(iii) if e is an idempotent, then $(1-e)\,e=0$.

This paper examines a subclass of near-rings in which idempotents behave in much the same manner as their ring counterparts. (In this paper near-ring will always mean left, zero-symmetric near-ring with identity; unless otherwise specifically stated the terminology used will conform to that in [12].).

This is the class of all near-rings N such that $(1-e)\,x = x - ex$, for each $x \in N$ and each idempotent $e \in N$ a near-ring in this class is said to have *Property $\mathcal{I}d$.* In Section 2 illustrative examples and basic results for near-rings with Property $\mathcal{I}d$ are given.

It is of interest to ask what important classes of near-rings do not enjoy Property $\mathcal{I}d$. In Section 3 the idempotents in the centralizer near-rings, $M_G(\Gamma)$, where G is a group of fixed-point-free (f.p.f) automorphisms on the group $(\Gamma,+)$ are described explicitly in set theoretic terms and from this it is seen that these near-rings do not have Property $\mathcal{I}d$ except in certain very special cases. Consequently we obtain results concerning 2-primitive near-rings with Property $\mathcal{I}d$.

In Section 4 von Neumann regular near-rings with Property $\mathcal{I}d$ are considered. In a near-ring N of this class the set of principal N-subgroups is closed under addition and hence very finitely generated N-subgroup is principal; each principal N-subgroup is a right ideal (in fact an annihilator) and is a right ideal direct summand. With the addition of an extra technical hypothesis it is then shown that the set of N-subgroups is closed under intersection. This establishes a strict analogy with the famous von Neumann result that the principal right ideals of a regular ring form a complemented modular lattice. It is shown that regular near-rings without $\mathcal{I}d$ need not have these extraordinary properties.

We use $\underline{\mathbf{r}}_N(S) = \{m \in N : Sm = 0\}$, for the right annihilator of the subset S of the near-ring N. If no ambiguity will arise, we use simple $\underline{\mathbf{r}}(S)$.

2. Property $\mathcal{I}d$

The subclass which we are led to examine is called to our attention by the following:

Proposition 2.1. *Let e be an idempotent in a near-ring N. The following are equivalent:*

(2.1.1) $\underline{\mathbf{r}}(e) \cap \underline{\mathbf{r}}(1-e) = 0$ *and* $(1-e)\,e = 0$,

(2.1.2) $\underline{\mathbf{r}}(1-e) = eN$,

(2.1.3) $(1-e)\,x = -ex + x$ *for each* $x \in N$,

(2.1.4) $(1-e)\,x = x - ex$ *for each* $x \in N$.

Proof. First observe that $1-e$ is an idempotent if and only if $(1-e)\,e = 0$. Then, given (2.1.1), $e \in \underline{\mathbf{r}}(1-e)$ and hence $eN \subseteq \underline{\mathbf{r}}(1-e)$. If $x \in \underline{\mathbf{r}}(1-e)$, then $e(x-ex) = 0 = (1-e)(x-ex)$ and $x - ex \in \underline{\mathbf{r}}(e) \cap \underline{\mathbf{r}}(1-e) = 0$; hence $x = ex \in eN$.

Given (2.1.2), $e = e^2 \in eN = \underline{\mathbf{r}}(1-e)$ and so $(1-e)\,e = 0$; consequently $1-e$ is also an idempotent. For any $x \in N$, since $(1-e)\,[x-(1-e)x] = 0$, it follows that $x - (1-e)\,x \in \underline{\mathbf{r}}(1-e)$. Thus there exists $y \in N$ such that $ey = x-(1-e)\,x$; but $ey = e^2y = e\,[x-(1-e)\,x] = ex$. Hence $x - (1-e)\,x = ex$ or $-ex + x = (1-e)\,x$.

Given (2.1.3), we have $(1-e)\,e = -e^2 + e = 0$ and if $x \in \underline{\mathbf{r}}(e) \cap \underline{\mathbf{r}}(1-e)$, then $0 = (1-e)\,x = -ex + x = x$. The implication (2.1.4)$\Longrightarrow$ (2.1.1) follows similarly. Noting that (2.1.1), (2.1.2), and (2.1.3) have been shown to be equivalent, the proof is concluded by showing that (2.1.1)$\Longrightarrow$ (2.1.4). Note that $e\,[(-ex+x)-(x-ex)] = 0$ and $(1-e)[(-ex+x)-(x-ex)] = -(1-e)ex + (1-e)x - [(1-e)x - (1-e)ex] = 0 - ex + x - [-ex + x - 0] = 0$. So $(-ex+x)-(x-ex) \in \underline{\mathbf{r}}(e) \cap \underline{\mathbf{r}}(1-e) = 0$ and hence $x - ex = -ex + x = (1-e)\,x$. $\Diamond$

The near-rings for which (2.1.1)–(2.1.4) hold for each idempotent e are exactly those with Property $\mathcal{I}d$. For brevity we say "N is a near-ring with $\mathcal{I}d$".

Corollary 2.2. *If e is an idempotent satisfying* (2.1.1) *through* (2.1.4), *then*

(2.2.1) $1-(1-e) = e$ *and* $1-e = -e+1$,

(2.2.2) $1-e$ *is an idempotent satisfying* (2.1.1) *through* (2.1.4). *Consequently, e is an idempotent satisfying* (2.1.1)–(2.1.4) *if and only if* $1-e$ *is also.*

Proof. Since $e\,[1-(1-e)-e] = 0$ and $(1-e)\,[1-(1-e)-e] = 0$, it follows that $1-(1-e)-e = 0$, or $1-(1-e) = e$ and $1-e = -e+1$.

Using $(1-e)\,e = 0$ we have that $1-e$ is an idempotent. From (2.2.1), $[1-(1-e)]\,x = ex$ for each $x \in N$. However, $-(1-e)\,x + x =$

$= -(x - ex) + x = ex$; so $[1 - (1 - e)]\, x = -(1 - e)\, x + x$ and $1 - e$ satisfies (2.1.3).

Conversely, if $1 - e$ satisfies (2.1.1)–(2.1.4), then by the previous argument $1-(1 - e)$ is an idempotent satisfying them also. Using $1-e$ as the idempotent in (2.2.1) we have $1-[1 - (1 - e)] = 1-e$, or $1+1-e-1 =$ $= 1 - e$, and hence $1 - e - 1 = -e$; transposing we get $e = 1 - (1 - e)$. Thus e satisfies (2.1.1)–(2.1.4). ◊

If N has $\mathcal{I}d$, then for each idempotent $e \in N$, the Peirce decomposition becomes: $N = eN \oplus (1 - e)\, N$, with $\underline{\mathbf{r}}(e) = (1 - e)\, N, \underline{\mathbf{r}}(1 - e) = eN$, and e and $1 - e$ are orthogonal idempotents. Elementwise, $m = em +$ $+ (1 - e)\, m$, for each $m \in N$, which does not hold for near-rings in general. Observe that the Peirce decomposition is now a direct sum of right ideals of N, and in particular one obtains a group direct sum instead of just a semidirect sum. Note that if a near-ring N has a Peirce decomposition as just described, for each idempotent e, then N will have $\mathcal{I}d$.

Near-rings without proper idempotents (i.e. the only idempotents are 0 and 1) trivially have Property $\mathcal{I}d$. This includes near-fields, integral near-rings, and local near-rings, (Maxson [9] showed that local near-rings have no proper idempotents. He also gave various examples of local near-rings in [10], [11]). It is worthwhile to recall that there are integral d.g. near-rings which are neither fields nor rings.

The construction in the next example illustrates that there are near-rings with $\mathcal{I}d$ which are relatively rich in idempotents.

Example 2.3. Let R be a commutative ring with identity and let M be the set of all formal power series over R whose lead coefficient is zero, i.e., those of the form $a_1x + a_2x^2 + \cdots\cdots$. If one defines addition pointwise and multiplication to be composition, i.e., $(x)\, p \circ q = ((x)\, p)\, q$, then $(M, +, \cdot)$ is a left zero-symmetric near-ring with identity. The subset M^* of all polynomials is a subnear-ring with the same identity element. Let $M_k, k = 1, 2, ...,$ be the set of all elements of M whose coefficients up through that of x^{k-1} are zero. The M_k are ideals of M. Similarly define the subsets M_k^* of M_k; these are ideals of M^*. A routine calculation shows that the idempotents in M and M^* are exactly the elements of the form ex where e is an idempotent in R. From this point it is easy to see that M, M^*, and each of the factor near-rings M/M_k and M^*/M_k^* have Property $\mathcal{I}d$. In order for these examples to have many idempotents, just choose R to be rich in idempotents; for example, take R to be Boolean.

From the examples given so far, and using the basic constructions to be discussed in the remainder of this section, one can build a large assortment of near-rings with $\mathcal{I}d$. It is worthwhile to consider the class,

$\mathcal{I}d$, of all near-rings with $\mathcal{I}d$. We show $\mathcal{I}d$ is closed under direct products, finite direct sums, and certain other processes. Observe that the class of all rings (with unity) is in $\mathcal{I}d$. Since a Boolean near-ring with unity is a ring, [14, p.300] , all such near-rings are in $\mathcal{I}d$. Of course, our main interest is in near-rings that are not rings.

Proposition 2.4. *Let $\Phi : N \to \overline{N}$ be a surjective near-ring homomorphism. If N has $\mathcal{I}d$ and if every cyclic subsemigroup of $(N, \cdot)$ is finite, then $\overline{N}$ has $\mathcal{I}d$. In particular, every homomorphic image of a finite near-ring with $\mathcal{I}d$ is also a near-ring with $\mathcal{I}d$.*

Proof. Let $u\Phi$ be any idempotent in $\overline{N}$, with $u\Phi \neq \overline{0}$. Since the multiplicative semigroup generated by u is finite, and since u cannot be nilpotent, some power of u is a nonzero idempotent. Let e be that idempotent and observe that $u\Phi = e\Phi$. For any $\overline{x} = x\Phi$ in $\overline{N}$, we then have: $(\overline{1} - c\Phi)\,\overline{x} = (1\Phi - e\Phi)\,(x\Phi) = ((1-e)\,x)\,\Phi = (x - ex)\,\Phi = x\Phi - e\Phi x\Phi$. So $\overline{N}$ has $\mathcal{I}d$. $\Diamond$

Question 2.5. Is $\mathcal{I}d$ closed under homomorphisms? The obstacle to using the proof scheme used in Prop. 2.4 is that new idempotents may be created in the image, and these might not lift.

Proposition 2.6. *Let $N_i, i \in I$, be near-rings. The direct product $N = \Pi N_i$, $i \in I$, has Property $\mathcal{I}d$ if and only if each N_i has Property $\mathcal{I}d$.*

Proof. If each N_i has $\mathcal{I}d$, then using the notation $< x_i >= x$ for the elements in the direct product, given any idempotent $e =< e_i >$, each e_i is an idempotent in N_i and $(1_i - e_i)\,x_i = x_i - e_i x_i$, which implies that $(1-e)\,x = x - ex$ and hence N has Property $\mathcal{I}d$. Conversely, if N has $\mathcal{I}d$, note that in the pre-image of an idempotent e_j under the j-th projection mapping on $N = \Pi N_i$, there is the idempotent $e =< \delta_i >$, where $\delta_i = 0$, if $i \neq j$, and $\delta_j = e$. Then from $(1-e)\,x = x - ex$ one obtains $(1_j - e_j)\,x_j = x_j - e_j x_j$, for each $x_j \in N_j$, by applying the j-th projection. $\Diamond$

Using similar arguments involving projection mappings we obtain the next two results.

Proposition 2.7. *If N is a subdirect product of near-rings with property $\mathcal{I}d$, and N has an identity, then N has property $\mathcal{I}d$. In particular, a finite direct sum of near-rings with $\mathcal{I}d$ has $\mathcal{I}d$ itself.*

Proposition 2.8. *Every direct summand of an near-ring with $\mathcal{I}d$ also has $\mathcal{I}d$.*

Subnear-rings containing the identity of a near-ring with $\mathcal{I}d$ must have $\mathcal{I}d$ themselves. Of course in general $\mathcal{I}d$ is not inherited by subnear-rings.

Proposition 2.9. *Let N be a near-ring with $\mathcal{I}d$.*

(2.9.1) *If S is a subnear-ring of N and S contains the identity for N, then S has $\mathcal{I}d$.*

(2.9.2) *If T is N-subgroup of N and the semigroup$(T,\cdot)$ has a left identity, u, then $T = \underline{\mathbf{r}}_N(1-u)$ and T is a right ideal direct summand of N. Consequently, T is a normal subgroup of $(N,+)$.*

Proof. (2.9.1) This part is immediate.

(2.9.2) Observe that $S = uN$. (This does not depend on $\mathcal{I}d$). Then using Property $\mathcal{I}d$ we obtain $S = \underline{\mathbf{r}}_N(1-u)$, and the rest follows immediately. $\Diamond$

We next show that several ring theoretic properties of idempotents hold for near-rings with property $\mathcal{I}d$, but that these properties do not hold in general for zero-symmetric near-rings with identity.

Proposition 2.10. *Let N be a near-ring with Property $\mathcal{I}d$ and e an idempotent in N. Then:*

(2.10.1) *$ex = xe$ if and only if $(1-e)x = x(1-e)$;*

(2.10.2) *if e commutes with all nilpotent elements in N, then e is central; and*

(2.10.3) *if e commutes with all idempotents in N, then e is central.*

Proof. (2.10.1) If $ex = xe$, then Property $\mathcal{I}d$ implies $(1-e)x = x - ex = = x - xe = x(1-e)$. Now $1-e$ is also an idempotent so that $(1-e)x = = x(1-e)$ implies $[1-(1-e)]x = x[1-(1-e)]$; but $1-(1-e) = e$ since N has $\mathcal{I}d$.

(2.10.2) Let $x \in N$. Observe that $[ex(1-e)]^2 = 0 = [(1-e)xe]^2$. Thus $ex - exe = ex(1-e) = e[ex(1-e)] = [ex(1-e)]e = 0$ and similarly $xe - exe = 0$. Hence $ex = exe = xe$ and e is central.

(2.10.3) First we show that $e + (1-e)xe$ is an idempotent for each $x \in N$. Since N has Property $\mathcal{I}d$, it suffices to show that $[e + (1 - -e)xe]^2 - [e + (1-e)xe]$ is a right annihilator of both e and $1-e$. The argument is a calculative one quite similar to that for rings because of the properties of near-rings with $\mathcal{I}d$. Since $1-e$ is an idempotent and $1-(1-e) = e$, the argument is symmetrical in e and $1-e$ and it then follows that $(1-e) + ex(1-e)$ is also an idempotent.

Since $e[(1-e) + ex(1-e)]e = 0$ and e commutes with the idempotent $1 - e + ex(1-e)$, we have

$$0 = e[(1-e) + ex(1-e)] = ex(1-e) = ex - exe.$$

So $ex = exe$. By (2.10.1) we see that the argument is symmetrical in e and $1-e$; that is, $(1-e)xe = 0$. So again invoking Property $\mathcal{I}d$ we have $xe - exe = 0$ and hence $xe = exe = ex$. $\Diamond$

Corollary 2.11. *Let N be a near-ring with $\mathcal{I}d$, then $e \in N$ is a central idempotent if and only if $1 - e$ is a central idempotent.*

Further examples showing that properties (2.10.1)–(2.10.3) do not hold in general for zero-symmetric near-rings with identity can be found in Clay's tables on near-rings defined on the dihedral group of order eight [3, Table 3 p.256].

3. $M_G(\Gamma)$ and 2-primitive near-rings

In this section we consider the centralizer near-rings of the form $M_G(\Gamma)$, where $(G, \circ)$ is a group of fixed-point free automorphisms on a group $(\Gamma, +)$. These are well-known for the role they play in the structure developed in terms of 2-primitive near-rings and the radical J_2. Here we describe in set theoretic terms all of the idempotents in such $M_G(\Gamma)$, and then use this to show that such near-rings are "almost never" in $\mathcal{I}d$. We then use the results obtained to obtain the structure of 2-primitive near-rings with Property $\mathcal{I}d$.

Example 3.1. Recall that an idempotent e in $M_0(\Gamma)$ is completely and uniquely determined by a subset Φ of Γ which contains zero (this subset being the fixed points of e) and a mapping $f : \Lambda \to \Phi$, where Λ is the set complement of Φ in Γ. With this in mind we turn to the idempotents in $M_G(\Gamma)$.

Let G be a group of f.p.f. automorphisms on $(\Gamma, +)$. Define Φ to be the union of the orbit $\{0\}$ and at least one other orbit of G. Let Λ be the complement of Φ in Γ. Define $\varphi e = \varphi$ for each $\varphi \in \Phi$. Given any orbit $O_z \subseteq \Lambda$, choose some representative $\lambda_z \in O_z$ and choose $\beta_z \in \Phi$. Then define e on O_z as $(g\lambda_z) e = g\beta_z$, $g \in G$. This will be well-defined on all of O_z because $G\lambda_z = O_z$ and G is a group of f.p.f. automorphisms on $(\Gamma, +)$. This defines e on all of Γ. It is easy to see that e is an idempotent and is in $M_G(\Gamma)$. Note that in order to obtain an idempotent different from the identity G must have at least three orbits in Γ.

Conversely, let e be an idempotent in $M_G(\Gamma)$, $e \neq 0, 1$. Since $e \in$ $\in M_0(\Gamma)$ we select the set Φ of fixed points of e and the complementary set Λ. If $\gamma_1, \gamma_2 \in \Gamma$ are equivalent with respect to the equivalence relation induced by G on Γ, written $\gamma_1 \sim \gamma_2$, then there exists $g \in G$ such that $g\gamma_1 = \gamma_2$ and consequently $\gamma_2 e = (g\gamma_1) e = g(\gamma_1 e)$, or $\gamma_2 e \sim \gamma_1 e$. Given $\varphi \in \Phi$, for any $g \in G, (g\varphi) e = g(\varphi e) = g\varphi$; so $g\varphi$ is also a fixed point. Thus Φ is a union of orbits and Λ is the union of the remaining orbits. If O is any orbit contained in Λ, select $\lambda_0 \in O$. Then $\lambda_0 e = \beta_0 \in \Phi$. For each $\lambda \in O$, there exists a unique $g \in G$ such that $g\lambda_0 = \lambda$. Then

$\lambda e = (g\lambda_0)\, e = g\,(\lambda_0\, e) = g\beta_0$. Thus all of the idempotents are uniquely described by specifying Φ, the orbits contained in the complementary set Λ, and by choosing a representative element from each orbit contained in Λ and its image in Φ.

Proposition 3.2. *If G is a group of f.p.f. automorphisms on $(\Gamma, +)$ and G has at least three orbits on Γ, then $M_G(\Gamma)$ does not have Property $\mathcal{I}d$. Consequently, if $|\Gamma| \geq 3$, then $M_0(\Gamma)$ does not have Property $\mathcal{I}d$.*

Proof. Let $\{0\}, O_1, O_2$ be distinct orbits of G on Γ. Define e as follows: let $O_1 \cup \{0\} = \Phi$, the set of fixed points of e, and let Λ be the complement of Φ in Γ; then $O_2 \subseteq \Lambda$. Choose $\lambda_0 \in O_2$ and $\beta_0 \in O_1$ and define $(g\lambda_0)\, e = g\beta_0$ for each $g \in G$. Finally if O_t is any other orbit of G contained in Λ select a representative $\lambda_t \in O_t$ and any non-zero $\beta_t \in \Phi$ and define $(g\lambda_0)\, e = g\beta_0$ for each $g \in G$. As we have seen this defines an idempotent e in $M_G(\Gamma)$. Note that $e \neq 0, 1$ and that $\gamma e = 0$ implies $\gamma = 0$. From the construction, $\lambda_0 - \lambda_0 e \neq 0$ and hence $(\lambda_0 - \lambda_0 e)\, e \neq 0$. However, if $M_G(\Gamma)$ has Property $\mathcal{I}d$, then $0 = \lambda_0\,(1 - e)\, e = (\lambda_0 - \lambda_0 e)\, e$.

Taking $G = \{1_\Gamma\}$ we obtain that $M_0(\Gamma)$ does not have Property $\mathcal{I}d$ if $|\Gamma| \geq 3$. $\Diamond$

Note that if G has only two orbits on Γ, then the only idempotents are 0 and 1 and in this case $M_G(\Gamma)$ does have Property $\mathcal{I}d$.

Example 3.3. Recall that if $(\Gamma, +)$ is a finite simple, nonabelian group, then $M_0(\Gamma) = E(\Gamma)$. This yields a large class of examples of d.g., near-rings which are von Neumann regular and simple, but do not have Property $\mathcal{I}d$. This is in sharp contrast to what happens for rings.

Proposition 3.4. *Let N be a 2-primitive near-ring with d.c.c. on right ideals. If N has Property $\mathcal{I}d$, then either N is a ring or N is a near-field. Consequently $(N, +)$ is commutative.*

Proof. If N is not a ring, then $N = M_G(\Gamma)$, where G is a group of f.p.f. automorphisms on the group $(\Gamma, +)$, [1, 2.5], [12, Th. 4.16]; also N contains a non-zero idempotent e which is itself contained in a minimal right ideal K. Since $N = M_G(\Gamma)$ has Property $\mathcal{I}d$, $e = 1$ and hence $K = N$.

If $f \in N$ and $\gamma \in \Gamma - \{0\}$, then $\gamma f = 0$ implies $\Gamma f = 0$; however, the set of all f which annihilate Γ is a right ideal of N and hence must be zero . So for an arbitrary $f \in N$ and an arbitrary $\gamma \in \Gamma - \{0\}$, define $\gamma f = \beta \neq 0$. Let $\sigma \in \Gamma - \{0\}$, then there exists $g \in G$ such that $g\beta = \sigma$. So $\sigma = g\beta = g\,(\gamma f) = (g\gamma)\, f$ and f is surjective. So very element of the multiplicative monoid $N - \{0\} = M_G(\Gamma) - \{0\}$ has a left inverse and consequently the monoid is a group. This establishes that N is a near-field; the commutativity of addition of such is well-known. $\Diamond$

Corollary 3.5. *Let N have Property $\mathcal{I}d$. If N has d.c.c. on right ideals and $J_2(N) = 0$, then $N = A \oplus B$, where A is a semi-simple Artinian ring and B is a finite direct sum of near-fields.*
Proof. This follows immediately from 2.8, 3.4, and standard results on semi-simple rings or near-rings. ◊

Observe that N such as given in 3.5 always have commutative additive group. Also, if in 3.5 N is assumed to be d.g., then N is a ring, since d.g. near-fields are skew-fields.

From 3.2-3.5 we see where not to look for near-rings with $\mathcal{I}d$. From another viewpoint, Property $\mathcal{I}d$ distinguishes between two important classes of simple near-rings: the near-fields, which have $\mathcal{I}d$, and certain $M_G(\Gamma)$, especially the $M_0(\Gamma)$, which do not. A question still to be resolved is which $M_G(\Gamma)$, G a group of automorphisms on Γ or more generally a monoid of endomorphisms on Γ, have Property $\mathcal{I}d$.

4. Von Neumann regular near-rings with $\mathcal{I}d$

Von Neumann regular near-rings, the strict analog of von Neumann regular rings, have been considered in some detail (see Pilz [14, p.330], or the most recent Near-Ring Newsletter containing a full bibliography). In this section a near-ring version of von Neumann's theorem concerning the lattice theoretic structure for the principal right ideals of a regular ring will be developed. For brevity we shall use the term regular in place of von Neumann regular.

An N-subgroup S of a near-ring N is said to be a *principal N-subgroup* if $S = xN$, for some $x \in N$.
Proposition 4.1. *Let N has Property $\mathcal{I}d$. Then N is regular if and only if every principal N-subgroup is a direct summand as a right ideal.*
Proof. It is well known that even without the assumption of Property $\mathcal{I}d$, N is regular if and only if very principal N-subgroup is generated by an idempotent [2]. So given any principal N-subgroup eN, one uses property $\mathcal{I}d$ to write $N = eN \oplus \underline{\mathbf{r}}(e)$, where $eN = \underline{\mathbf{r}}(1-e)$, as a direct sum of right ideals. Conversely, assume every principal N-subgroup is a direct summand as a right ideal. For any $x \in N, N = xN \oplus I$, where xN and I are right ideals of N. Write $1 = xy + z$, where $y \in N, z \in I$. Since the sum comes from the direct sum of right ideals we have $x = (xy+z)x = xyx + zx$. But $zx = -xyx + x \in xN$ and $zx \in I$ because I is a N-subgroup. So $zx = 0$ and $x = xyx$. Thus N is regular. ◊

Observe that in a von Neumann regular near-ring with $\mathcal{I}d$ very principal N-subgroup is a right ideal and an annihilator set.

Example 4.2. A principal N-subgroup of a regular near-ring need not be a direct summand as a right ideal; in fact it need not even be a right ideal. Let $f \in M_0(Z_3)$ be defined by $\gamma f = 0$ if $\gamma = 0$ and $\gamma f = 1$ otherwise. Then $fM_0(Z_3)$ is a principal $M_0(Z_3)$ subgroup in the simple regular near-ring $M_0(Z_3)$. However, it is not a right ideal.

Note that it is indeed right ideals which are the natural objects to consider and not normal N-subgroups because right ideals are submodules in the near-ring module N_N and hence behave correctly with respect to N-homomorphisms, whereas normal N-subgroups need not do so. However, by invoking property $\mathcal{I}d$ we do get some of the desired behavior for N-subgroups, as the next results show.

Proposition 4.3. *Let N be a near-ring with the property that if $N = = N_1 \oplus N_2$ as a direct sum of normal N-subgroups, then N_1 and N_2 have property $\mathcal{I}d$. If N satisfies either d.c.c. on a.c.c. on N-subgroups, then N is a finite direct sum of normal N-subgroups, each of which is a near-ring with no proper idempotents. Furthermore, if N is regular, then each of the direct summands is a near-field, and consequently $(N,+)$ is commutative.*

Proof. By hypothesis N has $\mathcal{I}d$. If N has no proper idempotents we are finished. Otherwise, there exists a proper idempotent e such that $N = eN \oplus (1-e)N$, and $eN = \underline{\mathbf{r}}_N(e), (1-e)N = \underline{\mathbf{r}}_N(e)$. Thus eN and $(1-e)N$ have $\mathcal{I}d$. Because of symmetry we can deal just with eN. So eN has $\mathcal{I}d$ and hence has a two-sided identity, which must be e. Apply the decomposition process to the near-ring eN. Note that if R is a right eN-subgroup direct summand for eN, then R is normal in $(N,+)$ also. If $r \in R$ and $n \in N$, then $rn = (re)n = r(en) \in R$; so R is a N-subgroup. Thus the repeated use of the decomposition process must eventually terminate under the assumption of either chain condition on N-subgroups. Complete termination occurs when all summands have no proper idempotents and then can no longer undergo a proper Peirce decomposition.

If N is also regular, then for any non-zero, non-identity m in a given summand M, (of the decomposition given above) there exists $a \in M$ such that ma and am are idempotents and $mam = m$. So $ma = am = 1_M$ and hence M is a near-field. Since the additive group of a near-field is commutative and $(N,+)$ is the direct sum of these commutative groups, we have $(N,+)$ is commutative. ◊

The question is opened as to whether normal N-subgroup can be replaced by right ideal in Prop. 4.3. In the situation where right ideal direct summands are themselves near-rings with $\mathcal{I}d$ one can show that

eN and $(1-e)\,N$ are orthogonal. In attempting to push through the proof of 4.3 in this case the difficulty is that a right ideal direct summand of eN need not be a right ideal of N itself. This serves to further illustrate the limitations on ring theoretic-like techniques in studying near-rings.

Proposition 4.4. *If N is a regular near-ring with $\mathcal{I}d$, then the sum of any two principal N-subgroups of N is again a principal N-subgroup.*

Proof. Without loss of generality consider e_1N, e_2N where e_1 and e_2 are idempotents. Then $e_iN = \underline{\mathbf{r}}_N\,(1-e_i)\,,\ i=1,2$ and $N = e_1N \oplus\ \underline{\mathbf{r}}_N(e_1)$. Thus e_1N+e_2N is a right ideal and an N-subgroup. Write $e_2 = e_1e_2 + {}+(1-e_1)\,e_2$. For any $x,y\in N$, $e_1x+e_2y = e_1x+[e_1e_2+(1-e_1)\,e_2]\,y$. Since the two terms in the bracket are from different right ideal direct summands, $[e_1e_2+(1-e_1)e_2]y = e_1e_2y+(1-e_1)e_2y$ and $e_1x+e_2y = {}= e_1e_2+e_1e_2y+(1-e_1)e_2y = e_1(e_2+e_2y)+(1-e_1)e_2y$. So $e_1N+e_2N \subseteq {}\subseteq e_1N+(1-e_1)e_2N$. For each $z,t\in N$, $e_1z+(1-e_1)e_2t = e_1z-e_1e_2t+e_2t$, using $\mathcal{I}d$, and hence $e_1z+(1-e_1)e_2t = e_1(z-e_2t)+e_2t$, or $e_1N + {}+(1-e_1)\,e_2N \subseteq e_1N+e_2N$. Hence $e_1N+(1-e_1)\,e_2N = e_1N+e_2N$.

There exists $a\in N$ such that $[(1-e_1)e_2]a[(1-e_1)e_2] = (1-e_1)e_2$ and $e_2' = (1-e_1)e_2a$ is an idempotent. Then $e_1e_2' = 0$ and $e_2' \in\ \underline{\mathbf{r}}(e_1)$. Also, $(1-e_1)\,e_2N = e_2'N$, so $e_1N+e_2N = e_1N+e_2'N$. Let $b = -e_2'e_1+e_1+e_2' = {}= (1-e_2')\,e_2+e_2'$. Then for each $x,y\in N$, by using $\mathcal{I}d$ and the fact that elements distribute over sums from distinct right ideal direct summands we have:

$$b(e_1x+e_2'y) = be_1x+be_2'y = [(1-e_2')e_1+e_2']e_1x+[(1-e_2')e_1+$$
$$+e_2']e_2'y = [(1-e_2')e_1x+e_2'e_1x]+[(1-e_2')e_1e_2'y+e_2'e_2'y] = e_1x+e_2'y.$$

So $e_1N+e_2'N\subseteq bN$. Now for each $n\in N$,

$$bn = [(1-e_2')e_1+e_2']n = (1-e_2')e_1n+e_2'n =$$
$$= e_1n-e_2'e_1n+e_2'n = e_1n+e_2'[-e_1n+n].$$

This establishes that $e_1N+e_2'N = bN$ and hence $e_1N+e_2N = bN$.◊

Corollary 4.5. *If N is a regular near-ring with $\mathcal{I}d$, then any finitely generated N-subgroup is a principal N-subgroup.*

Proof. This follows from 4.4 by induction on the number of generators.◊

Observe that the sum of two principal $M_0\,(\Gamma)$-subgroups of $M_0\,(\Gamma)$ need not be a principal $M_0\,(\Gamma)$-subgroup; the $M_0\,(\Gamma)$ are regular near-rings [2].

Proposition 4.6. *Let N be a regular near-ring with Property $\mathcal{I}d$. If for each two element set S of N, $\underline{\mathbf{r}}_N\,(S)$ is a principal N-subgroup, then the principal N-subgroups of N form a complemented modular lattice with respect to addition and intersection.*

Proof. Let $L(N)$ be the set of principal N-subgroups of N. From 4.4 it is closed under addition. Closure under intersection follows from the hypothesis on the $\underline{\mathbf{r}}_N(S)$. So $L(N)$ is a sublattice of the lattice of all normal subgroups of $(N,+)$ and hence it is modular. That it is complemented follows from 4.1. $\Diamond$

The condition on the $\underline{\mathbf{r}}(S)$ is clearly a blemish to be removed. Nevertheless, 4.6 raises hope that an interesting and fruitful connection between regular near-rings and lattice theory, continuous geometries, and certain algebras of operators can be established in a manner analogous to that for von Neumann regular rings. This might prove useful in the development of quantum mechanics based on nonlinear operators.

Certain paths are definitely closed, however. For convenience call a class of near-rings *sterile* if all of its members are rings. Then several classes which one might wish to investigate in relation to regular near-rings are sterile, for example: *-regular near-rings, Baer *-near-rings, von Neumann near algebras, and near rings of matrices over a regular near-ring. The sterility of the first three classes follows from the fact that a zero symmetric near-ring (not necessarily with identity) supports an involution only if it is distributive [6], and a distributive near-ring which has an identity is a ring. The sterility of the latter class becomes apparent when one notes that (i) for $n > 1$, the set of $n \times n$ matrices over a near-ring N (zero symmetric, but not necessarily with identity) is itself a near-ring only if N is n-distributive [8]; and (ii) a regular n-distributive near-ring is a ring [5]. (The phrase "matrices over a near-ring" refers to the concept as used in [5] and [8], not to the construction due to Meldrum and Van der Walt [13]). In a subsequent paper we will show how the barrier erected by the involution-identity-regularity contretemps can be circumvented to some extent to obtain near-ring versions of certain "ring-of-operator " type results. One method to be used is the more general definition of involution found in [4]. Another possible route is to use structural matrix near-rings.

It is easy to see that a subdirect product of regular near-rings with Property $\mathcal{I}d$ has Property $\mathcal{I}d$. So, in particular, a subdirect product of near-fields is a regular near-ring with Property $\mathcal{I}d$. In the next proposition a converse is given which improves upon that found in [7, Th. 4.4].

Proposition 4.7. *Let N be a regular near-ring with Property $\mathcal{I}d$, Assume either of the following hold:*

(4.7.1) *Every idempotent commutes multiplicatively with every other idempotent,* (4.7.2) *Every idempotent commutes multiplicatively with every nilpotent element.*

Then:

(a) *if N is subdirectly irreducible, then N is a near-field;*

(b) *N is isomorphic to a subdirect product of near-fields and hence $(N, +)$ is commutative;*

(c) *if N is d.g., then N is isomorphic to a subdirect product of skew fields.*

Proof. From 2.10 every idempotent of N is central. A subdirectly irreducible regular near-ring whose idempotents are central is a near-field [11]. Since every near-ring is isomorphic to a subdirect product of subdirectly irreducible near-rings, (b) follows immediately. Since *d.g.* near-fields are skewfields, (c) follows from (b). ◇

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