

# ON THE INTEGRABILITY OF WALSH-FOURIER TRANSFORM

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**Dedicated to Professor Ferenc Schipp on his 60th birthday**

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**Abstract:** In this paper we introduce a Hardy space by means of the stopped dyadic maximal function on  $[0, \infty)$ . We show that this Hardy space has an atomic structure and identify the atoms. Then, using the Hardy norm, a Sidon type inequality is proved for the generalized Walsh-Dirichlet kernels, the periodic version of which was due to Schipp [7], [8]. Finally, we apply this inequality to construct a sufficient condition for the integrability of Walsh-Fourier transform.

## Introduction

The *binary expansion* of  $x \in [0, \infty)$  is defined as

$$x = \sum_{j=-\infty}^{\infty} x_j 2^{-j-1},$$

where  $x_j = 0$  or  $1$ . In case when there are two expansions of this form, i.e. in case of dyadic rationals, we take the one that terminates in 0's. The sequence  $(x_j, j \in \mathbb{Z})$  is called the *binary form* of  $x$ . For any two nonnegative numbers  $x, y$  their *dyadic sum* is defined by

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$$(1) \quad x \dot{+} y = \sum_{j=-\infty}^{\infty} |x_j - y_j| 2^{-j-1}.$$

The functions

$$(2) \quad \psi_y(x) = (-1)^{\sum_{j=-\infty}^{\infty} x_j y^{-j-1}} \quad (0 \leq x, y < \infty)$$

are called *generalized Walsh functions*. It is clear that

$$(3) \quad \psi_x(y) = \psi_y(x) \quad (0 \leq x, y < \infty),$$

and

$$(4) \quad \psi_t(x)\psi_t(y) = \psi_t(x \dot{+} y) \quad (0 \leq x, y < \infty, x \dot{+} y \text{ dyadic irrational}).$$

The *generalized Dirichlet kernels* are defined as

$$D_t(y) = \int_0^t \psi_u(y) du \quad (0 \leq t, y < \infty).$$

It is known (see [1] or [10]) that

$$(5) \quad D_{2^n}(y) = \begin{cases} 2^n & y \in [0, 2^{-n}) \\ 0 & y \notin [0, 2^{-n}) \end{cases}$$

for  $n \in \mathbb{Z}$ , and

$$(6) \quad |D_t(y)| \leq \frac{3}{y} \quad (0 \leq y, t < \infty, y \neq 0).$$

For any  $0 \leq a < b \leq \infty$  and  $1 \leq p \leq \infty$  let  $L^p[a, b)$  stand for the usual function space with the corresponding norm.

For each  $f \in L^1[0, \infty)$  the *Walsh-Fourier transform*  $\hat{f}$  and the *Walsh-Dirichlet integrals*  $S_t f$  ( $0 \leq t < \infty$ ) are defined as

$$\hat{f}(y) = \int_0^{\infty} f(t)\psi_y(t) dt, \quad S_t f(y) = \int_0^t \hat{f}(x)\psi_x(y) dx \quad (0 \leq x, y < \infty).$$

Clearly,

$$(7) \quad S_t f(y) = (f * D_t)(y) = \int_0^{\infty} f(u)D_t(y \dot{+} u) du \quad (0 \leq y < \infty).$$

The intervals of the form  $[k2^n, (k+1)2^n)$  ( $k \in \mathbb{N}$ ,  $n \in \mathbb{Z}$ ) are called *dyadic intervals*. The collection of all dyadic intervals will be denoted by  $\mathcal{I}$ . Let  $I_n(x)$  ( $0 \leq x < \infty$ ,  $n \in \mathbb{Z}$ ) stand for the dyadic interval containing  $x$  whose length is  $2^{-n}$ . The length of  $I \in \mathcal{I}$  will be denoted by  $|I|$ .

In view of (5) and (7) we have that

$$S_{2^n} f(x) = 2^n \int_{I_n(x)} f \quad (f \in L^1[0, \infty), 0 \leq x < \infty).$$

Consequently,

$$(8) \quad \lim_{n \rightarrow \infty} S_{2^n} f(x) = f(x)$$

for any  $x$  at which  $f$  is continuous.

Let  $f \in L^1_{loc}[0, \infty)$ , i.e. let  $f$  be integrable over every finite interval. Then define the *stopped dyadic maximal function*  $\mathcal{M}f$  by

$$\mathcal{M}f(x) = \sup_{0 \notin I_n(x)} 2^n \left| \int_{I_n(x)} f \right| \quad (0 \leq x < \infty).$$

The corresponding Hardy space, i.e. the collection of locally integrable functions whose stopped dyadic maximal function is integrable, will be denoted by  $H$ . Set  $\|f\|_H = \|\mathcal{M}f\|_1$  ( $f \in H$ ). Obviously,  $H \subset L^1[0, \infty)$ .

Throughout this paper  $C$  will denote an absolute positive constant not necessarily the same in different occurrences.

## Results

Let  $\chi_A$  stand for the characteristic function of  $A \subset [0, \infty)$ . Now we introduce the concept of  $H$ -atoms. Let  $b \in L^\infty[0, \infty)$ .  $b$  will be called an  $H$ -atom if either  $b = 2^{-(n-1)} \chi_{[2^{n-1}, 2^n)}$  with some  $n \in \mathbb{Z}$  ( $H$ -atom of first type), or there exists  $I \in \mathcal{I}$  such that  $0 \notin I$ ,  $\text{supp } b \subset I$ ,  $\int_I b = 0$ , and  $\|b\|_\infty \leq |I|^{-1}$  ( $H$ -atom of second type).

We will show that  $H$  has an atomic structure. Namely, each function in  $H$  can be decomposed as a sum of  $H$ -atoms. Its advantage is that several statements with respect to  $H$  are enough to be proved for  $H$ -atoms. This is a real benefit since  $H$ -atoms are easier to work with.

**Theorem 1.**  $f \in H$  if and only if there exist real numbers  $\lambda_\ell$  and  $H$ -atoms  $b_\ell$  ( $\ell \in \mathbb{N}$ ) such that  $f = \sum_{\ell=0}^\infty \lambda_\ell b_\ell$ , and  $\sum_{\ell=0}^\infty |\lambda_\ell| < \infty$ . Moreover,

$$\|f\|_H \approx \inf \sum_{\ell=0}^\infty |\lambda_\ell|,$$

where the infimum is taken over all such decompositions. ( $f = \sum_{\ell=0}^\infty \lambda_\ell b_\ell$  is understood in  $L^1[0, \infty)$  norm.)

In the following theorem, in the proof of which the atomic structure of  $H$  will be used, we will prove a so called Sidon type inequality for the generalized Walsh-Dirichlet kernels. The corresponding Sidon type

inequalities for the trigonometric and the Walsh-Dirichlet kernels are due to Schipp, see [7] and [8]. We note that Schipp proved such inequalities for several other systems, such as Ciesielski, UDMD systems etc. For instance, a version for Legendre polynomials was shown by Schipp and Szili in [9]. It is known that Sidon type inequalities have applications in several areas. Among them are the study of convergence and approximation properties of strong means of Fourier series, and the construction of conditions that imply integrability and  $L^1$ -convergence of orthogonal series. For the relation between Sidon type inequalities and strong convergence and approximation we refer to the joint papers [4], [5] of Schipp and the author. For integrability and  $L^1$ -convergence conditions see [6] by Móricz and Schipp, and [2], [3] by the author. The promised result for the generalized Walsh-Dirichlet kernels is

**Theorem 2.** *Let  $f \in H$ . Then*

$$\int_0^\infty \left| \int_0^\infty f(t) D_t(y) dt \right| dy \leq C \|f\|_H.$$

From the proof of Th. 2 we can deduce the following relation between  $H$  and the  $L^p[0, \infty)$  spaces.

**Corollary.** *If  $f \in L^p_{loc}$  for some  $p > 1$  then  $\chi_{[a,b]} f \in H$  for any  $0 < a < b < \infty$ .*

In our next theorem we show how the inequality in Th. 2 can be applied for constructing a condition that guarantees the integrability of Walsh-Fourier transform. For the case of Walsh-Fourier series see [2], [6] and [7].

**Theorem 3.** *Let  $g : [0, \infty) \mapsto \mathbb{R}$  be absolutely continuous with*

$$\lim_{t \rightarrow \infty} g(t) = 0.$$

*Suppose that  $g' \in H$ . Then there exists  $f \in L^1[0, \infty)$  such that  $\hat{f} = g$ , and the inversion formula holds.*

## Proofs

**Proof of Theorem 1.** The proof will be deduced from the corresponding property of the dyadic Hardy space  $\mathbf{H}$ . The dyadic maximal function  $h^*$  of an  $h \in L^1[0, 1)$  is defined as

$$h^*(x) = \sup_{n \geq 0} \left| \int_{I_n(x)} h \right| \quad (0 \leq x < 1).$$

Then  $h$  is said to belong to  $\mathbf{H}$  if  $h^* \in L^1[0, 1)$ , and  $\|h\|_{\mathbf{H}} = \int_0^1 |h^*|$ . It is known [10] that  $\mathbf{H}$  has an atomic structure. Namely, an  $a \in L^\infty[0, 1)$  is called a dyadic atom if either  $a = \chi_{[0,1)}$  or there exists an  $I \in \mathcal{I}$  for which  $\text{supp } a \subset I \subset [0, 1)$ ,  $\int_I a = 0$ , and  $\|a\|_\infty \leq |I|^{-1}$ . Then for any  $h \in \mathbf{H}$  there exist real numbers  $\mu_j$  and dyadic atoms  $a_j$  ( $j \in \mathbb{N}$ ) such that  $h = \sum_{j=0}^\infty \mu_j a_j$ . Moreover,

$$\|h\|_{\mathbf{H}} \approx \inf \sum_{j=0}^\infty |\mu_j| \quad (h \in \mathbf{H}),$$

where the infimum is taken over all dyadic decompositions of  $h$ .

Let  $f \in L^1[0, \infty)$ . Clearly, if  $f = \sum_{\ell=0}^\infty \lambda_\ell b_\ell$  is an atomic decomposition of  $f$ , i.e. the  $\lambda_\ell$ 's are real numbers and the  $b_\ell$ 's are  $H$ -atoms, then  $f \in H$  and  $\|f\|_H \leq \sum_{\ell=0}^\infty |\lambda_\ell|$ .

In order to prove the other direction set

$$T_k : L^1[2^k, 2^{k+1}) \mapsto L^1[0, 1), \quad T_k g(x) = g(2^k + 2^k x) \quad (0 \leq x < 1, k \in \mathbb{Z}).$$

Thus  $T_k$  is one-to-one and

$$\begin{aligned} (T_k g)^*(x) &= \sup_{\ell \geq 0} 2^\ell \left| \int_{I_\ell(x)} T_k g \right| = \sup_{\ell \geq 0} 2^\ell 2^{-k} \left| \int_{I_{\ell-k}(2^k + 2^k x)} g \right| = \\ &= \sup_{0 \notin I_n(2^k + 2^k x)} 2^n \left| \int_{I_n(2^k + 2^k x)} g \right| = \\ &= \mathcal{M}g(2^k + 2^k x) \quad (0 \leq x < 1, k \in \mathbb{Z}). \end{aligned}$$

Hence  $\mathcal{M}g \in L^1[2^k, 2^{k+1})$  if and only if  $(T_k g)^* \in L^1[0, 1)$ , and  $\int_{2^k}^{2^{k+1}} \mathcal{M}g = 2^k \|T_k g\|_{\mathbf{H}}$ .

On the other hand it is easy to see that  $b$  is an atom with  $\text{supp } b \subset [2^k, 2^{k+1})$  if and only if  $2^k T_k b$  is a dyadic atom. Consequently, if  $T_k g = \sum_{j=0}^\infty \mu_j a_j$  is an atomic decomposition of  $T_k g$  then  $g = \sum_{j=0}^\infty \lambda_j b_j$  with  $\lambda_j = 2^k \mu_j$  and  $b_j = 2^{-k} T_k^{-1} a_j$  is an atomic decomposition of  $g$ . Moreover,  $\int_{2^k}^{2^{k+1}} \mathcal{M}g = 2^k \|T_k g\|_{\mathbf{H}} \approx 2^k \inf \sum_{j=0}^\infty |\mu_j| = \inf \sum_{j=0}^\infty |\lambda_j|$ .

Let  $f \in H$ . Then  $\mathcal{M}f \in L^1[0, \infty)$ , i.e.

$$\|f\|_H = \sum_{k=-\infty}^\infty \int_{2^k}^{2^{k+1}} \mathcal{M}f < \infty.$$

Since  $\chi_{[2^k, 2^{k+1})} \mathcal{M}f = \mathcal{M}\chi_{[2^k, 2^{k+1})} f \in L^1[2^k, 2^{k+1})$  we have that  $\chi_{[2^k, 2^{k+1})} f$  has an atomic decomposition

$$\chi_{[2^k, 2^{k+1})} f = \sum_{j=0}^{\infty} \lambda_{jk} b_{jk},$$

where  $\text{supp } b_{jk} \subset [2^k, 2^{k+1})$  ( $j \in \mathbb{N}$ ,  $k \in \mathbb{Z}$ ). Consequently,

$$f = \sum_{k=-\infty}^{\infty} \sum_{j=0}^{\infty} \lambda_{jk} b_{jk}$$

is an atomic decomposition of  $f$ .

Moreover, taking the infimum over all such decompositions we have

$$\|f\|_H = \int_0^{\infty} \mathcal{M} = \sum_{k=-\infty}^{\infty} \int_{2^k}^{2^{k+1}} \mathcal{M} f \approx \sum_{k=-\infty}^{\infty} \inf \sum_{j=0}^{\infty} |\lambda_{jk}| \quad (f \in H). \quad \diamond$$

**Proof of Corollary.** Let  $f \in L_{\text{loc}}^p$  for some  $p > 1$ . Then  $f \in L^p[2^k, 2^{k+1})$  ( $k \in \mathbb{Z}$ ). Therefore  $T_k(\chi_{[2^k, 2^{k+1})} f) \in L^p[0, 1)$ . It is known (see e.g. [10]) that  $L^p[0, 1) \subset \mathbf{H}$  ( $p > 1$ ). Hence,  $T_k(\chi_{[2^k, 2^{k+1})} f) \in \mathbf{H}$ , i.e.  $\int_{2^k}^{2^{k+1}} \mathcal{M} f < \infty$ . Since  $\chi_{[a,b)} f \in L_{\text{loc}}^p$  for any  $0 < a < b < \infty$ , and there are  $n, m \in \mathbb{Z}$  such that  $2^n \leq a < b \leq 2^m$  we have that

$$\|\chi_{[a,b)} f\|_H = \int_0^{\infty} \mathcal{M}(\chi_{[a,b)} f) = \sum_{k=n}^{m-1} \int_{2^k}^{2^{k+1}} \mathcal{M}(\chi_{[a,b)} f) < \infty.$$

Consequently,  $\chi_{[a,b)} f \in H$ .  $\diamond$

The proof of Th. 2 will be based on the concept of atomic decomposition and on the following two lemmas. The first one is of interest of its own.

**Lemma 1.** *Let  $0 \leq t < \infty$ . Then the following decomposition holds true for the generalized Dirichlet kernel:*

$$D_t = \psi_t \sum_{j=-\infty}^{\infty} t_j \psi_{2^{-j-1}} D_{2^{-j-1}}.$$

**Proof of Lemma 1.** Let the binary form of  $0 \leq t < \infty$  be  $(t_j, j \in \mathbb{Z})$ . Define  $t^{(k)}$  ( $k \in \mathbb{Z}$ ) as  $t^{(k)} = \sum_{j=-\infty}^k t_j 2^{-j-1}$ . Then

$$D_t(y) = \int_0^t \psi_x(y) dx = \sum_{k=-\infty}^{\infty} \int_{t^{(k-1)}}^{t^{(k)}} \psi_x(y) dx.$$

Suppose that  $t_k = 1$ . By (1) it is clear that,  $t^{(k-1)} + x = t^{(k-1)} + x$  if  $0 \leq x < 2^{-k}$ . Then we use (3) and (4) to obtain

$$\begin{aligned} \int_{t^{(k-1)}}^{t^{(k)}} \psi_x(y) dx &= \int_0^{2^{-k-1}} \psi_{t^{(k-1)+x}}(y) dx = \psi_{t^{(k-1)}}(y) \int_0^{2^{-k-1}} \psi_x(y) dx = \\ &= \psi_{t^{(k-1)}}(y) \psi_{2^{-k-1}}(y) \psi_{2^{-k-1}}(y) D_{2^{-k-1}}(y) = \\ &= \psi_{t^{(k)}}(y) \psi_{2^{-k-1}}(y) D_{2^{-k-1}}(y). \end{aligned}$$

By definition  $t-t^{(k)} = \sum_{j=k+1}^{\infty} t_j 2^{-j-1}$ , and  $\psi_{t-t^{(k)}}(y) = (-1)^{\sum_{j=k+1}^{\infty} t_j y^{-j-1}}$ . Hence,  $\psi_{t-t^{(k)}}(y) = 1$  for any  $y < 2^{k+1}$ , i.e.  $\psi_{t-t^{(k)}}$  is constant 1 on the support of  $D_{2^{-k-1}}$ . Therefore,

$$\int_{t^{(k-1)}}^{t^{(k)}} \psi_x(y) dx = \psi_{t^{(k)}}(y) t_k \psi_{2^{-k-1}}(y) D_{2^{-k-1}}(y) \quad (0 \leq y < \infty).$$

Consequently,

$$D_t = \psi_t \sum_{k=-\infty}^{\infty} t_k \psi_{2^{-k-1}} D_{2^{-k-1}}. \diamond$$

**Lemma 2.** Let  $h \in L_{loc}^{\infty}[0, \infty)$ . Then

$$\int_0^{\infty} \frac{1}{2^n} \left| \int_0^{2^n} h(t) D_t(y) dt \right| dy \leq C \| \chi_{[0, 2^n)} h \|_{\infty} \quad (n \in \mathbb{N}).$$

**Proof of Lemma 2.** By Lemma 1 and (5) we have

$$\begin{aligned} \int_0^{\infty} \frac{1}{2^n} \left| \int_0^{2^n} h(t) D_t(y) dt \right| dy &= \\ &= \int_0^{\infty} \frac{1}{2^n} \left| \int_0^{2^n} h(t) \psi_t(y) \sum_{k=-n}^{\infty} t_k \psi_{2^{-k-1}}(y) D_{2^{-k-1}}(y) dt \right| dy = \\ &= \frac{1}{2^n} \int_0^{\infty} \left| \sum_{k=-n}^{\infty} \psi_{2^{-k-1}}(y) D_{2^{-k-1}}(y) \int_0^{2^n} t_k h(t) \psi_y(t) dt \right| dy \leq \\ &\leq \frac{1}{2^n} \sum_{k=-n}^{\infty} 2^{-k-1} \int_0^{2^{k+1}} \left| \int_0^{2^n} t_k h(t) \psi_y(t) dt \right| dy. \end{aligned}$$

Set  $g_k(y) = \text{sgn} \int_0^{2^n} t_k h(t) \psi_y(t) dt$  ( $-n \leq k < \infty, 0 \leq y < \infty$ ). Then by the Fubini theorem, the Cauchy-Schwarz inequality and the Bessel inequality we have

$$\begin{aligned}
\int_0^{2^{k+1}} \left| \int_0^{2^n} t_k h(t) \psi_y(t) dt \right| dy &= \int_0^{2^n} t_k h(t) \int_0^{2^{k+1}} g_k(y) \psi_y(t) dy dt = \\
&= \int_0^{2^n} t_k h(t) (\widehat{\chi_{[0,2^{k+1}]g_k}})(t) dt \leq \\
&\leq \left( \int_0^{2^n} |h(t)|^2 dt \right)^{1/2} \left( \int_0^{2^n} |(\widehat{\chi_{[0,2^{k+1}]g_k}})(t)|^2 dt \right)^{1/2} \leq \\
&\leq 2^{n/2} \|\chi_{[0,2^n]} h\|_\infty \|\chi_{[0,2^{k+1}]}\|_2 = 2^{n/2} 2^{(k+1)/2} \|\chi_{[0,2^n]} h\|_\infty.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_0^\infty \frac{1}{2^n} \left| \int_0^{2^n} h(t) D_t(y) dt \right| dy &\leq \\
&\leq 2^{-n/2} \|\chi_{[0,2^n]} h\|_\infty \sum_{k=-n}^\infty 2^{-(k+1)/2} \leq C \|\chi_{[0,2^n]} h\|_\infty. \diamond
\end{aligned}$$

**Proof of Theorem 2.** Let  $b$  be an  $H$ -atom of first type, i.e.  $b = 2^{-(n-1)} \chi_{[2^{n-1}, 2^n]}$  with some  $n \in \mathbb{Z}$ . Then by Lemma 2 we have

$$\int_0^\infty \left| \int_0^\infty b(t) D_t(y) dt \right| dy \leq C 2^{n-1} \|b\|_\infty \leq C.$$

Now let  $b$  be an  $H$ -atom of second type. Then there exist  $k \in \mathbb{N}$  and  $n \in \mathbb{Z}$  such that

$$(9) \quad \text{supp } b \subset [k2^n, (k+1)2^n], \quad \int_{k2^n}^{(k+1)2^n} b = 0, \quad \|b\|_\infty \leq 2^{-n}.$$

By definition  $\psi_{k2^n+u} = \psi_{k2^n} \psi_u$  for any  $0 \leq u < 2^n$ . Therefore,  $D_{k2^n+t} = D_{k2^n} + \psi_{k2^n} D_t$  ( $0 \leq t < 2^n$ ). Hence by (9) and Lemma 2 we have

$$\begin{aligned}
\int_0^\infty \left| \int_0^\infty b(t) D_t(y) dt \right| dy &= \int_0^\infty \left| \int_0^{2^n} b(k2^n + t) D_{k2^n+t}(y) dt \right| dy = \\
&= \int_0^\infty \left| D_{k2^n}(y) \int_{k2^n}^{(k+1)2^n} b(t) dt + \psi_{k2^n} \int_0^{2^n} b(k2^n + t) D_t(y) dt \right| dy = \\
&= \int_0^\infty \left| \int_0^{2^n} b(k2^n + t) D_t(y) dt \right| dy \leq C 2^n \|b\|_\infty \leq C.
\end{aligned}$$

Consequently, Th. 2 holds for  $H$ -atoms. The proof can be completed by using the concept of atomic decomposition. Indeed, if  $f = \sum_{\ell=0}^\infty \lambda_\ell b_\ell$  is an atomic decomposition of  $f \in H$  then the sublinearity implies



$$\begin{aligned} \int_0^\infty \left| \int_0^\infty f(t) D_t(y) dt \right| dy &\leq \\ &\leq \sum_{\ell=0}^\infty |\lambda_\ell| \int_0^\infty \left| \int_0^\infty b_\ell(t) D_t(y) dt \right| dy \leq C \sum_{\ell=0}^\infty |\lambda_\ell|. \end{aligned}$$

Hence we conclude by Th. 1 that  $\int_0^\infty \left| \int_0^\infty f(t) D_t(y) dt \right| dy \leq C \|f\|_H$  ( $f \in H$ ).  $\diamond$

**Proof of Theorem 3.** Since  $\lim_{t \rightarrow \infty} g(t) = 0$ , and  $D_0 \equiv 0$  we have by integration by parts that

$$\lim_{t \rightarrow \infty} \int_0^t g(u) \psi_u(y) du = - \lim_{t \rightarrow \infty} \int_0^t g'(u) D_u(y) du \quad (0 \leq y < \infty).$$

The existence of the limit on the right side follows from  $g' \in L^1[0, \infty)$  and from (6). Moreover,  $\lim_{t \rightarrow \infty} \int_0^t g'(u) D_u(y) du = \int_0^\infty g'(u) D_u(y) du$ . Then

$$f : [0, \infty) \mapsto \mathbb{R}, \quad f(y) = \lim_{t \rightarrow \infty} \int_0^t g(u) \psi_u(y) du = - \int_0^\infty g'(u) D_u(y) du$$

is well defined. Th. 2 implies

$$\int_0^\infty |f(y)| dy = \int_0^\infty \left| \int_0^\infty g'(u) D_u(y) du \right| dy \leq C \|g'\|_H < \infty.$$

Consequently,  $f \in L^1[0, \infty)$ .

In order to see  $\hat{f} = g$  we will use Fubini's theorem, and the techniques of integration by parts and differentiation of parametric integrals. Indeed,

$$\begin{aligned} \hat{f}(y) &= \lim_{n \rightarrow \infty} \int_0^{2^n} f(u) \psi_y(u) du = - \lim_{n \rightarrow \infty} \int_0^{2^n} \psi_y(u) \int_0^\infty g'(t) D_t(u) dt du \\ &= - \lim_{n \rightarrow \infty} \int_0^\infty g'(t) \int_0^{2^n} \psi_y(u) D_t(u) du dt \\ &= \lim_{n \rightarrow \infty} \int_0^\infty g(t) \int_0^{2^n} \psi_u(y) \psi_u(t) du dt \\ &= \lim_{n \rightarrow \infty} \int_0^\infty g(t) D_{2^n}(y + t) dt \\ &= \lim_{n \rightarrow \infty} S_{2^n} g(y) \quad (0 \leq y < \infty). \end{aligned}$$

Since  $g$  is continuous we conclude, see (8), that  $\lim_{n \rightarrow \infty} S_{2^n} g(y) = g(y)$  ( $0 \leq y < \infty$ ). Consequently,  $\hat{f} = g$ .

Finally, the inversion formula is immediate by the definition of  $f$ .  $\diamond$

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