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AN ADDITION TO FACTORIZATION OF INEQUALITIES

László **Leindler**

Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, H-6720 Szeged, Hungary

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Abstract: In this paper we are concerned with factorization of inequalities. We extend one of our earlier factorization results in [6] from x^p to more general functions.

The well-known Hardy-inequality

$$\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n}|x_k|\Big)^p \le \Big(\frac{p}{p-1}\Big)^p\sum_{n=1}^{\infty}|x_n|^p,\ p>1, \tag{1}$$

has several extensions and generalizations. For details see e.g. [1] and [3]. In a recent monograph [2] G. Bennett introduced a new way of looking at inequalities. His method is a certain "factorization" of inequalities. To demonstrate his idea let us take the mentioned Hardy's inequality. This inequality is equivalent, in its qualitative version, to the inclusion

$$\ell^p \subset \operatorname{ces}(p),$$

where $\operatorname{ces}(p)$ is defined to be the set of all sequences $\mathbf{x} := (x_1, x_2, \ldots)$ that satisfy

$$\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n}|x_k|\Big)^p < \infty;$$

and ℓ^p denotes the usual ℓ^p-space.

G. Bennett asked how much room there is between ℓ^p and ces (p). To study this problem it is natural to consider the multipliers from ℓ^p into ces (p), namely those sequences, $\mathbf{z}$, with the property that $\mathbf{y} \cdot \mathbf{z} \in$ ces (p) whenever $\mathbf{y} \in \ell^p$. The set, Z, of all such multipliers clearly satisfies

(2) $$\ell^p \cdot Z \subseteq \text{ces}\,(p).$$

What surprising is that Z may be described in very simple terms, and that the inclusion (2) turns out to be an identity. These assertions are proved by the following theorem of G. Bennett [2].

Theorem A. *Let $p > 1$ be fixed. A sequence $\mathbf{x}$ belongs to* ces (p) *if and only if it admits a factorization,*

(3) $$\mathbf{x} = \mathbf{y} \cdot \mathbf{z}, \quad (x_n = y_n \cdot z_n),$$

with

(4) $$\mathbf{y} \in \ell^p \quad \textit{and} \quad \sum_{k=1}^{n} |z_k|^{p^*} = O(n), \quad p^* = \frac{p}{p-1}.$$

Th. A may be stated succinctly:

$$\text{ces}\,(p) = \ell^p \cdot g(p^*) \ (p > 1),$$

introducing the sequence spaces

$$g(p) := \{\mathbf{x} : \sum_{k=1}^{n} |x_k|^p = O(n)\}.$$

It easy to show that Th. A contains Hardy's inequality. To see this, suppose that $\mathbf{x} \in \ell^p$ is given. We may factorize $\mathbf{x}$ as in (3) and (4) by taking $\mathbf{y} = \mathbf{x}$ and $\mathbf{z} = \mathbf{1} = (1, 1, \dots)$; thus Th. A implies that $\mathbf{x} \in$ ces(p).

We ([4], [5]) have also proved four general inequalities. One of them reduces to Hardy's inequality with the choice $\lambda_n = n^{-p}$. It reads as follows:

Theorem B. *Let $a_n \geq 0$ and $\lambda_n > 0$ be given. Then, using the notations*

$$A_{m,n} := \sum_{i=m}^{n} a_i \quad \textit{and} \quad \Lambda_{m,n} := \sum_{i=m}^{n} \lambda_i \quad (1 \leq m \leq n \leq \infty),$$

we have

(5) $$\sum_{n=1}^{\infty} \lambda_n A_{1,n}^p \leq p^p \sum_{n=1}^{\infty} \lambda_n^{1-p} \Lambda_{n,\infty}^p a_n^p,$$

(6) $$\sum_{n=1}^{\infty} \lambda_n A_{n,\infty}^p \leq p^p \sum_{n=1}^{\infty} \lambda_n^{1-p} \Lambda_{1,n}^p a_n^p$$

for any $p \geq 1$; and when $0 < p \leq 1$, the signs of inequalities (5) *and* (6) *are reversed.*

In some previous papers we also factorized the inequalities (5) and (6) with $p > 1$ similarly as the inequality (1) was improved by Th. A.

Now we shall recall only a special case of one of our results. Since the aim of the present note is only to extend the validity of this fractional statement by replacing the functions x^p by more general functions $\varphi(x)$.

In order to formulate our assertions exactly, we need some further definitions and notations. Given $p > 1$, we shall write $\varphi \in \Phi(p)$ if φ is a non-negative function on $[0, \infty)$, $\varphi(0) = 0$, $\varphi(x)x^{-p}$ is non-decreasing. We also define, for $p > 0$ and with $\Lambda_n := \Lambda_{n,\infty}$, the sets:

$$\lambda(p) := \Big\{\mathbf{x} : \sum_{n=1}^{\infty} \lambda_n \big(\sum_{k=1}^{n} |x_k|\big)^p < \infty \Big\},$$

$$\Lambda(p) := \Big\{\mathbf{x} : \sum_{k=1}^{n} |x_k|^p = O(\Lambda_n^{1-p})\Big\}.$$

We intend to generalize the following result, which is a special case of the first part of Th. 1 in [6].

Theorem C. *Let $p > 1$ and let $\lambda := \{\lambda_n\}$ be a given sequence of non-negative terms having infinitely many positive ones.*

If a sequence $\mathbf{x}$ belongs to $\lambda(p)$ then it admits a factorization (3) *with*

$$\text{(7)} \qquad \mathbf{y} \in \ell^p \quad \textit{and} \quad \mathbf{z} \in \Lambda(p^*) \quad \Big(p^* = \frac{p}{p-1}\Big),$$

furthermore

$$\sum_{n=1}^{\infty} |y_n|^p \le K \sum_{n=1}^{\infty} \lambda_n \Big(\sum_{k=1}^{n} |x_k|\Big)^p.$$

We remark that a certain converse of this part of Th. C is also proved in [6], but now it is out of our current interest. Roughly speaking the neglected converse states that under certain restrictions on the sequence λ the factorization (3) with (7) implies that $\mathbf{x} \in \lambda(p)$.

Now we consider the problem of factorization of the set

$$\lambda(\varphi) := \Big\{\mathbf{x} : \sum_{n=1}^{\infty} \lambda_n \varphi\big(\sum_{k=1}^{n} |x_k|\big) < \infty\Big\},$$

where $\varphi \in \Phi(p)$, $p > 1$. We also define the set:

$$\ell(\varphi) := \{\mathbf{x} : \sum_{n=1}^{\infty} \varphi(|x_n|) < \infty\},$$

and agree that the constants K to be appearing in inequalities may vary from occurance to occurance, and they denote positive constants. Now we can formulate our result.

Theorem. *Let $p > 1$ and let $\lambda := \{\lambda_n\}$ be a given sequence of non-negative terms having infinitely many positive ones.*

If a sequence $\mathbf{x}$ *belongs to* $\lambda(\varphi)$ *then it admits a factorization* (3) *with*

$$\mathbf{y} \in \ell(\varphi) \quad \textit{and} \quad \mathbf{z} \in \Lambda(p^*);$$

furthermore

$$\sum_{n=1}^{\infty} \varphi(|y_n|) \le K \sum_{n=1}^{\infty} \lambda_n \varphi\left(\sum_{k=1}^{n} |x_k|\right). \tag{8}$$

It is clear that if $\varphi(x) = x^p$ then our theorem reduces to Th. C.

We consider our current result as the first small step in generalizing Th.C when we replace the function x^p by more general functions. We mention that our Th. can be deduced from Th. C, too, but now we give a direct proof being shorter than that of Th. C. We also remark that our new proof would be usable to the proof of Th. C, too.

Proof of the Theorem. We assume that $\mathbf{x} \ne \mathbf{0} := (0, 0, \dots)$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$, otherwise the assertions are trivial.

First we define a new sequence:

$$b_n := \sum_{k=n}^{\infty} \lambda_k \varphi\left(\sum_{i=1}^{k} |x_i|\right)^{1/p^*} \qquad \left(p^* = \frac{p}{p-1}\right).$$

It is easy to see that $\mathbf{b} := \{b_n\}$ is a decreasing sequence of positive terms and b_n tends to zero, namely, by Hölder's inequality,

$$b_n \le \left(\sum_{k=n}^{\infty} \lambda_k\right)^{1/p} \left(\sum_{k=n}^{\infty} \lambda_k \varphi\left(\sum_{i=1}^{k} |x_i|\right)\right)^{1/p^*}.$$

Next we denote the inverse function of $\varphi(x)$ by $\varphi^{-1}(x)$ and factorize $\mathbf{x}$ as follows:

$$\mathbf{x} = \mathbf{y} \cdot \mathbf{z} \ (x_n = y_n z_n),$$

where

$$y_n := \varphi^{-1}(|x_n| b_n), \quad \text{and} \quad z_n := \begin{cases} \dfrac{x_n}{y_n}, & \text{if } y_n > 0 \\ 0, & \text{if } y_n = 0. \end{cases}$$

Then, using Hölder's inequality and, in the last step, $\varphi \in \Phi(p)$, we get that

$$\begin{aligned}\sum_{n=1}^{\infty}\varphi(y_n)=\sum_{n=1}^{\infty}|x_n|b_n=\sum_{n=1}^{\infty}|x_n|\sum_{k=n}^{\infty}\lambda_k\varphi\left(\sum_{i=1}^{k}|x_i|\right)^{1/p^*}=\\ =\sum_{k=1}^{\infty}\lambda_k\varphi\left(\sum_{i=1}^{k}|x_i|\right)^{1/p^*}\sum_{n=1}^{k}|x_n|\le K\sum_{k=1}^{\infty}\lambda_k\varphi\left(\sum_{i=1}^{k}|x_i|\right),\end{aligned}\tag{9}$$

and this proves (8) and $\mathbf{y}\in\ell(\varphi)$.

The relation (9) also yields that $|x_n|b_n$ tends to zero and thus, by $\varphi\in\Phi(p)$,

$$(|x_n|b_n)^{1/p}\le K\varphi^{-1}(|x_n|b_n).$$

This implies that if $z_k\neq 0$ then

$$|z_k|=\frac{|x_k|}{\varphi^{-1}(|x_k|b_k)}\le K|x_k|^{1/p^*}b_k^{-1/q}.\tag{10}$$

Using (10), the definition of b_n and finally $\varphi\in\Phi(p)$, we get that

$$\begin{aligned}\sum_{k=1}^{m}|z_k|^{p^*}\le K\sum_{k=1}^{m}|x_k|b_k^{-p^*/p}\le\\ \le K\sum_{k=1}^{m}|x_k|\left(\varphi\left(\sum_{i=1}^{m}|x_i|\right)\Lambda_m^{p^*}\right)^{-1/p}\le K\Lambda_m^{-p^*/p}=K\Lambda_m^{1-p^*}.\end{aligned}$$

This clearly proves that $\mathbf{z}\in\Lambda(p^*)$.

Thus the proof is complete. ◊

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