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THE TWO-PARAMETER CESÀRO OPERATORS

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Abstract: In the one dimensional case is already proved (see [5]) that the dyadic Cesàro operator is bounded on $L^p[0,1)$ $(1 \le p < \infty)$ and on the dyadic Hardy space $H^1[0,1)$ and is not bounded on the spaces VMO and on $L^\infty[0,1)$. In the present paper we show similary results in the two dimensional case.

1. Preliminaries

We shall denote the set of non-negative integers by $\mathbb{N}$, the set of positive integers by $\mathbb{P}$, the set of real numbers by $\mathbb{R}$, and the set of dyadic rationals in the unit interval $[0,1]$ by $\mathbb{Q}$. In particular, each element of $\mathbb{Q}$ has the form $p/2^n$ for some $p, n \in \mathbb{N}$, $0 \le p \le 2^n$. Furthermore, let $\mathbf{I} := [0,1)$ be the unit interval.

For any set $\mathbf{X} \ne \emptyset$ let $\mathbf{X}^1 := \mathbf{X}$ and denote by $\mathbf{X}^2$ the cartesian product $\mathbf{X} \times \mathbf{X}$. Thus $\mathbb{N}^2$ is the collection of integral latice points in the first quadrant, and $\mathbf{I}^2$ is the unit square.

We shall use the following partial ordering in $\mathbb{R}^2$. For $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$ let $x \le y$ iff $x_1 \le y_1$ and $x_2 \le y_2$. We set $|x| = |x_1| +$

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$+|x_2|$. For $n=(n_1,n_2)\in\mathbb{N}^2$ it will be used the notation $n-1=(n_1-1,$ $n_2-1)$.

The dyadic addition of x and y is defined (see [6]) by

$$x\dot{+}y=\sum_{k=0}^{\infty}|x_k-y_k|2^{-k-1} \tag{1.1}$$

for $x,y\in\mathbf{I}$. If $x,y\in\mathbf{I}^2$ then by definition let $x\dot{+}y:=(x_1\dot{+}y_1,x_2\dot{+}y_2)$.

By a dyadic interval in $\mathbf{I}$ we mean one of the form $[p/2^n,(p+1)/$ $/2^n)$ for some $p,n\in\mathbb{N}$, $0\le p<2^n$. Given $n\in\mathbb{N}$ and $x\in\mathbf{I}$ let $I_n(x)$ denote the dyadic interval of length 2^{-n} which contains x. Denote the collection of dyadic intervals by $\mathcal{I}$.

Let $\mathcal{I}^2$ be the collection of dyadic intervals in $\mathbf{I}^2$, i.e. the sets of the form $I=I_1\times I_2$, where $I_1,I_2\in\mathcal{I}$. It is clear, that the dyadic intervals in $\mathbf{I}^2$ containing $x=(x_1,x_2)\in\mathbf{I}^2$ are of the form

$$I_n(x):=I_{n_1}(x_1)\times I_{n_2}(x_2), \tag{1.2}$$

where $n:=(n_1,n_2)\in\mathbb{N}^2$. We denote by $f_1\times f_2$ the Kronecker-product of the functions $f_j:\mathbf{I}\to\mathbb{R}$ $(j=(1,2)$, i.e.

$$(f_1\times f_2)(x)=f_1(x_1)\cdot f_2(x_2)\qquad (x=(x_1,x_2)\in\mathbf{I}^2).$$

Especially for $f_1=f_2=f$ we set $f^{(2)}:=f\times f$.

The symbol $L^p(\mathbf{I}^2)$, $1\le p\le\infty$ stands for the usual Lebesgue L^p-space on $\mathbf{I}^2$.

The atomic σ-algebra generated by the two dimensional dyadic intervals of the form $I=K\times L$ with $|K|=2^{-p}$ and $|L|=2^{-q}$ will be denoted by $\mathcal{A}^{(p,q)}$. For $n\in\mathbb{N}^2$ let $L(\mathcal{A}^n)$ be the set of the $\mathcal{A}^n$-measurable function defined on $\mathbf{I}^2$. Set

$$\mathcal{A}^n_- :=\mathcal{A}^{(n_1-1,n_2-1)}\qquad (n=(n_1,n_2)\in\mathbb{N}^2),$$

where

$$\mathcal{A}^{(-1,-1)}:=\mathcal{A}^{(0,0)}\quad\text{and}\quad \mathcal{A}^{(-1,i)}:=\mathcal{A}^{(0,i)},\ \mathcal{A}^{(i,-1)}:=\mathcal{A}^{(i,0)}\ (i\in\mathbb{N}).$$

The conditional expectation of the function $f\in L^1(\ I^2)$ with respect to $\mathcal{A}^n$ $(n\in\mathbb{N}^2)$ is denoted by E_nf and can be given in the form

$$(E_nf)(x)=\frac{1}{|I_n(x)|}\int_{I_n(x)} f(s,t)dsdt\quad (x\in\mathbf{I}^2,n\in\mathbb{N}^2). \tag{1.3}$$

Extending (1.3) we set

(1.4a) $$(E_{(\infty,k)}f)(x) := \frac{1}{|I_k(x_2)|}\int\limits_{I_k(x_2)} f(x_1,s)ds$$

$$(x=(x_1,x_2)\in \mathbf{I}^2,\ k\in\mathbb{N})$$

(1.4b) $$(E_{(k,\infty)}f)(x) := \frac{1}{|I_k(x_1)|}\int\limits_{I_k(x_1)} f(t,x_2)dt$$

$$(x=(x_1,x_2)\in \mathbf{I}^2,\ k\in\mathbb{N}).$$

A sequence of functions $\mathbf{f}=(f_n, n\in\mathbb{N}^2)$ defined on $\mathbf{I}^2$ is called a dyadic martingale if f_n belongs to $L(\mathcal{A}^n)$ and

(1.5) $$E_n f_m = f_n \quad \text{for all} \quad n\le m \quad \text{and} \quad n,m\in\mathbb{N}^2.$$

If $0<p\le\infty$, $f_n\in L^p(\mathbf{I}^2)$ $(n\in\mathbb{N}^2)$ and

$$\|\mathbf{f}\|_p := \sup_{n\in\mathbb{N}^2}\|f_n\|_p < \infty,$$

then $\mathbf{f}$ is a so-called L^p-bounded martingale.

Let $f\in L^1(\mathbf{I}^2)$ and define the sequence $\mathbf{f}=(f_n, n\in\mathbb{N}^2)$ by

(1.6) $$f_n := E_n f \quad (n\in\mathbb{N}^2).$$

It easy to see that $\mathbf{f}$ is a martingale. Martingales of this type are called regular.

The map $f\mapsto \mathbf{f} := (E_n f, n\in\mathbb{N}^2)$ is norm-preserving from L^p onto the space of L^p-bounded martingales if $1<p<\infty$ and consequently the two spaces can be identified. In a similar way, we can identify $L^1(\mathbf{I}^2)$ with the space of uniformly integrable martingales (see [11], [12]).

The martingale maximal function f^* is given by

(1.7) $$f^* := \sup_{n\in\mathbb{N}^2}|f_n|.$$

To define the martingale transform introduce the martingale difference sequence in two-dimensional case by

(1.8)

$$d_{0,0} := f_{0,0}, \quad d_{k,0} := f_{k,0}-f_{k-1,0}, \quad d_{0,k} := f_{0,k}-f_{0,k-1} \quad (k\in\mathbb{P})$$

$$d_n := f_{(n_1,n_2)}-f_{(n_1-1,n_2)}-f_{(n_1,n_2-1)}+f_{(n_1-1,n_2-1)} \quad (n=(n_1,n_2)\in\mathbb{P}^2).$$

Obviously,

$$f_n = \sum_{k\le n} d_k.$$

Moreover, if $\alpha = (\alpha_n, n\in\mathbb{N}^2)$ and $\alpha_n\in\mathcal{A}^n_-$ $(n\in\mathbb{N}^2)$, then the sequence

$$f_n^\alpha := \sum_{k\le n} \alpha_k d_k, \quad \mathbf{f}^\alpha := (f_n^\alpha, n \in \mathbb{N}^2) \tag{1.9}$$

is also a martingale and it is called the transform of f by the sequence α.

We introduce a set of function sequences to define special martingale transforms. To this end set

$$\mathcal{T} := \{\tau = (\tau_n, n \in \mathbb{N}^2) : \tau_n(x) \in \{0,1\},\ \tau_n \in L(\mathcal{A}_-^n) \text{ and } \tau_n \ge \tau_m \quad \text{if} \quad n \le m\}. \tag{1.10}$$

For $0 < p < \infty$ denote by $\mathcal{H}^p$ the set of martingales $\mathbf{f} = (f_n, n \in \in \mathbb{N}^2)$ for which

$$\|\mathbf{f}\|_{\mathcal{H}^p} := \|f^*\|_p < \infty. \tag{1.11}$$

It is easy to see that if $p > 1$ then (1.11) implies that f is uniformly integrable and consequently $\mathcal{H}^p$ can be identified by a subspace of $L^1(\mathbf{I}^2)$.

For any $Y \subseteq L^1(\mathbf{I}^2)$ denote by Y_0 the set

$$Y_0 := \{f \in Y : E_{(n,0)}f = E_{(0,n)}f = 0 \quad (n \in \mathbb{N})\}.$$

The dual space of $\mathcal{H}_0^1$ is the $\mathcal{BMO}$ space which is defined by

$$\|f\| := \sup_{\tau\in\mathcal{T}} |\{\wedge\tau = 0\}|^{-1/2} \|f - f^\tau\|_2, \tag{1.12}$$

where $\wedge\tau := \inf_{n\in\mathbb{N}^2} \tau_n$, $f := (E_n f, n \in \ N^2)$ and $f \in L_0^2(\mathbf{I}^2)$ (see [2], [12]).

Feffermann's inequality implies

$$\left| \int_{\mathbf{I}^2} f(x)\phi(x)dx \right| \le C\|f\|_{\mathcal{H}^1}\|\phi\|_{\mathcal{BMO}}, \quad (f \in L^\infty, \phi \in \mathcal{BMO}) \tag{1.13}$$

where C is an absolute constant (see [11]).

The closure of the set of the dyadic step functions in the $\mathcal{BMO}$-norm is the $\mathcal{VMO}$ space. It is well-known (see [11]) that the dual space of the $\mathcal{VMO}$ space is $\mathcal{H}_0^1$.

We study the double Walsh series

$$\sum_{j\in\mathbb{N}^2} a_j w_j(x), \tag{1.14}$$

where $(a_j, j \in \mathbb{N}^2)$ is a null sequence of real numbers, and $w_j = w_{j_1} \times \times w_{j_2}$ $(j = (j_1, j_2) \in \mathbb{N}^2)$ is the two dimensional Walsh orthonormal system generated by the Walsh-Paley system. Thus, series (1.14) are considered on the unit square $\mathbf{I}^2$.

The pointwise convergence of series (1.14) will be taken in Pringsheim's sense (see [13], vol.2., ch.17). In other words, if we form the rectangular partial sums

$$S_n(x) = \sum_{j_1=0}^{n_1-1} \sum_{j_2=0}^{n_2-1} a_j w_j(x)$$

$$(j = (j_1, j_2),\ n = (n_1, n_2),\ x = (x_1, x_2),\ n_1, n_2 \geq 1),$$

then let both n_1 and n_2 tend to infinity independently of one another, and assign the limit $f(x)$ (if it exists) to the series (1.14) as its sum.

It is known that $S_{2^n} f = E_n f$ $(n \in \mathbb{N}^2)$, where $E_n f$ is defined in (1.3) (see [6]). In the case $n_1 = \infty$ or $n_2 = \infty$ we use the notations

$$S_{(2^{n_1},\infty)} f := E_{(n_1,\infty)} f, \qquad S_{(\infty,2^{n_2})} f := E_{(\infty,n_2)} f,$$

where $E_{(\infty,n_1)}$,$E_{(n_2,\infty)}$ are defined in (1.4 a) and (1.4 b). Furthermore we introduce the operators

$$\Delta_n f := S_{(2^{n_1},\infty)} f + S_{(\infty,2^{n_2})} f - S_{2^n} f.$$

Let D_n denote the two dimensional Walsh-Dirichlet kernel of order $n = (n_1, n_2)$, i.e.,

$$(1.15) \qquad \begin{aligned} D_n := D_{n_1} \times D_{n_2} = \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} w_k = \\ = \sum_{k=(0,0)}^{n-1} w_k \qquad (k = (k_1, k_2) \in \mathbb{N}^2,\ n \in \mathbb{P}^2). \end{aligned}$$

In the one dimensional case the dyadic difference quotient is defined as

$$\mathbf{d}_n f(x) := \sum_{j=0}^{n-1} 2^{j-1}(f(x) - f(x + 2^{-j-1}))$$

for each f defined on $[0, 1)$, $n \in \mathbb{P}$ and $x \in [0, 1)$ (see [6]). The two dimensional variant is defined as follows. For each function f given on the unit square $\mathbf{I}^2$ and for $n \in \mathbb{P}^2$ set

$$(1.16) \qquad \begin{aligned} \mathbf{d}_n f(x) := \sum_{j=(0,0)}^{n-1} 2^{|j|-2}(f(x_1, x_2) - f(x_1 + 2^{-j_1-1}, x_2) - \\ - f(x_1, x_2 + 2^{-j_2-1}) + f(x_1 + 2^{-j_1-1}, x_2 + 2^{-j_2-1})) \end{aligned}$$

where $x = (x_1, x_2) \in \mathbf{I}^2$. We shall say that f is dyadically differentiable at x if

$$\mathbf{d}f(x) := \lim_{\min(n_1,n_2)\to\infty} \mathbf{d}_n f(x) \tag{1.17}$$

exists and is finite, and call $\mathbf{d}f$ the two dimensional dyadic derivative of f at x (see [7]).

It is easy to see that for $f = g \times h$ and $n = (n_1, n_2)$

$$\mathbf{d}_n f = \mathbf{d}_{n_1} g \times \mathbf{d}_{n_2} h,$$

and consequently (see [6])

$$\mathbf{d}_n w_m = m_1 \cdot m_2 \cdot w_m \quad (m \le 2^n;\ m, n \in \mathbb{N}^2). \tag{1.18}$$

Obviously, it follows by (1.17) and (1.18) that the Walsh functions w_n $(n \in \mathbb{N}^2)$ are dyadic differentiable and

$$\mathbf{d}w_m = m_1 \cdot m_2 \cdot w_m \quad (m = (m_1, m_2) \in \mathbb{N}^2).$$

The inverse operator of $\mathbf{d}$, i.e., the dyadic antiderivative (or integral) can be given by the convolution

$$(Jf)(x) := (f * W^{(2)})(x) = \int_{\mathbf{I}^2} f(t) W^{(2)}(x \dotplus t)\,dt$$

$(f \in L^1(\mathbf{I}^2), t = (t_1, t_2), x = (x_1, x_2) \in \mathbf{I}^2)$, where $W^{(2)}(x) = W \times W$ and $W = \sum_{j=1}^{\infty} w_j / j$.

It is known (see [3]) that

$$W \in L^1(\mathbf{I}), \quad \|W\|_1 = O(1) \quad (k \in \mathbb{N}).$$

Consequently $W^{(2)} \in L^1(\mathbf{I}^2)$. Furthermore it follows from the one dimensional case, that $\|\mathbf{d}_n W\|_1 = O(1)$ $(n \in \mathbb{P}^2)$.

Lemma 1. *We can write $\mathbf{d}_m W^{(2)}$ in the following form:*

$$\mathbf{d}_m W^{(2)}(x) = D_{2^m}(x) + R_m(x) \quad (m = (m_1, m_2)), \tag{1.19}$$

where $\hat{R}_m(k) = 0$ if $k = (k_1, k_2) \in \mathbb{N}$ and $k_1 \le 2^{m_1}$ or $k_2 \le 2^{m_2}$ respectively and $\|R_m\|_1 = O(1)$.

Proof. It is known that if $n \in \mathbb{N}$ then $\mathbf{d}_n W(x) = D_{2^n}(x) + R_n(x)$, where for $\hat{R}_n(k) = 0$ for $k \le 2^m$ and $\|R_n\|_1 = O(1)$ (see [5]). From this it follows also the two dimensional equality, since

$$\mathbf{d}_n W^{(2)} = (\mathbf{d}_{m_1} W) \times (\mathbf{d}_{m_2} W) = (D_{2^{m_1}} + R_{m_1}) \times (D_{2^{m_2}} + R_{m_2}) =$$
$$= D_{2^m} + R_{m_1} \times D_{2^{m_2}} + D_{2^{m_1}} \times R_{m_2} + R_{m_1} \times R_{m_2} = D_{2^m} + R_m.$$

Hence that if $k_j \le 2^{m_j}$ for $j = 1$ or $j = 2$ then $\hat{R}_m(k) = 0$ and also our statement is established. ◊

We will introduce a modified form of the one dimensional operator $\mathbf{d}_k$:

$$(1.20)\qquad \begin{aligned}(\mathbf{d}_n^- f)(x) := \sum_{j=0}^{n-1} 2^{j-1}\big(f(x) - f(x \dotplus 2^{-j-1})\big) - \\ - 2^{n-1}\big(f(x) - f(x \dotplus 2^{-n-1}\big) \quad (n \in \mathbb{N}, x \in \mathbb{R}).\end{aligned}$$

Since $\mathbf{d}_k^- f = \mathbf{d}_k f - (\mathbf{d}_{k+1} f - \mathbf{d}_k f) = 2\mathbf{d}_k f - \mathbf{d}_{k+1} f$, therefore $\|\mathbf{d}_k^- W\|_1 = O(1)$ $(k \in \mathbb{P})$.

It is known that in the one dimensional case the Walsh-Dirichlet kernel can be written in the following form (see [5])

$$(1.21)\qquad D_n(t) = \mathbf{d}_{s-1}^- w_n(t) \quad (n \in \mathbb{N}),$$

if $t \in [2^{-s}, 2^{-s+1})$, $s = 1, 2, \dots$. In the two dimensional case, that is, if $t \in [2^{-s_1}, 2^{-s_1+1}) \times [2^{-s_2}, 2^{-s_2+1})$ and $n \in \mathbb{N}^2$ we get by means of the equality

$$(1.22)\qquad D_n(t) = \big(D_{n_1} \times D_{n_2}\big)(t) \quad (t \in \mathbb{R}^2, n \in \mathbb{N}^2)$$

that

$$(1.23)\qquad D_n(t) = \mathbf{d}_{s_1-1}^- w_{n_1}(t_1) \cdot \mathbf{d}_{s_2-1}^- w_{n_2}(t_2) =: \mathbf{d}_{s-1}^- w_n(t).$$

Using the dyadic addition we introduce the so-called dyadic translation operators τ_x for any $x \in \mathbf{I}^2$ as

$$(1.24)\qquad \big(\tau_x f\big)(t) = f(x \dotplus t) \quad (x, t \in \mathbf{I}^2),$$

where $f : \mathbf{I}^2 \to \mathbb{R}$ is an arbitrary function (see [6]). Dyadic translations are norm preserving in L^p spaces, i.e. for all $f \in L^p(\mathbf{I}^2)$ and $x \in \mathbf{I}^2$ we have $\tau_x f \in L^p(\mathbf{I}^2)$ and $||\tau_x f||_p = ||f||_p$.

A Banach space $\mathbf{X} \subseteq L^1(\mathbf{I}^2)$ with the norm $||\cdot||_X$ is called a homogeneous Banach space if the set $\mathcal{P}$ of double dyadic step functions is dense in $\mathbf{X}$, $||f||_1 \le ||f||_X$ $(f \in \mathbf{X})$ and the norm $||\cdot||$ is translation invariant, i.e. if $f \in \mathbf{X}$ and $x \in \mathbf{I}^2$ then $\tau_x f \in \mathbf{X}$ and $||\tau_x f||_X = ||f||_X$. It is known (see [6]) that $L^p(\mathbf{I}^2)$ $(1 \le p < \infty)$ and $\mathcal{H}^1$ are homogeneous Banach spaces.

If $\mathbf{X}$ is a homogeneous Banach space and if $f \in L^1(\mathbf{I}^2)$ and $g \in \mathbf{X}$ then $f * g \in \mathbf{X}$. Moreover,

$$(1.25)\qquad ||f * g||_X \le ||f||_1 ||g||_X$$

is true (see [6]).

Denote by $\mathbf{X}'$ the dual space of any homogeneous Banach space $\mathbf{X}$. If $f \in L^1$ and $g \in \mathbf{X}' \cap L^1$, then

$$\|f * g\|_{X'} \leq \|f\|_1 \|g\|_{X'} \tag{1.26}$$

is also true (see [1]).

2. The two dimensional dyadic Cesàro operator on $L^1(\mathbf{I}^2)$

Let $f \in L^1(\mathbf{I}^2)$ be an integrable function with Walsh-Fourier series

$$f \sim \sum_{k \in \mathbb{N}^2} \hat{f}(k) w_k.$$

We will assume that $\hat{f}(0,l) = \hat{f}(l,0) = 0$ $(l \in \mathbb{N})$ and denote the class of such functions by L_0^1. Now, we shall define the two dimensional dyadic Cesàro operator. First we prove that there exists a unique $g \in L_0^1$ such that

$$\begin{aligned} \hat{g}(n) &= \frac{1}{n_1 \cdot n_2} \sum_{i=(0,0)}^{n-1} \hat{f}(i) \quad (n \in \mathbb{P}^2), \\ \hat{g}(0,l) &= \hat{g}(l,0) = 0 \quad (l \in \mathbb{N}) \end{aligned} \tag{2.1}$$

is satisfied (see Th. 1). The map $\mathcal{C} : L_0^1 \to L_0^1$ defined by $\mathcal{C}f := g$ is called the two dimensional dyadic Cesàro operator. The aim of this paper is to investigate some properties of Cesàro operators in several subspaces of $L^1(\mathbf{I}^2)$.

First of all we will give a representation of $\mathcal{C}$ in a form of integral-operator

$$(\mathcal{C}f)(x) = \int_{\mathbf{I}^2} f(t) M(x,t) dt \quad (x \in \mathbf{I}^2, f \in L_0^1),$$

where the kernel M can be expressed by the modified dyadic difference operators $\mathbf{d}_n^-$ as follows:

$$M(x,t) = \sum_{r=(1,1)}^{(\infty,\infty)} \chi_r(t) (\mathbf{d}_{r-1}^- W^{(2)})(x+t). \tag{2.2}$$

Here χ_r is the characteristic function of the rectangle $[2^{-r_1}, 2^{-r_1+1}) \times$ $\times [2^{-r_2}, 2^{-r_2+1})$. It will be proved that series (2.2) converges in $L^1(\mathbf{I}^4)$ norm and for almost all $(x,t) \in \mathbf{I}^4$.

Define the integral operator $\mathcal{M}^{(2)}$ by

$$(2.3)\qquad (\mathcal{M}^{(2)}f)(x) := \int_{\mathbf{I}^2} f(t)M(x,t)dt \quad (x \in \mathbf{I}^2, f \in L_0^1),$$

where M is given by (2.2).

Denote $\mathcal{P}_n$ the set of the two dimensional Walsh polynomials p of order less than 2^n and set

$$\mathcal{P} = \bigcup_{n=(0,0)}^{(\infty,\infty)} \mathcal{P}_n,$$

$$\mathcal{P}_* = \{p \in \mathcal{P} : p(0,0) = 0, \hat{p}(0,k) = \hat{p}(k,0) = 0, k = 0,1,\dots\}.$$

Then for any $p = \sum_{k=(0,0)}^{n} c_k w_k$, $p \in \mathcal{P}_*$ we have

$$\sum_{k=(0,0)}^{N} \hat{p}(k) = \sum_{k=(0,0)}^{n} c_k = p(0,0) = 0$$

if $N \geq n$, $n \in \mathbb{N}^2$, therefore the operator

$$\mathcal{C}p = \sum_{k=(1,1)}^{(\infty,\infty)} \frac{\hat{p}(0,0) + \dots + \hat{p}(k-1)}{k_1 \cdot k_2} w_k$$

is well defined, maps $\mathcal{P}_*$ into $\mathcal{P}$ and satisfy

$$(\mathcal{C}p)\hat{}(n) = \frac{\hat{p}(0,0) + \dots + \hat{p}(n-1)}{n_1 \cdot n_2} \quad (n \in \mathbb{P}^2).$$

We show that $\mathcal{C}$ and the integral operator $\mathcal{M}^{(2)}$ coincides on $\mathcal{P}_*$ and $\mathcal{M}^{(2)}$ has the required properties.

Theorem 1. *Let $\mathcal{M}^{(2)}$ denote the integral operator defined by* (2.3). *Then*

(1) *$\mathcal{M}^{(2)}$ is a bounded linear operator from L_0^1 into itself,*

(2) *for all $f \in L_0^1$ the function $g := \mathcal{M}^{(2)}f \in L_0^1$ satisfies* (2.1).

Proof. Part (1). It is easy to see that

$$\mathcal{M}^{(2)}(x,t) = (M^{(1)} \times M^{(1)})(x,t) \quad (x,t \in \mathbf{I}^2),$$

where

$$M^{(1)}(x,t) = \sum_{s=1}^{\infty} \chi_s^{(1)}(t)(\mathbf{d}_{s-1}^{-}W^{(1)})(x+t)(x,t \in \mathbf{I}),$$

and $\chi_s^{(1)}(t)$ denotes the characteristic function of the interval $[2^{-s}, 2^{-s+1})$. Because of this the last sum converges in $L^1(\mathbf{I}^2)$-norm and also almost everywhere (see [5]), so the series (2.2) converges in $L^1(\mathbf{I}^4)$-norm and the operator $\mathcal{M}^{(2)}$ is bounded.

To prove (2) first we show that $\mathcal{M}^{(2)}f = \mathcal{C}f$ for every $f \in \mathcal{P}^*$. If $f \in \mathcal{P}^*$, then $f, \mathcal{C}f \in \mathcal{P}_N$ for some $N \in \mathbb{N}^2$. Consequently $\mathcal{C}f$ can be written in the form

$$(\mathcal{C}f)(x) = \sum_{k=(1,1)}^{(2^{N_1},2^{N_2})} \frac{\hat{f}(0,0) + \cdots + \hat{f}(k-1)}{k_1 \cdot k_2} w_k(x) =$$
$$= \sum_{k=(1,1)}^{(2^{N_1},2^{N_2})} \frac{w_k(x)}{k_1 \cdot k_2} \sum_{i=(0,0)}^{k-1} \int_{\mathbf{I}^2} f(t) w_i(t) dt =$$
$$= \int_{\mathbf{I}^2} f(t) \sum_{k=(1,1)}^{(2^{N_1},2^{N_2})} \frac{D_k(t) w_k(x)}{k_1 \cdot k_2} dt.$$

Hence by (1.23) we get for the kernel for $t \in [2^{-N_1}, 1) \times [2^{-N_2}, 1)$

$$\sum_{s=(1,1)}^{(N_1,N_2)} \chi_s(t) \sum_{k=(1,1)}^{(2^{N_1},2^{N_2})} \frac{w_k(x)}{k_1 \cdot k_2} (\mathbf{d}_{s-1}^{-} w_k)(t).$$

Thus for $f \in \mathcal{P}^*$ we have

$$(\mathcal{C}f)(x) = \int_{\mathbf{I}^2} f(t) M(x,t) dt = (\mathcal{M}^{(2)} f)(x).$$

This means that $\mathcal{C}$ coincides with $\mathcal{M}^{(2)}$ on the dense set $\mathcal{P}^*$ of L_0^1. We will show that $g = \mathcal{M}^{(2)} f$ satisfies (2.1) for all $f \in L_0^1$. To this end consider the the functions

$$\phi_k(f) = (\mathcal{M}^{(2)} f)\widehat{\ }(k), \qquad \psi_k(f) = \frac{\hat{f}(0,0) + \cdots + \hat{f}(k-1)}{k_1 \cdot k_2} \quad (k \in \mathbb{P}^2)$$

on L_0^1. It is easy to check that both are bounded linear functionals on L_0^1, for all $k \in \mathbb{N}^2$ and they coincide on $\mathcal{P}^*$. Since $\mathcal{P}^*$ is dense in $L_0^1(\mathbf{I}^2)$, our statement is established. $\Diamond$

3. Main results

As in the one dimensional case (see [5]) we will define a class of operators denoted by $\mathcal{N}$. Each element of $\mathcal{N}$ is given by a sequence of two dimensional dyadic convolution-operators $\Phi_n f = f * \phi_n$, $n \in \mathbb{N}^2$, where $\phi_n(x) = (\phi_{n_1}^{(1)} \times \phi_{n_2}^{(1)})(x)$, $x \in \mathbf{I}^2$ and $\phi_k^{(1)}$ $(k \in \mathbb{N})$ are integrable functions. Namely, let $\Phi \in \mathcal{N}$ be defined as

$$(3.1) \qquad \Phi f = \sum_{n\in\mathbb{N}^2} \Phi_n(\chi_n f) \quad (f \in L_0^1(\mathbf{I}^2)),$$

where $\chi_n = \chi_{n_1}^{(1)} \times \chi_{n_2}^{(1)}$, $n = (n_1, n_2) \in \mathbb{N}^2$. The convolution-operator Φ_n maps the class of the Walsh-polynomials into itself and

$$(3.2) \qquad \langle \Phi_n f, g\rangle = \langle f, \Phi_n g\rangle \qquad (f \in \mathcal{P}, g \in L^1(\mathbf{I}^2), n \in \mathbb{N}^2),$$

where $\langle f, g\rangle = \int_{\mathbf{I}^2} f(t)g(t)dt$ is the usual inner product of f and g. For the maximal operator of the sequence Φ_n $(n \in \mathbb{N}^2)$ we will use the notation $\Phi^* f = \sup_{n\in\mathbb{N}^2} |\Phi_n|$.

Theorem 2. *Let $\Phi \in \mathcal{N}$ be given as above, $1 < p < \infty$ and $1/p + +1/p' = 1$. Then*

(1) *if the generating sequence of ϕ_n $(n \in \mathbb{N}^2)$ satisfies*

$$(3.3) \qquad M := \sup_{n\in\mathbb{N}^2} ||\phi_n||_1 < \infty,$$

then Φ is a bounded linear operator from L^1 into itself, and

$$(3.4) \qquad ||\Phi f||_1 \le M||f||_1 \quad (f \in L_0^1(\mathbf{I}^2)).$$

(2) *Suppose that Φ^* is bounded from $L^{p'}$ into itself:*

$$||\Phi^* g||_{p'} \le M^*||g||_{p'} \quad (g \in L^{p'}(\mathbf{I}^2)).$$

Then Φ is a bounded linear operator from L^p into itself and

$$(3.5) \qquad ||\Phi f||_p \le M^*||f||_p \quad (f \in L^p).$$

Proof. Using the triangle inequality, (1.26) and (3.3) we get

$$||\Phi f||_1 \le \sum_{n\in\mathbb{N}^2} ||(\chi_n f) * \phi_n||_1 \le \sum_{n\in\mathbb{N}^2} ||\chi_n f||_1 ||\phi_n||_1 \le$$
$$\le M \sum_{n\in\mathbb{N}^2} ||\chi_n f||_1 = M \cdot ||f||_1,$$

and (1) is proved.

Part (2): Let $f \in \mathcal{P}$, $g \in L^{p'}(\mathbf{I}^2)$ with $||g||_{p'} \le 1$ and consider the inner product of Φf and g. It follows by (3.2) that

$$\langle \Phi f, g\rangle = \sum_{n\in\mathbb{N}^2} \langle \Phi_n(\chi_n f), g\rangle = \sum_{n\in\mathbb{N}^2} \langle \chi_n f, \Phi_n g\rangle.$$

If we take the absolute value of this inner product, and apply Hölder's inequality we get

$$|\langle \Phi f, g\rangle| \le \sum_{n\in\mathbb{N}^2} \langle |\chi_n f|, \Phi^* g\rangle = \langle |f|, \Phi^* g\rangle \le$$
$$\le ||f||_p ||\Phi^* g||_{p'} \le M^* ||g||_{p'} ||f||_p.$$

Taking the supremum with respect to $g \in L^{p'}$, $||g||_{p'} \le 1$ it follows that

$$||\Phi f||_p = \sup\left\{|\langle \Phi f, g\rangle| \ : \ g \in L^{p'},\ ||g||_{p'} \le 1\right\} \le M^* ||f||_p,$$

which proves our statement. ◊

Corollary 1. *The Cesàro operator C*

(1) *is bounded linear operator from L^p into itself if $1 \le p < \infty$,*
(2) *is not bounded from L^∞ into itself.*

Proof. By (2.2) the Cesàro operator belongs to $\mathcal{N}$ with generator sequence $\mathbf{d}_n^- W^{(2)}$. It is known (see [11], [12]) that the maximal operator given by this sequence is a bounded operator from $L^{p'}$ to itself, $1 < p' < \infty$ and by Lemma 1 (3.3) holds, too. If we apply Th. 2 to the Cesàro operator we get part (1).

Part (2) follows immediately from the one dimensional case (see [5]). ◊

Theorem 3. *Let $\Phi \in \mathcal{N}$ an operator for which*

(1) *the function sequence ϕ_n satisfies* (3.3) *and*

$$\hat{\phi}_n(k) = 0 \quad (0 \le k_1 < 2^{n_1} \ \textit{or}\ 0 \le k_2 < 2^{n_2}), n \in \mathbb{N}^2), \ \textit{or} \tag{3.6}$$

(2) $\phi_n = D_{2^n} \quad (n \in \mathbb{N}^2)$.

Then Φ is a bounded linear operator from the dyadic Hardy space $\mathcal{H}^1$ into itself, and

$$||\Phi f||_{\mathcal{H}^1} \le M_1 ||f||_{\mathcal{H}^1} \qquad (f \in \mathcal{H}^1)$$

with a constant M_1 dependly only on M in (3.3).

To the proof of Th. 3 we need

Lemma 2. *If $f \in L^1$ then the following inequality is true*

$$\sum_{n\in\mathbb{N}^2} ||\chi_n (f - \Delta_n f)||_{\mathcal{H}^1} \le 4 \cdot ||f||_{\mathcal{H}^1}. \tag{3.7}$$

Proof. Let us consider the following function

$$g_n(x) := \chi_n(x)(f(x) - \Delta_n f(x)) =$$
$$= \chi_n(x)\Big(f(x) - \frac{1}{|J_{n_1}|}\int_{J_{n_1}} f(s,x_2)ds - \frac{1}{|J_{n_2}|}\int_{J_{n_2}} f(x_1,t)dt +$$
$$+ \frac{1}{|J_n|}\int_{J_n} f(s,t)dsdt\Big) \quad (x \in \mathbf{I}^2, n \in \mathbb{N}^2),$$

where

$$J_{n_1} = [2^{-n_1}, 2^{-n_1+1}), \quad J_{n_2} = [2^{-n_2}, 2^{-n_2+1}), \quad J_n = J_{n_1} \times J_{n_2}.$$

It is easy to prove that

$$\int_{J_{n_1}} g_n(s,x_2)ds = \int_{J_{n_2}} g_n(x_1,t)dt = 0.$$

From this follows that if $x \in J_n$ then for $m_1 \le n_1$ or $m_2 \le n_2$ we get $S_{2^m} g_n(x) = 0$. If $x \notin J_n$ then we get for all m that $S_{2^m} g_n(x) = 0$. Furthermore, if $x \in J_1 \times J_2$ and $m \ge n$ then

$$S_{2^m} S_{2^n}(\chi_n f) = \chi_n S_{2^n} f,$$
$$S_{2^m} S_{(2_1^n,\infty)}(\chi_n f) = \chi_n S_{(2^{n_1},2^{m_2})} f,$$
$$S_{2^m} S_{(\infty,2_2^n)}(\chi_n f) = \chi_n S_{(2^{m_1},2^{n_2})} f,$$

and so

$$|S_{2^m} g_n| \le \chi_n(|S_{2^m} f| + |S_{(2^{m_1},2^{n_2})} f| + |S_{(2^{n_1},2^{m_2})} f| + |S_{2^n} f|).$$

Taking the supremum over m we get

$$g_n^* \le 4\chi_n f^*,$$

and so our statement is proved. ◊

Proof of the Theorem 3. Statement (1): Because $\mathcal{P}$ is dense in $\mathcal{H}^1$, it is sufficient to prove that the Th.3 holds for $f \in \mathcal{P}$. Let $f \in \mathcal{P}$, $g \in \mathcal{BMO}$, with $\|g\|_{\mathcal{BMO}} \le 1$, and consider the inner product of Φf and g. It follows by (3.2) that

$$\langle \Phi f, g\rangle = \sum_{n\in\mathbb{N}^2} \langle \Phi_n(\chi_n f), g\rangle = \sum_{n\in\mathbb{N}^2} \langle \chi_n f, \Phi_n g\rangle,$$

where by (3.6) $(\Phi_n g)^\frown(k) = \hat{\phi}_n(k) \cdot \hat{g}(k) = 0$ for all $0 \le k_1 < 2^{n_1}$ or $0 \le k_2 < 2^{n_2}$. Therefore $\langle \chi_n f, \Phi_n g\rangle = 0$ for all n large enough, and $S_{2^n}(\Phi_n g) = 0$, $S_{(2^{n_1},\infty)}(\Phi_n g) = 0$ and $S_{(\infty,2^{n_2})}(\Phi_n g) = 0$. Consequently for all $h \in L^1$ and $n \in \mathbb{N}^2$

$$\langle S_{2^n}h, \Phi_n g\rangle = \langle h, S_{2^n}(\Phi_n g)\rangle = 0,$$
$$\langle S_{(2^{n_1},\infty)}h, \Phi_n g\rangle = \langle h, S_{(2^{n_1},\infty)}(\Phi_n g)\rangle = 0,$$
$$\langle S_{(\infty,2^{n_2})}h, \Phi_n g\rangle = \langle h, S_{(\infty,2^{n_2})}(\Phi_n g)\rangle = 0.$$

Applying these equalities for $h = \chi_n f$ we get

$$\langle \chi_n f, \Phi_n g\rangle = \langle \chi_n f + S_{2^n}(\chi_n f) - S_{(2^{n_1},\infty)}(\chi_n f) - S_{(\infty,2^{n_2})}(\chi_n f), \Phi_n g\rangle = \\ = \langle \chi_n(f - \Delta_n f), \Phi_n g\rangle.$$

From (1.13), (1.26) it follows by Lemma 2 for $\mathbf{X} = \mathcal{H}^1$ that

$$|\langle \Phi f, g\rangle| \leq \sum_{n\in\mathbb{N}^2} |\langle \chi_n(f - \Delta_n f), \Phi_n g\rangle| \leq \\ \leq C \sum_{n\in\mathbb{N}^2} \|\chi_n(f - \Delta_n f)\|_{\mathcal{H}^1} \|\Phi_n g\|_{\mathcal{BMO}} \leq \\ \leq C \sum_{n\in\mathbb{N}^2} \|\chi_n(f - \Delta_n f)\|_{\mathcal{H}^1} \|\phi_n\|_1 \|g\|_{\mathcal{BMO}} \leq \\ \leq 4MC\|f\|_{\mathcal{H}^1}\|g\|_{\mathcal{BMO}}.$$

If we take the supremum over those functions g for which $g \in \mathcal{BMO}$ and $\|g\|_{\mathcal{BMO}} \leq 1$ we will prove part (1) with $M_1 = 4MC$, where the absolute constant C is from the Feffermann's inequality.

Statement (2): Since $\Phi_n f = (\chi_n f) * D_{2^n} = 0$ outside the interval $J_n = [2^{-n_1}, 2^{-n_1+1}) \times [2^{-n_2}, 2^{-n_2+1})$ and $\Phi_n f = S_{2^n} f$ on J_n, thus Φ is of the form

$$\Phi f = \sum_{n\in\mathbb{N}^2} \chi_n S_{2^n} f.$$

Hence it is easy to see that

$$|S_{2^m}\Phi f| \leq f^*$$

for all $m \in \mathbb{N}^2$, and so $(\Phi f)^* \leq f^*$. ◊

Using Th. 3 for the Cesàro operator we get by Lemma 1 and (2.2)

Corollary 2. *The Cesàro operator is bounded linear operator from $\mathcal{H}^1$ into itself.* ◊

4. The two dimensional dyadic Copson operator

The Cesàro operator isn't selfadjoint operator, so if we take the adjoint of C we get a new operator with new statements. This adjoint operator is called the two dimensional dyadic Copson operator.

Theorem 4. *Let* $\mathbf{X} = L^p$ $(1 \le p < \infty)$ *or* $\mathbf{X} = \mathcal{H}^1$ *and denote* $\mathbf{X}^* = L^{p'}$ $(1/p + 1/p' = 1)$ *or* $\mathbf{X}^* = \mathcal{BMO}$ *the dual space of* $\mathbf{X}$. *Then the Copson operator* $\mathcal{C}^* : \mathbf{X}^* \to \mathbf{X}^*$ *is a bounded linear operator and satisfies*

$$(4.1) \qquad (\mathcal{C}^*\phi)^\frown(n) = \sum_{k=n+1}^{(\infty,\infty)} \frac{\hat{\phi}(k)}{k_1 \cdot k_2}, \quad (n \in \mathbb{N}^2, \phi \in \mathbf{X}^*).$$

The boundedness of the $\mathcal{C}^*$ *follows from* Cor. 1 *and from* Cor. 2.

Proof. The linear functionals of $\mathbf{X}$ have the form

$$(4.2) \qquad \langle f, \phi \rangle = \lim_{\min(n_1,n_2)\to\infty} \int_{\mathbf{I}^2} E_n f(t)\phi(t)dt \quad (f \in \mathbf{X}, \phi \in \mathbf{X}^*)$$

if $\mathbf{X} = \mathcal{H}^1$, and

$$(4.3) \qquad \langle f, \phi \rangle = \int_{\mathbf{I}^2} f(t)\phi(t)dt \quad (f \in \mathbf{X}, \phi \in \mathbf{X}^*)$$

if $\mathbf{X} = L^p$ $(1 \le p < \infty)$ (see [11]). Since $\|E_n f - f\|_p \to 0$ if $\min(n_1, n_2) \to \infty$ and $1 \le p < \infty$, therefore (4.2) holds for $\mathbf{X} = L^p$ too. If $g, h \in L^1$, then

$$(4.4) \qquad \int_{\mathbf{I}^2} E_n g(t)h(t)dt = \sum_{k=(0,0)}^{(2^{n_1-1},2^{n_2-1})} \hat{g}(k)\hat{h}(k).$$

From the definition of the adjoint operator, from (4.2) and (4.4) we get

$$(\mathcal{C}^*\phi)^\frown(n) = \lim_{\min(N_1,N_2)\to\infty} \int_{\mathbf{I}^2} (S_{2^N} w_n)(t)(\mathcal{C}^*\phi)(t)dt = \langle w_n, \mathcal{C}^*\phi \rangle =$$

$$= \langle \mathcal{C} w_n, \phi \rangle = \lim_{\min(N_1,N_2)\to\infty} \sum_{k=(0,0)}^{(2^{N_1-1},2^{N_2-1})} (\mathcal{C} w_n)^\frown(k)\hat{\phi}(k).$$

Since by (2.1)

$$(\mathcal{C} w_n)^\frown(k) = \begin{cases} 0, & \text{if } k_1 \le n_1 \text{ or } k_2 \le n_2 \\ \frac{1}{k_1 \cdot k_2}, & \text{if } k > n, \end{cases}$$

therefore we get

$$(\mathcal{C}^*\phi)^\frown(n) = \lim_{\min(N_1,N_2)\to\infty} \sum_{k=n+1}^{(2^{N_1-1},2^{N_2-1})} \frac{\hat{\phi}(k)}{k_1 \cdot k_2}.$$

Since $\mathbf{X}^* \subset \mathcal{H}^1$, therefore by a Hardy type inequality (see [11]) we get

$$\sum_{k\in\mathbb{P}^2} \frac{|\hat{\phi}(k)|}{k_1 \cdot k_2} \leq c\|\phi\|_{\mathcal{H}^1} < \infty,$$

where c is an absolute constant. That is, (4.1) holds which complete the proof of Th. 4. ◊

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