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ON PROPERTIES OF APPROXIMATE FIBRATIONS, I

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Abstract: This continuation of our paper "Equivalence of shape fibrations and approximate fibrations" explores further some basic properties of shape and approximate fibrations of arbitrary topological spaces using relations. Our method is to use relations with smaller and smaller images of points. The paper is self-reliant and does not require extensive knowledge of relations.

1. Introduction

In the paper "Equivalence of shape fibrations and approximate fibrations" [4], the author has introduced an intrinsic definition of approximate fibrations using relations with smaller and smaller images of points. These approximate fibrations are formally similar to approximate fibrations of Coram and Duvall [5]. However, by replacing single-valued continuous functions with relations i. e., multi-valued functions with non-empty images of points, we achieved greater flexibility. The use of multi-valued maps in shape theory was initiated by J. Sanjurjo [16].

The main result in [4] proves that the approximate fibrations defined either with functions or with relations agree with Mardešić's shape fibrations [13]. The goal in this paper is to explore basic properties of approximate fibrations. Our results are shape theoretic versions of standard theorems about fibrations. Here we shall consider the following topics: fibrations as approximate $(\mathcal{X}, \tau)$-fibrations; dependence of approximate $(\mathcal{X}, \tau)$-fibrations on the class $\mathcal{X}$ of spaces and on the triple τ of classes of relations; generalizations of approximate $(\mathcal{X}, \tau)$-fibrations suggested by Chapter III in [10] and Š. Ungar's notion of a shape bundle [18]; approximate fibrations with dense images; and restrictions, compositions, and products of approximate fibrations.

In the second part of this paper we shall deal with other properties of approximate fibrations such as: approximate fibrations and (shape) dominations; preservation of (shape) pathwise connectedness, (shape) stability, and (shape) contractibility under approximate fibrations; and the almost unique path lifting property.

Of course, some of these themes have been studied by Coram and Duvall for their approximate fibrations and also for shape fibrations by Mardešić, Rushing, Jani, Keesling, Yagasaki, Haxhibeqiri, and others. However, it appears that with our approach the assumptions in theorems are weaker and our proofs resemble more the proofs of corresponding results for fibrations.

Absolutely no expertise on relations is necessary to follow this paper. Anybody unfamiliar or uncomfortable with relations should replace them with functions and exercise slightly more care at places where inverses appear to get special functional versions of our results. For this the reference [3] might be useful. However, since the inverse of a function is more often a relation rather than a function, insisting on functions is not natural for our approach because it makes statements and proofs more complicated and less general.

2. Recalls from [4]

In this section we shall recall several important definitions and results from [4] in order to make this paper self-reliant as much as possible.

For the definition of approximate fibrations we need numerable coverings, small relations, and the notion of closeness for relations.

2.1. Numerable coverings. An open covering σ of a space Y is *numerable* if it has a partition of unity [1]. Let $\text{Cov}(Y)$ denote the collection of all numerable coverings of a topological space Y. With respect to the refinement relation $\geqslant$ the set $\mathrm{Cov}(Y)$ is a directed set.

Let $\sigma \in \mathrm{Cov}(Y)$. Let σ^+ denote the set of all numerable coverings of Y refining σ while σ^* denotes the set of all numerable coverings τ of Y such that the star $st(\tau)$ of τ refines σ. Similarly, for a natural number n, σ^{*n} denotes the set of all numerable coverings τ of Y such that the n-th star $st^n(\tau)$ of τ refines σ.

2.2. Relations. Let X and Y be topological spaces. By a *relation* $F : X \to Y$ we mean a rule which associates a non-empty subset $F(x)$ of Y to every point x of X.

For a relation $F : X \to Y$ and a subset A of X, let $F(A) = \cup\{F(x) | x \in A\}$. Let F^{-1} denote the relation from $F(X)$ into X defined by $F^{-1}(y) = \{x \in X \mid y \in F(x)\}$, for every $y \in F(X)$.

Let $\mathcal{G}$ be a class of relations. In order to state that a relation $F : X \to Y$ is from the class $\mathcal{G}$ we shall say that F is a *$\mathcal{G}$-relation.*

We reserve $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{M}$ for classes of all relations, single-valued relations, and single-valued continuous relations, respectively. We shall use *relation*, *function*, and *map* instead of $\mathcal{R}$-relation, $\mathcal{S}$-relation, and $\mathcal{M}$-relation, respectively. Unless stated otherwise, we shall use μ, ϱ, and σ to denote the triples $(\mathcal{M}, \mathcal{M}, \mathcal{M})$, $(\mathcal{R}, \mathcal{R}, \mathcal{R})$, and $(\mathcal{S}, \mathcal{S}, \mathcal{S})$, respectively. The class of all topological spaces is denoted by $\mathcal{T}$ while $\mathcal{P}$ stands for the class of all polyhedra.

2.3. Small relations. Let $F : X \to Y$ be a relation and let $\alpha \in \mathrm{Cov}(X)$ and $\beta \in \mathrm{Cov}(Y)$. We shall say that F is an *(α, β)-relation* provided for every $A \in \alpha$ there is a $B_A \in \beta$ with $F(A) \subset B_A$.

Now, we define that F is a *β-relation* provided there is an $\alpha \in \mathrm{Cov}(X)$ such that F is an (α, β)-relation. On the other hand, F is called a *weak β-relation* provided for every $x \in X$ there is a $B_x \in \beta$ with $F(x) \subset B_x$.

We shall frequently use the obvious property of maps $f : X \to Y$ that they are σ-relations for every $\sigma \in \mathrm{Cov}(Y)$. Moreover, if $\sigma \in \mathrm{Cov}(Y)$ and $\alpha \in \mathrm{Cov}(X)$ refines $f^{-1}(\sigma)$, then f is an (α, σ)-relation.

The fact that a relation F is at the same time from the class $\mathcal{G}$ of relations and that it is a (weak) β-relation will be expressed by saying that it is a *(weak) $\beta\mathcal{G}$-relation.* The term *$(\alpha, \beta)\mathcal{G}$-relation* has an analogous meaning.

The following lemma will be used to estimate the size of the composition of relations. Recall that the composition $G \circ F$ of the relations

$F : X \to Y$ and $G : Y \to Z$ is the relation from X into Z defined for every $x \in X$ by

$$G \circ F(x) = \{z \in Z \mid (\exists y \in Y) \quad y \in F(x) \,\&\, z \in G(y)\}.$$

Lemma 2.1. *Let α, β, and γ be numerable coverings of spaces X, Y, and Z. Let $G : Y \to Z$ be a (β, γ)-relation, let $F : X \to Y$ be a relation, and let $H : X \to Z$ denote the composition of F and G.*

1. *If F is an (α, β)-relation, then H is an (α, γ)-relation.*
2. *If F is a (weak) β-relation, then H is a (weak) γ-relation.*

2.4. Proximities for relations. Let F, $G : X \to Y$ be relations and let $\sigma \in \mathrm{Cov}(Y)$. We shall say that F and G are *σ-close* and we write $F \overset{\sigma}{=} G$ provided for every x in X there is an $S_x \in \sigma$ with $F(x) \cup G(x) \subset \subset S_x$.

On the other hand, let F, $G : X \to Y$ be relations, let $\alpha \in \in \mathrm{Cov}(X)$, and let $\sigma \in \mathrm{Cov}(Y)$. We shall say that F and G are *(α, σ)-near* and we write $F \overset{\alpha,\sigma}{=} G$ provided for every member A of the covering α there is a member S_A of σ with $F(A) \cup G(A) \subset S_A$. Moreover, F and G are *σ-near* and we write $F \overset{\sigma}{\equiv} G$ provided there is a numerable covering α of X such that F and G are (α, σ)-near.

Observe that σ-near relations are also σ-close. The next lemma shows that the converse is almost true for σ-relations.

Lemma 2.2. *Let σ be a numerable covering of a space Y. If two σ-relations F and G from a space X into Y are σ-close, then they are also $st(\sigma)$-near.*

2.5. Homotopy. Let F and G be relations from a space X into a space Y. It is customary to call a relation H from the product $X \times I$ of X and the unit segment $I = [0, 1]$ into Y such that $F(x) = H(x, 0)$ and $G(x) = H(x, 1)$ for every $x \in X$ a *homotopy* that *joins* F and G. We say that F and G are *homotopic* and we write $F \simeq G$.

Let β be a numerable covering of Y. If a homotopy $H : X \times I \to \to Y$ is a (weak) β-relation, then we shall say that H is a *(weak) β-homotopy*, that F and G are *(weakly) β-homotopic*, and we shall write $F \overset{\beta}{\simeq}_w G$ for weak homotopy and $F \overset{\beta}{\simeq} G$ for homotopy.

2.6. Fibrations and approximate fibrations. Let $\tau = (\mathcal{G}, \mathcal{H}, \mathcal{K})$ be a triple of classes of relations and let $\mathcal{X}$ be a class of topological spaces. A map $p : E \to B$ is said to be an *$(\mathcal{X}, \tau)$-fibration* provided for every $\alpha \in \mathrm{Cov}(B)$ and every $\delta \in \mathrm{Cov}(E)$ there are $\beta \in \alpha^+$ and $\varepsilon \in \delta^+$

such that for every $X \in \mathcal{X}$, every $\varepsilon\mathcal{G}$-relation $G : X \to E$, and every $\beta\mathcal{H}$-relation $H : X \times I \to B$ with $H_0 = p \circ G$ there is a $\delta\mathcal{K}$-relation $K : X \times I \to E$ with $K_0 = G$ and $p \circ K = H$.

Observe that a map $p : E \to B$ is a *Hurewicz fibration* if and only if it is a $(\mathcal{T}, \mu)$-fibration while it is a Serre fibration if and only if it is a $(\mathcal{P}, \mu)$-fibration.

A map $p : E \to B$ is called an *approximate* $(\mathcal{X}, \tau)$*-fibration* provided for every $\alpha \in \mathrm{Cov}(B)$ and every $\delta \in \mathrm{Cov}(E)$ there is a $\beta \in \in \mathrm{Cov}(B)$ and an $\varepsilon \in \mathrm{Cov}(E)$ such that for every member X of $\mathcal{X}$, every $\varepsilon\mathcal{G}$-relation $G : X \to E$, and every $\beta\mathcal{H}$-relation $H : X \times I \to B$ with $H_0 \overset{\beta}{=} p \circ G$ there is a $\delta\mathcal{K}$-relation $K : X \times I \to E$ with $K_0 \overset{\delta}{=} G$ and $p \circ K \overset{\alpha}{=} H$.

An approximate $(\mathcal{T}, \varrho)$-fibration will be called simply an *approximate fibration.* We shall prove later in §5 that approximate fibrations agree with approximate $(\mathcal{T}, \sigma)$-fibrations and with shape fibrations [4, §8].

There are three other forms called *approximate* $(\mathcal{X}, \tau)$C*-fibrations, approximate* $(\mathcal{X}, \tau)$D*-fibrations*, and *approximate* $(\mathcal{X}, \tau)$CD*-fibrations.* We get them from the above definition by replacing either only the first condition, only the second condition, or both the first and the second conditions on closeness of relations with the equality of relations, respectively.

Observe that a map $p : E \to B$ is an approximate fibration in the sense of Coram and Duvall [5] if and only if it is an approximate $(\mathcal{T}, \mu)$CD-fibration.

Finally, by replacing in the above definitions the relation of closeness with the relation of nearness we get four additional versions which we denote in the same way using the word proximate instead of the word approximate. It is most fortunate that these two groups coincide and that for a map $p : E \to B$ and for ϑ either $(\mathcal{R}, \mathcal{R}, \mathcal{R})$ or $(\mathcal{S}, \mathcal{S}, \mathcal{S})$ the four versions of approximate $(\mathcal{T}, \vartheta)$-fibrations coincide (see [4, Theorems 6.1 and 6.2]).

2.7. Proximate and approximate movability. Let $\mathcal{F}$ and $\mathcal{G}$ be classes of relations and let $\mathcal{X}$ be a class of topological spaces. Let $\omega = = (\mathcal{F}, \mathcal{G})$. A space B is called *proximately* $(\mathcal{X}, \omega)$*-movable* provided for every $\sigma \in \mathrm{Cov}(B)$ there is a $\tau \in \mathrm{Cov}(B)$ such that for every $X \in \mathcal{X}$ and every $\tau\mathcal{F}$-relation $F : X \to B$ there is a $\sigma\mathcal{G}$-relation $G : X \to B$ with $F \overset{\sigma}{=} G$. On the other hand, a space B is *approximately* $(\mathcal{X}, \omega)$*-movable* provided for every $\sigma \in \text{Cov}(B)$ there is a $\tau \in \text{Cov}(B)$ so that for every X in $\mathcal{X}$, every $\tau\mathcal{F}$-relation $F : X \to B$, and every $\varrho \in \text{Cov}(B)$ there is a $\varrho\mathcal{G}$-relation $G : X \to B$ with $F \overset{\sigma}{=} G$.

Recall [13] that an *approximate polyhedron* is a topological space Y with the property that for every $\sigma \in \text{Cov}(Y)$ there are a polyhedron P and maps $u : Y \to P$ and $d : P \to Y$ with $id_Y \overset{\sigma}{=} d \circ u$. The following is Th. 5.1 in [4].

Theorem 2.1. *Let $\omega = (\mathcal{R}, \mathcal{M})$. Let $\mathcal{T}$ be the class of all topological spaces. A space Y is an approximate polyhedron iff it is approximately $(\mathcal{T}, \omega)$-movable.*

3. Fibrations vs. approximate fibrations

In this section we shall explore under what conditions will a fibration be an approximate fibration. These conditions include the assumption that the base space B is an approximate $(\mathcal{X}, \mu)$-plank which we define in the following definition.

Let $\tau = (\mathcal{G}, \mathcal{H}, \mathcal{K})$ be a triple of classes of relations. Let $\mathcal{X}$ be a class of topological spaces. A space Y is called an *approximate* $(\mathcal{X}, \tau)$-*plank* provided for every $\alpha \in \text{Cov}(Y)$ there is a $\beta \in \text{Cov}(Y)$ such that for every space X from the class $\mathcal{X}$, every $\beta\mathcal{G}$-relation $G : X \to Y$, and every $\beta\mathcal{H}$-homotopy $H : X \times I \to Y$ with $H_0 \overset{\beta}{=} G$ there is an $\alpha\mathcal{K}$-homotopy $H : X \times I \to Y$ with $K_0 = G$ and $K \overset{\alpha}{=} H$.

Theorem 3.1. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be classes of relations and let $\mathcal{X}$ be a class of spaces. Let $\lambda = (\mathcal{A}, \mathcal{B}, \mathcal{C})$, $\nu = (\mathcal{A}, \mathcal{D}, \mathcal{C})$, and $\pi = (\mathcal{E}, \mathcal{B}, \mathcal{D})$. If $p : E \to B$ is an $(\mathcal{X}, \nu)$-fibration, the space B is an approximate $(\mathcal{X}, \pi)$-plank, and the class $p \circ \mathcal{A}$ is contained in the class $\mathcal{E}$, then the map p is also an approximate $(\mathcal{X}, \lambda)$-fibration.*

Proof. Let $\alpha \in \text{Cov}(B)$ and $\delta \in \text{Cov}(E)$ be given. Since p is an $(\mathcal{X}, \nu)$-fibration, there are $\xi \in \alpha^+$ and $\eta \in \delta^+$ such that for every space X in $\mathcal{X}$, every $\eta\mathcal{A}$-relation $G : X \to E$, and every $\xi\mathcal{B}$-homotopy $D : X \times I \to B$ with $D_0 = p \circ G$ there is a $\delta\mathcal{C}$-homotopy $K : X \times I \to E$ with $K_0 = G$ and $p \circ K = D$.

We utilize now the assumption that B is an approximate $(\mathcal{X}, \pi)$-plank to select a $\beta \in \text{Cov}(B)$ such that for every space X from the class $\mathcal{X}$, every $\beta\mathcal{E}$-relation $L : X \to B$, and every $\beta\mathcal{B}$-homotopy $H : X \times I \to B$ with $H_0 \overset{\beta}{=} L$ there is a $\xi\mathcal{D}$-homotopy $D : X \times I \to B$

with $D_0 = L$ and $D \overset{\xi}{=} H$. Let $\varepsilon \in \mathrm{Cov}(E)$ be a common refinement of η and $p^{-1}(\beta)$. $\Diamond$

Example 3.1. Let $E = A \cup B$, where A and B are subsets of $\mathbb{R}^3$ given by

$$\{(0, y, 1) \mid -1 \le y \le 1\}$$

and

$$\{(0, y, 0) \mid -1 \le y \le 1\} \cup \left\{(x, \sin\frac{1}{x}, 0) \mid 0 < x \le \frac{2}{\pi}\right\}.$$

Let $p : E \to B$ be a projection given by $p(x, y, z) = (x, y, 0)$ for every $(x, y, z) \in E$. It is easy to check that p is a Hurewicz fibration, both E and B are approximate polyhedra, and p is not a shape fibration. This example shows that Cor. 3 in [13] is not true. The reason for this according to Th. (3.31) is that the space B is an approximate polyhedron which is not an approximate $(\mathcal{T}, \varrho)$-plank.

In order to get some examples of approximate $(\mathcal{N}, \mu)$-planks, where $\mathcal{N}$ denotes the class of all normal spaces, we need the following two definitions.

In the literature there is another notion of size for homotopy based on the idea that tracks of points are included in members of a given covering on the codomain.

Let X and Y be spaces and let $\beta \in \mathrm{Cov}(Y)$. A relation $H : X \times I \to Y$ will be called a *weak track β-homotopy* provided for every $x \in X$ there is an $S_x \in \beta$ with $H(\{x\} \times I) \subset S_x$. Two relations F and G are *weakly track β-homotopic* and we write $F \overset{\beta}{\simeq}_w G$ provided there is a weak track β-homotopy $H : X \times I \to Y$ joining them.

Let $\tau = (\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a triple of classes of relations. Let $\mathcal{X}$ be a class of spaces. A space Y is called an *approximate $(\mathcal{X}, \tau)$-bridge* provided for every $\alpha \in \mathrm{Cov}(Y)$ there is a $\beta \in \mathrm{Cov}(Y)$ such that for every space $X \in \mathcal{X}$, every $\beta\mathcal{A}$-relation $F : X \to Y$, and every $\beta\mathcal{B}$-relation $G : X \to Y$ such that $G \overset{\beta}{=} F$ there is a weak track $\alpha\mathcal{C}$-homotopy $H : X \times I \to Y$ with $H_0 = F$ and $H_1 = G$.

Observe that [11, p. 111] implies that every absolute neighbourhood retract for the class of all metrizable spaces is an approximate $(\mathcal{T}, \mu)$-bridge.

Lemma 3.1. *Every approximate $(\mathcal{N}, \mu)$-bridge Y is an approximate $(\mathcal{N}, \mu)$-plank.*

Proof. Let a numerable covering σ of Y be given. Let $\eta \in \sigma^*$. Since the space Y is an approximate $(\mathcal{N}, \mu)$-bridge, there is a $\tau \in \mathrm{Cov}(Y)$ such that for every normal space X and all maps $f : X \to Y$ and $g : X \to Y$ with $f \overset{\tau}{=} g$ there is a weak track η-homotopy $k : X \times I \to Y$ with $k_0 = f$ and $k_1 = g$. Then τ is the required numerable covering.

Indeed, consider a normal space X and maps $f : X \to Y$ and $g : X \times I \to Y$ such that $f \overset{\tau}{=} g_0$. Let a map $k : X \times [-1, 0] \to Y$ be a weak track η-homotopy joining f and g_0. Define a homotopy $m : X \times [-1, 1] \to Y$ as k on $X \times [-1, 0]$ and as g on $X \times [0, 1]$.

The collection $\delta = g^{-1}(\eta)$ is a numerable covering of the product $X \times I$. By [7, p. 358], there is a numerable covering ξ of X and a function $r : \xi \to \{2, 3, 4, \dots\}$ such that every set $U \times [(i-1)/rU, (i+1)/rU]$ is contained in some $D_U^i \in \delta$, where $U \in \xi$ and $i = 1, \dots, rU - 1$. Let $\pi = \{\pi_U\}$ be a locally finite numeration of ξ. Define a function $p : X \to (0, 1)$ by the rule $p(x) = \inf\{1/rU \mid \pi_U(x) \neq 0\}$, for every $x \in X$. Since the function p is lower semi-continuous and the space X is a normal space, by [[]p. 442]eng, there is a map $q : X \to (0, 1)$ such that $g(\{x\} \times [0, q(x)])$ is contained in some member of η for each $x \in X$.

Define a map $h : X \times I \to Y$ by the formula

$$h(x, t) = \begin{cases} m(x, 2t/q(x) - 1), & 0 \le t \le q(x)/2, \\ m(x, 2t - q(x)), & q(x)/2 \le t \le q(x), \\ m(x, t), & q(x) \le t \le 1. \end{cases}$$

Then $h(x, 0) = m(x, -1) = f(x)$. If $0 \le t \le q(x)/2$, then $s = 2t/q(x) - 1$ lies in $[-1, 0]$, so $h(x, t) = m(x, s)$ and $g(x, 0)$ are contained in some member of η. On the other hand, $g(x, 0)$ and $g(x, t)$ are also in some member of η. Hence, a member of σ contains both $h(x, t)$ and $g(x, t)$. The case $q(x)/2 \le t \le q(x)$ is verified similarly. $\Diamond$

4. Dependence on classes of spaces

This section examines how the definition of an approximate $(\mathcal{X}, \tau)$-fibration depends on the class $\mathcal{X}$. The answer is offered by the following concept of approximate domination for classes of spaces.

Let X be a space, let $\mathcal{Y}$ be a class of spaces, and let $\pi = (\mathcal{A}, \mathcal{B})$ be a pair of classes of relations. Let σ be a numerable covering of X. We say that the space X is *approximately* (π, σ)*-dominated* by the class $\mathcal{Y}$ provided there is a space Y in $\mathcal{Y}$ and a $\sigma\mathcal{A}$-relation $A : Y \to X$ such that for every $\tau \in \mathrm{Cov}(Y)$ there is a $\tau\mathcal{B}$-relation $B : X \to Y$ with $A \circ B \overset{\sigma}{=} \mathrm{id}_X$.

If a space X is approximately (π, σ)-dominated by $\mathcal{Y}$ for every $\sigma \in \mathrm{Cov}(X)$, then the space X is said to be *approximately* π-*dominated* by the class $\mathcal{Y}$.

A class $\mathcal{X}$ of spaces is *approximately* π-*dominated* by the class of spaces $\mathcal{Y}$ provided every member of $\mathcal{X}$ is approximately π-dominated by $\mathcal{Y}$.

By replacing the closeness relation with the nearness relation we shall get another version of previous notions. It follows from Lemma 2.2 that this version agrees with the original.

For classes $\mathcal{A}$ and $\mathcal{B}$ of relations, let $\mathcal{A} \circ \mathcal{B}$ and $\mathcal{A} \times \mathcal{B}$ denote all compositions $A \circ B$ and all products $A \times B$ where A is a relation in $\mathcal{A}$ and B is a relation in $\mathcal{B}$.

Theorem 4.1. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{V}$ be classes of relations and let $\mathcal{X}$ and $\mathcal{Y}$ be classes of topological spaces. Let $\mathcal{G} = \mathcal{C} \circ (\mathcal{V} \times \mathrm{id}_I)$, $\mathcal{F} = \mathcal{B} \circ (\mathcal{D} \times \mathrm{id}_I)$, $\mathcal{E} = \mathcal{A} \circ \mathcal{D}$, $\lambda = (\mathcal{A}, \mathcal{B}, \mathcal{G})$, $\nu = (\mathcal{E}, \mathcal{F}, \mathcal{C})$, and $\pi = (\mathcal{D}, \mathcal{V})$. If a map $p : E \to B$ is an approximate $(\mathcal{Y}, \nu)$-fibration and the class $\mathcal{X}$ is approximately π-dominated by the class $\mathcal{Y}$, then p is also an approximate $(\mathcal{X}, \lambda)$-fibration.*

Proof. Let an $\alpha \in \mathrm{Cov}(B)$ and a $\delta \in \mathrm{Cov}(E)$ be given. Let $\xi \in \alpha^*$ and $\pi \in \delta^*$. Since p is a proximate $(\mathcal{Y}, \nu)$-fibration, there is a $\beta \in \xi^+$ and $\varepsilon \in \pi^+$ with the property that for every space Y in $\mathcal{Y}$, every $\varepsilon\mathcal{E}$-relation $S : Y \to E$, and every $\beta\mathcal{F}$-relation $F : Y \times I \to B$ with $F_0 \overset{\beta}{\equiv} p \circ S$ there is a $\pi\mathcal{C}$-relation $C : Y \times I \to E$ such that $C_0 \overset{\pi}{\equiv} S$ and $F \overset{\xi}{\equiv} p \circ C$.

Let a space X in $\mathcal{X}$, an $\varepsilon\mathcal{A}$-relation $A : X \to E$, and a $\beta\mathcal{B}$-relation $T : X \times I \to B$ with $T_0 \overset{\beta}{\equiv} p \circ A$ be given. Pick a numerable covering η of X and a stacked numerable covering ϱ of $X \times I$ over η such that A is an (η, ε)-relation, $T_0 \overset{\eta,\beta}{\equiv} p \circ A$, and B is a (ϱ, β)-relation. Since the class $\mathcal{X}$ is approximately π-dominated by the class $\mathcal{Y}$, there is a space Y in $\mathcal{Y}$ and an $\eta\mathcal{D}$-relation $D : Y \to X$ such that for every $\kappa \in \mathrm{Cov}(Y)$ there is a $\kappa\mathcal{V}$-relation $V : X \to Y$ with $\mathrm{id}_X \overset{\eta}{\equiv} D \circ V$. Put $S = A \circ D$ and $F = T \circ (D \times \mathrm{id}_I)$. Observe that S is an $\varepsilon\mathcal{E}$-relation, F is a $\beta\mathcal{F}$-relation, and $F_0 \overset{\beta}{\equiv} p \circ S$. By assumption, there is a $\pi\mathcal{C}$-relation $C : Y \times I \to E$ such that $S \overset{\pi}{\equiv} C_0$ and $F \overset{\xi}{\equiv} p \circ C$. Pick a numerable covering κ of Y and a stacked numerable covering $\zeta \in \varrho^+$ of $Y \times I$ over κ such that $S \overset{\kappa,\pi}{\equiv} C_0$, $F \overset{\zeta,\xi}{\equiv} p \circ C$, and C is a (ζ, π)-relation. Choose a $\kappa\mathcal{V}$-relation V as above and put $G = C \circ (V \times \mathrm{id}_I)$. Notice that $G : X \times I \to E$ is a $\pi\mathcal{G}$-relation and therefore also a $\delta\mathcal{G}$-relation. Moreover, $G_0 = C_0 \circ V \overset{\pi}{=} S \circ V = A \circ D \circ V \overset{\varepsilon}{=} A$ and

$$p \circ G = p \circ C \circ (V \times \mathrm{id}_I) \overset{\xi}{=} F \circ (V \times \mathrm{id}_I) = T \circ ((D \circ V) \times \mathrm{id}_I) \overset{\beta}{=} T.$$

Hence, $A \overset{\alpha}{=} G_0$ and $T \overset{\alpha}{=} p \circ G$. ◊

Theorem 4.2. *Let ω be a pair $(\mathcal{R}, \mathcal{M})$. The class $\mathcal{T}$ of all topological spaces is approximately ω-dominated by the class $\mathcal{P}$ of all polyhedra.*

Proof. Let a topological space X and a numerable covering σ of X be given. Let $\tau \in \sigma^\star$. Let $N(\tau)$ be the nerve of the covering τ, let $P = |N(\tau)|$ be a geometric realization of $N(\tau)$, and let $q : X \to P$ be a canonical map. Define $D : P \to X$ by the following rule: If an $x \in P$ lies in the interior of a simplex of P with vertices $U_1, \ldots, U_n$, then $D(x) = U_1 \cap \ldots \cap U_n$. The relation D is clearly a σ-relation and $\mathrm{id}_X \overset{\sigma}{=} D \circ q$. ◊

Corollary 4.1. *A map $p : E \to B$ is an approximate $(\mathcal{P}, \varrho)$-fibration if and only if it is an approximate $(\mathcal{T}, \varrho)$-fibration.*

5. Dependence on classes of relations

The following result explores in what way does the definition of an approximate $(\mathcal{X}, \tau)$-fibration depend on the triple $\tau = (\mathcal{A}, \mathcal{B}, \mathcal{C})$ of classes of relations.

Recall that for a class $\mathcal{X}$ of spaces, $\mathcal{X} \times I$ denotes the class of all products of members of $\mathcal{X}$ with the unit closed segment I.

Theorem 5.1. *Let $\lambda = (\mathcal{A}, \mathcal{B}, \mathcal{C})$ and $\nu = (\mathcal{D}, \mathcal{E}, \mathcal{F})$ be triples of classes of relations. Let $\eta = (\mathcal{A}, \mathcal{D})$, $\kappa = (\mathcal{B}, \mathcal{E})$, and $\tau = (\mathcal{F}, \mathcal{C})$. Let $\mathcal{X}$ be a class of spaces and let $\mathcal{Y} = \mathcal{X} \times I$. If $p : E \to B$ is an approximate $(\mathcal{X}, \nu)$-fibration from a space E which is both proximately $(\mathcal{X}, \eta)$-movable and proximately $(\mathcal{Y}, \tau)$-movable into a proximately $(\mathcal{Y}, \kappa)$-movable space B, then p is also an approximate $(\mathcal{X}, \lambda)$-fibration.*

Proof. Let numerable coverings α and δ of spaces B and E be given. Let $\xi \in \alpha^\star$ and $\pi \in \delta^\star \cap p^{-1}(\xi)^\star$. Since E is proximately $(\mathcal{Y}, \tau)$-movable, there is a $\varrho \in \pi^+$ such that for every space $X \in \mathcal{X}$, every $\varrho\mathcal{F}$-relation $F : X \times I \to E$ is π-close to a $\pi\mathcal{C}$-relation. Now we use the assumption that the map p is an approximate $(\mathcal{X},\ \nu)$-fibration to select $\sigma \in \varrho^+$ and $\zeta \in \xi^+$ such that for every $X \in \mathcal{X}$, every $\sigma\mathcal{D}$-relation $D : : X \to E$, and every $\zeta\mathcal{E}$-relation $S : X \times I \to B$ with $S_0 \stackrel{\zeta}{=} p \circ D$ there is a $\varrho\mathcal{F}$-relation $F : X \times I \to E$ which satisfies $F_0 \stackrel{\varrho}{=} D$ and $S \stackrel{\xi}{=} p \circ F$. Let $\omega \in \zeta^\star$ and $\mu \in \sigma^+ \cap p^{-1}(\omega)^\star$. Since E is also proximately $(\mathcal{X},\ \eta)$-movable, there is an $\varepsilon \in \lambda^+$ such that for every space $X \in \mathcal{X}$, every $\varepsilon\mathcal{A}$-relation $A : X \to E$ is μ-close to a $\mu\mathcal{D}$-relation. Finally, we utilize the fact that B is a proximately $(\mathcal{Y},\ \kappa)$-movable space to pick a $\beta \in \omega^+$ so that for every space $X \in \mathcal{X}$, every $\beta\mathcal{B}$-relation $T : X \times I \to B$ is ω-close to an $\omega\mathcal{E}$-relation. Then β and ε are numerable coverings that we were looking for. ◊

From Theorems 5.1 and 2.1 we get the following two corollaries.

Corollary 5.1. *Let $\mathcal{X}$ be a class of spaces. Let $\pi = (\mathcal{R},\ \mathcal{M})$. If $p : : E \to B$ is an approximate $(\mathcal{X},\ \varrho)$-fibration and E is an approximate polyhedron or approximately $(\mathcal{X} \times I,\ \pi)$-movable, then p is also an approximate $(\mathcal{X},\ \mu)$-fibration.*

Corollary 5.2. *Let $\mathcal{X}$ be a class of spaces. If a map $p : E \to B$ is an approximate $(\mathcal{X},\ \mu)$-fibration and both E and B are approximate polyhedra, then the map p is also an approximate $(\mathcal{X},\ \varrho)$-fibration.*

Theorem 5.2. *Let $\mathcal{X}$ be a class of spaces. A map $p : E \to B$ is an approximate $(\mathcal{X},\ \varrho)$-fibration if and only if it is an approximate $(\mathcal{X},\ \sigma)$-fibration.*

Proof. ($\Longrightarrow$): Let $\alpha \in \mathrm{Cov}(B)$ and $\delta \in \mathrm{Cov}(E)$ be given. Since p is an approximate $(\mathcal{X},\ \varrho)$-fibration, there are $\beta \in \alpha^+$ and $\varepsilon \in \delta^+$ such that for every space X in $\mathcal{X}$, every ε-relation $G : X \to E$, and every β-relation $H : X \times I \to B$ with $p \circ G \stackrel{\beta}{=} H_0$, there is a δ-relation $K : X \times I \to E$ with $K_0 \stackrel{\delta}{=} G$ and $p \circ K \stackrel{\alpha}{=} H$.

Consider a space X from the class $\mathcal{X}$, an ε-function $g : X \to E$, and a β-function $h : X \times I \to B$ with $h_0 \stackrel{\beta}{=} p \circ g$. Select a relation K as above. Let $k : X \times I \to E$ be a single-valued selection of the relation K. Clearly, k is a δ-function, $k_0 \stackrel{\delta}{=} g$, and $p \circ k \stackrel{\alpha}{=} h$.

($\Longleftarrow$): Let $\alpha \in \mathrm{Cov}(B)$ and $\delta \in \mathrm{Cov}(E)$ be given. Let $\pi \in \in \alpha^\star$ and $\tau \in \delta^\star$. Since p is an approximate $(\mathcal{X},\ \sigma)$-fibration, there are $\beta \in \pi^+$ and $\varepsilon \in \tau^+$ such that for every space X from the class $\mathcal{X}$, every ε-function $g : X \to E$, and every β-function $h : X \times I \to B$ with $p \circ g \stackrel{\beta}{=} h_0$, there is a τ-function $k : X \times I \to E$ with $k_0 \stackrel{\tau}{=} g$ and $p \circ k \stackrel{\pi}{=} h$.

Consider a space X in $\mathcal{X}$, an ε-relation $G : X \to E$, and a β-relation $H : X \times I \to B$ with $H_0 \stackrel{\beta}{=} p \circ G$. Let $g : X \to E$ and $h : X \times I \to B$ be single-valued selection functions for G and H, respectively. By assumption, there is a τ-function $k : X \times I \to E$ with $k_0 \stackrel{\tau}{=} g$ and $p \circ k \stackrel{\pi}{=} h$. Clearly, $k_0 \stackrel{\delta}{=} G$ and $p \circ k \stackrel{\alpha}{=} H$. ◊

6. Generalizations of approximate fibrations

In this section we shall introduce some classes of maps related to approximate fibrations and explore their relationships.

Let $\pi = (\mathcal{H}, \mathcal{K})$, $\tau = (\mathcal{F}, \mathcal{H}, \mathcal{K})$, and $\chi = (\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{K})$ for classes $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{K}$ of relations. Let $\mathcal{X}$ be a class of spaces and let $\mathcal{X}_p$ be a class of pairs (X, A) consisting of a space X and a subspace A of X.

A map $p : E \to B$ is an *approximate* $(\mathcal{X}, \pi)$*-bundle* provided for every $\delta \in \mathrm{Cov}(E)$ and every $\alpha \in \mathrm{Cov}(B)$ there is a $\beta \in \mathrm{Cov}(B)$ such that for every space X in $\mathcal{X}$ and every $\beta\mathcal{H}$-relation $H : X \to B$ there is a $\delta\mathcal{K}$-relation $K : X \to E$ with $p \circ K \stackrel{\alpha}{=} H$.

A map $p : E \to B$ is an *approximate* $(\mathcal{X}_p, \tau)$*-bundle* provided for every $\delta \in \mathrm{Cov}(E)$ and every $\alpha \in \mathrm{Cov}(B)$ there is a $\beta \in \mathrm{Cov}(B)$ and an $\varepsilon \in \mathrm{Cov}(E)$ such that for every pair (X, A) in $\mathcal{X}_p$, every $\varepsilon\mathcal{F}$-relation $F : A \to E$, and every $\beta\mathcal{H}$-relation $H : X \to B$ with $p \circ F \stackrel{\beta}{=} H|_A$ there is a $\delta\mathcal{K}$-relation $K : X \to E$ with $K|_A \stackrel{\delta}{=} F$ and $p \circ K \stackrel{\alpha}{=} H$.

A map $p : E \to B$ is an *approximate* $(\mathcal{X}_p, \chi)$*-fibration* provided for every $\delta \in \mathrm{Cov}(E)$ and every $\alpha \in \mathrm{Cov}(B)$ there is a $\beta \in \mathrm{Cov}(B)$ and an $\varepsilon \in \mathrm{Cov}(E)$ such that for every pair (X, A) in $\mathcal{X}_p$, every $\varepsilon\mathcal{F}$-relation $F : X \to E$, every $\varepsilon\mathcal{G}$-homotopy $G : A \times I \to E$ with $G_0 \stackrel{\varepsilon}{=} F|_A$, and every $\beta\mathcal{H}$-homotopy $H : X \times I \to B$ with $p \circ F \stackrel{\beta}{=} H_0$ and $p \circ G \stackrel{\beta}{=} H|_{A \times I}$, there is a $\delta\mathcal{K}$-homotopy $K : X \times I \to E$ with $K_0 \stackrel{\delta}{=} F$, $K|_{A \times I} \stackrel{\delta}{=} G$, and $p \circ K \stackrel{\alpha}{=} H$.

For a class $\mathcal{X}$ of spaces, let $\mathcal{X}_{\emptyset p}$ and $\mathcal{X}_{hp}$ denote classes of pairs $\{(X, \emptyset) \mid X \in \mathcal{X}\}$ and $\{(X \times [0, 1], X \times \{0\}) \mid X \in \mathcal{X}\}$.

Observe that a map $p : E \to B$ is an approximate $(\mathcal{X}, \tau)$-fibration if and only if it is an approximate $(\mathcal{X}_{hp}, \tau)$-bundle. Also, it is an approximate $(\mathcal{X}, \pi)$-bundle if and only if it is an approximate $(\mathcal{X}_{\emptyset p}, \tau)$-bundle. Moreover, the map p is an approximate $(\mathcal{X}, \tau)$-fibration if and only if it is an approximate $(\mathcal{X}_{\emptyset p}, \chi)$-fibration.

In the first result in this section we shall show that approximate $(\mathcal{X}, \sigma)$-fibrations with P-dense image for the class $\mathcal{X}$ of spaces with trivial shape is an approximate $(\mathcal{X}, \varrho)$-bundle, where $\sigma = (\mathcal{S}, \mathcal{S}, \mathcal{S})$ and $\varrho = (\mathcal{S}, \mathcal{S})$. The analogous property is shared by fibrations (see [17, p. 74]).

A subspace A is *P-dense* in a space B provided for every $x \in B$ and every $\sigma \in \mathrm{Cov}(B)$ there is a member S of σ such that $x \in S$ and $S \cap A \neq \emptyset$. In other words, A is P-dense in B provided the star $st(A, \sigma)$ of A with respect to every numerable covering of B agrees with all of B.

Observe that a dense subset is also P-dense. In the next lemma we shall show that for normal spaces the converse is also true.

Lemma 6.1. *Every P-dense subset A of a normal space B is dense.*

Proof. Let $x \in B$ and let U be an open neighbourhood of x in B. Since B is regular, there is an open neighbourhood V of x in B such that the closure $\overline{V}$ of V is contained in U. Let $\sigma = \{U, B \setminus \overline{V}\}$. This is a finite open covering of B. Since the space B is normal, σ is a numerable covering of B [1]. By assumption, there is a member S of σ such that $x \in S$ and $S \cap A \neq \emptyset$. But, the first requirement on S can be fulfilled only for $S = U$. Hence, $U \cap A \neq \emptyset$ so that A is indeed dense in B. $\lozenge$

Let $\mathcal{G}$ be a class of relations. Recall that a space E is *$\mathcal{G}$-contractible* provided for every $\delta \in \mathrm{Cov}(E)$ there is a $\delta\mathcal{G}$-homotopy $h : E \times I \to E$ such that h_0 is the identity map on E and h_1 is a constant map of E into a point of E. We use *contractible* for $\mathcal{M}$-contractible. One can show (see [cer-tv]) that a space has trivial shape if and only if it is either $\mathcal{S}$-contractible or $\mathcal{R}$-contractible.

Theorem 6.1. *If $\mathcal{X}$ is a class of $\mathcal{S}$-contractible spaces, then every approximate $(\mathcal{X}, \sigma)$-fibration $p : E \to B$ with a P-dense image is an approximate $(\mathcal{X}, \sigma)$-bundle.*

Proof. Let an $\alpha \in \mathrm{Cov}(B)$ and a $\delta \in \mathrm{Cov}(E)$ be given. Select a $\beta \in \mathrm{Cov}(B)$ and an $\varepsilon \in \mathrm{Cov}(E)$ using the assumption that p is an approximate $(\mathcal{X}, \sigma)$-fibration.

Consider a space X in $\mathcal{X}$ and a β-function $h : X \to B$. Pick a $\xi \in \mathrm{Cov}(X)$ such that h is a (ξ, β)-function. Since X is $\mathcal{S}$-contractible, there is a $\xi\mathcal{S}$-homotopy $m : X \times I \to X$ such that m_0 is a constant map of X into a point x of X and m_1 is the identity map on X. The composition $H = h \circ m$ is a $\beta\mathcal{S}$-homotopy such that H_0 is the constant map of X into the point $b = h(x)$ and $H_1 = h$.

Since the image $p(E)$ is P-dense in B, there is a member A of β such that $b \in A$ and $A \cap p(E) \neq \emptyset$. Choose an $e \in E$ with $p(e) \in A$.

Let $F : X \to E$ be a constant map into the point e. Then $p \circ F \overset{\beta}{=} H_0$. By assumption, there is a $\delta\mathcal{S}$-homotopy $K : X \times I \to E$ with $p \circ K \overset{\alpha}{=} H$. The relation $k = K_1$ satisfies $p \circ k = p \circ K_1 \overset{\alpha}{=} H_1 = h$. Hence, $p \circ k \overset{\alpha}{=} h$. ◊

In analogy with approximate $(\mathcal{X}, \tau)$-fibrations the notion of an approximate $(\mathcal{X}_p, \tau)$-bundle has its C, D, and CD versions. The following theorem is similar to Th. (6.2) in [4]. It can be proved with same techniques.

Recall that a subset A of a space X is *P-embedded* provided for every $\alpha \in \mathrm{Cov}(A)$ there is a $\beta \in \mathrm{Cov}(X)$ such that the restriction $\beta|_A$ of β to A refines α (see [1]).

Theorem 6.2. *Let ϑ be either $(\mathcal{R}, \mathcal{R}, \mathcal{R})$ or $(\mathcal{S}, \mathcal{S}, \mathcal{S})$. Let $\mathcal{X}_p$ be a class of pairs (X, A), where X is a space and A is a P-embedded subset of X. For a map $p : E \to B$ the four notions of approximate $(\mathcal{X}_p, \vartheta)$-bundle coincide.*

The situation with stronger forms of approximate $(\mathcal{X}_p, \chi)$-fibrations is slightly more complicated. We shall need later the following CD version and the next theorem which is just a part of a more elaborate result resembling the previous theorem.

Let $\omega \coloneqq (\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{K})$ and let $\mathcal{X}_p$ be a class of topological pairs. A map $p : E \to B$ is an *approximate $(\mathcal{X}_p, \omega)CD$-fibration* provided for every $\delta \in \mathrm{Cov}(E)$ and every $\alpha \in \mathrm{Cov}(B)$ there is a $\beta \in \mathrm{Cov}(B)$ and an $\varepsilon \in \mathrm{Cov}(E)$ such that for every pair (X, A) in $\mathcal{X}_p$, every $\varepsilon\mathcal{F}$-relation $F : X \to E$, every $\varepsilon\mathcal{G}$-homotopy $G : A \times I \to E$ with $G_0 = F|_A$, and every $\beta\mathcal{H}$-homotopy $H : X \times I \to B$ with $p \circ F = H_0$ and $p \circ G = H|_{A \times I}$, there is a $\delta\mathcal{K}$-homotopy $K : X \times I \to E$ such that $K_0 = F$, $K|_{A \times I} = G$, and $p \circ K \overset{\alpha}{=} H$.

Theorem 6.3. *Let ω be either $(\mathcal{R}, \mathcal{R}, \mathcal{R}, \mathcal{R})$ or $(\mathcal{S}, \mathcal{S}, \mathcal{S}, \mathcal{S})$. Let $\mathcal{X}_p$ be a class of pairs (X, A), where X is a space and A is a P-embedded subset of X. A map $p : E \to B$ is an approximate $(\mathcal{X}_p, \omega)$-fibration if and only if it is an approximate $(\mathcal{X}_p, \omega)$ CD-fibration.*

For a simplicial complex X and each $m \geq 0$, let X^m denote the m-dimensional skeleton of X.

Lemma 6.2. *Let X be a finite simplicial complex and let A be a subcomplex of X. Let Y be a space and let ξ be a numerable covering of Y. Suppose that $h : X^{m-1} \to Y$ is a ξ-relation, $g : A \times I \to Y$ is a $\xi\mathcal{R}$-homotopy with $g(x, t) = h(x)$ for every x in A^{m-1} and every t in I, and for every m-simplex σ of X not in A there is a $\xi\mathcal{R}$-homotopy*

$g^\sigma : \sigma \times I \to Y$ such that $g^\sigma(x, t) = h(x)$ for every x in the boundary $\partial\sigma$ of σ and every t in I. Then the relation $k : (A \cup X^m) \times I \to Y$ obtained by pasting g on $A \times I$ and g^σ's on $\sigma \times I$ for each m-simplex σ of X not in A is a $st(\xi)$-relation.

Proof. Pick an $\eta > 0$ such that g on $A \times I$ and g^σ on $\sigma \times I$ for each m-simplex σ of X not in A takes η-close points to a same member of ξ. Let T be a barycentric subdivision of $K_m = A \cup X^m$ with the property that each simplex of T has diameter less than η. Let $\star^T$ be the open covering of K_m by open stars of vertices of T. Let $\pi > 0$ be the Lebesgue number for the open covering $\star^T \times \omega$, where ω is an open covering of I with sets of diameter less than η.

Now, if the points (x, t), $(y, s) \in K_m \times I$ are π-close, then there is a vertex v of T and a $U \in \omega$ such that both x and y lie in the open star $\star_v^T$ of v in T and both t and s lie in U. Hence, there are m-simplexes σ and τ of T with v as a joint vertex such that $x \in \sigma$ and $y \in \tau$. Since diameters of σ and τ are less than η, there are members M_x, M_y, and M_0 of ξ such that $k(x, t)$, $k(v, t) \in M_x$, $k(y, s)$, $k(v, s) \in M_y$, and $k(v, t)$, $k(v, s) \in M_0$. Hence, $k(x, t)$, $k(y, s) \in st(M_0, \xi) \in st(\xi)$. ◊

Let $\mathcal{Q}$ be the class of all finite polyhedra and let $\mathcal{Q}_p$ be the class of all finite polyhedral pairs. For an integer $n \geqslant 0$, let $\mathcal{Q}^n$ denote all members of $\mathcal{Q}$ with dimension at most n while $\mathcal{Q}_p^n$ denotes all members (X, A) of $\mathcal{Q}_p$ with $dim(X \setminus A) \leq n$. Also, let $sdr\,\mathcal{Q}_p^n$ denote all members (X, A) of $\mathcal{Q}_p^n$ such that A is a strong deformation retract of X. Recall that Δ^n denotes the n-simplex and S^{n-1} the boundary $(n-1)$-sphere.

Using Lemma (6.2) and an inductive argument from Chapter III of Hu's book [10] one can prove the following theorem.

Theorem 6.4. *Let ϑ denote either $(\mathcal{R}, \mathcal{R}, \mathcal{R})$ or $(\mathcal{S}, \mathcal{S}, \mathcal{S})$ and let ω denote either $(\mathcal{R}, \mathcal{R}, \mathcal{R}, \mathcal{R})$ or $(\mathcal{S}, \mathcal{S}, \mathcal{S}, \mathcal{S})$, respectively. For an arbitrary map $p : E \to B$ and an integer $n \geqslant 0$, let us make the following statements.*

- (i)$_n$ *The map p is an approximate $(\mathcal{Q}^n, \vartheta)$-fibration.*
- (ii)$_n$ *For each $0 \leq m \leq n$, the map p is an approximate $(\{\Delta^m\}, \vartheta)$-fibration.*
- (iii)$_n$ *For each $0 \leq m \leq n$, the map p is an approximate $(\{(\Delta^m, S^{m-1})\}, \omega)$-fibration.*
- (iv)$_n$ *The map p is an approximate $(\mathcal{Q}_p^n, \omega)$-fibration.*
- (v)$_n$ *The map p is an approximate $(sdr\,\mathcal{Q}_p^n, \vartheta)$-bundle.*

Then $(i)_n \Rightarrow (ii)_n \Rightarrow (iii)_n \Rightarrow (iv)_n \Rightarrow (v)_n$ and $(v)_{n+1} \Rightarrow (i)_n$.

7. Approximate fibrations with dense images

It is well-known that a fibration into a pathwise connected space must be onto. This section gives an analogue of this statement for approximate fibrations.

Let $\mathcal{D}$ be a class of relations. Let E be a space and let X and Y be subsets of E. A $\mathcal{D}$-relation $\lambda : I \to E$ will be called a *$\mathcal{D}$-path (in E)*. If $X \cap \lambda(0) \neq \emptyset$ and $Y \cap \lambda(1) \neq \emptyset$, then we say that λ is a *$\mathcal{D}$-path from X to Y* or that it *joins* X and Y. When $X = \{x\}$ and $Y = \{y\}$ for $x, y \in E$ and a $\mathcal{D}$-path λ joins X and Y, then we say that λ joins x and y or that it is a $\mathcal{D}$-path from x to y. For $\mathcal{M}$-path, $\mathcal{S}$-path, and $\mathcal{R}$-path we use shorter names *path*, *track*, and *trail*, respectively.

A space B is *$\mathcal{D}$-pathwise connected* provided for every pair $x, y \in \in B$ and every $\sigma \in \mathrm{Cov}(B)$ there is a $\sigma\mathcal{D}$-path $H : I \to B$ joining x and y.

The following theorem is a shape theory version of the fact that a fibration into a pathwise connected space is a surjection.

Let $\mathcal{C}$ denote the class of all constant single-valued functions. Let $\mathcal{J}$ denote a class consisting only of a single element space $\{*\}$.

Theorem 7.1. *Let $\mathcal{D}$ be a class of relations. Let $\tau = (\mathcal{C}, \mathcal{D}, \mathcal{R})$. If $p : E \to B$ is an approximate $(\mathcal{J}, \tau)$-fibration and B is $\mathcal{D}$-pathwise connected, then $p(E)$ is P-dense in B.*

Proof. Let a point $x \in B$ and a numerable covering α of B be given. Let $\delta = p^{-1}(\alpha)$. Let $X = \{*\}$. Since $p : E \to B$ is an approximate $(\mathcal{J}, \tau)$-fibration, there is a $\beta \in \mathrm{Cov}(B)$ with the property that for every map $g : X \to E$, and every $\beta\mathcal{D}$-relation $H : X \times I \to B$ with $H_0 \stackrel{\beta}{=} p \circ g$ there is a δ-relation $K : X \times I \to E$ such that $g \stackrel{\delta}{=} K_0$ and $H \stackrel{\alpha}{=} p \circ K$.

Let $z \in E$ and $y = p(z)$. Since B is $\mathcal{D}$-pathwise connected, there is a $\beta\mathcal{D}$-relation $M : I \to B$ such that $y \in M(0)$ and $x \in M(1)$. Define $g : X \to E$ and $H : X \times I \to B$ by $g(*) = z$ and $H(*, t) = M(t)$ for every $t \in I$. Clearly, g is a constant map from a member of the class $\mathcal{X}$ into E, H is a $\beta\mathcal{D}$-relation, and $H_0 \stackrel{\beta}{=} p \circ g$.

By assumption, there is a δ-relation $K : X \times I \to E$ such that $g \stackrel{\delta}{=} K_0$ and $H \stackrel{\alpha}{=} p \circ K$. In particular, $H_1 \stackrel{\alpha}{=} p \circ K_1$. But, $x \in H_1(X)$ and $p \circ K_1(X)$ is a subset of $p(E)$. The relation $H_1 \stackrel{\alpha}{=} p \circ K_1$ implies that some member A of α must contain both $H_1(*)$ and $p \circ K_1(*)$. Hence, A contains x and it intersect the set $p(E)$. In other words, $p(E)$ is P-dense in B. $\Diamond$

Corollary 7.1. *Let B be a trackwise connected normal space. If $p:$ $: E \to B$ is an approximate fibration, then $p(E)$ is dense in B.*

8. Approximate fibrations and pathwise connectedness

The main goal in this section is to prove for approximate fibrations an analogue for the following property of fibrations. If a fibration has a pathwise connected base and some fiber is pathwise connected, then its total space is also pathwise connected (see [17,p. 104]).

A map $p : E \to B$ is *normal* provided for every $\alpha \in \mathrm{Cov}(E)$ there is a $\beta \in \mathrm{Cov}(B)$ such that $p^{-1}(st(b, \beta)) \subset st(p^{-1}(b), \alpha)$ for every $b \in$ $\in B$.

Let $f : E \to B$ be a map and let A be a subset of B. We shall say that f is *normal at A* provided for every $\delta \in \mathrm{Cov}(E)$ there is an $\alpha \in \mathrm{Cov}(B)$ such that

$$f^{-1}(st(A, \alpha)) \subset st(f^{-1}(A), \delta).$$

On the other hand, f is *strongly normal at A* provided for every neighbourhood N of $f^{-1}(A)$ in E there is an $\alpha \in \mathrm{Cov}(B)$ such that $f^{-1}(st(A, \alpha)) \subset N$.

Observe that a normal map $f : E \to B$ is normal at each subset of B.

Theorem 8.1. *Let $p : E \to B$ be an approximate $(\mathcal{J}, \sigma)$-fibration. If the base B is trackwise connected and there is a point $b \in p(E)$ such that p is normal at $\{b\}$ and the fiber $F_b = p^{-1}(b)$ is trackwise connected, then the total space E is also trackwise connected.*

Proof. Let $d \in F_b$. It suffices to show that for every $\xi \in \mathrm{Cov}(E)$ and every $e \in E$ there is a ξ-track $\omega : I \to E$ such that $\omega(0) = d$ and $\omega(1) = e$.

Let an $e \in E$ and a $\xi \in \mathrm{Cov}(E)$ be given. Let $\delta \in \xi^\star$. Since p is normal at $\{b\}$, there is an $\alpha \in \mathrm{Cov}(B)$ such that $p^{-1}(st(b, \alpha)) \subset st(F_b, \delta)$. Let $X = \{*\}$. We now utilize the assumption that p is an approximate $(\mathcal{J}, \sigma)$-fibration to select a $\beta \in \mathrm{Cov}(B)$ such that for every function $f : X \to E$ and every $\beta\mathcal{S}$-homotopy $h : X \times I \to B$ with $p \circ f = h_0$, there is a $\delta\mathcal{S}$-homotopy $k : X \times I \to E$ with $k_0 = f$ and $p \circ k \overset{\alpha}{=} h$.

Let $c = p(e)$. Since the space B is trackwise connected, there is a $\beta\mathcal{S}$-homotopy $h : X \times I \to B$ such that $h_0(*) = c$ and $h_1(*) = b$. Define a function $f : X \to E$ by $f(*) = e$. Observe that $p \circ f = h_0$.

By assumption, there is a $\delta\mathcal{S}$-homotopy k as above. Since points $h_1(*)$ and $(p \circ k_1)(*)$ lie in the same member of the covering α, a point $k_1(*)$ lies in the set $st(F_b, \delta)$. Pick a member D of δ such that $k_1(*) \in D$ and $D \cap F_b \neq \emptyset$. Let $a \in D \cap F_b$. Since F_b is trackwise connected, there is a $(\delta|F_b)$-track $\mu : X \times I \to F_b$ such that $\mu_0(*) = d$ and $\mu_1(*) = a$. Define a function $\omega : I \to E$ by the rule

$$\omega(t) = \begin{cases} \mu(*, 2t), & 0 \leq t \leq \frac{1}{2}, \\ k(*, 2-2t), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then ω is a ξ-track joining d and e. $\Diamond$

In order to state a version of the above theorem for the approximate $(\mathcal{J}, \mu)CD$-fibrations, where μ denotes the triple $(\mathcal{M}, \mathcal{M}, \mathcal{M})$, we must assume that some fiber has the following property.

A subset F of a space E is *neighbourhood pathwise connected in* E provided there is a neighbourhood N of F in E such that for every $x \in N$ there is a path $\lambda : I \to E$ with $\lambda(0) = x$ and $\lambda(1) \in F$.

Observe that in a (locally) pathwise connected space E each subset is neighbourhood pathwise connected in E.

Theorem 8.2. *Let $p : E \to B$ be an approximate $(\mathcal{J}, \mu)CD$-fibration. If the base B is pathwise connected and there is a point $b \in p(E)$ such that p is strongly normal at $\{b\}$ and the fiber $F_b = p^{-1}(b)$ is neighbourhood pathwise connected in E, then the total space E is also pathwise connected.*

9. Compositions of approximate fibrations

In this section we shall see that approximate fibrations behave well with respect to compositions in close analogy with fibrations. The first result explores when will the composition of two approximate fibrations be an approximate fibration, while the two theorems following it will provide partial converses.

Theorem 9.1. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be classes of relations. Let $\lambda = (\mathcal{A}, \mathcal{E}, \mathcal{C})$, $\omega = (\mathcal{A}, \mathcal{B}, \mathcal{C})$, and $\nu = (\mathcal{D}, \mathcal{B}, \mathcal{E})$. Let $\mathcal{X}$ be a class of topological spaces. If a map $f : S \to T$ is an approximate $(\mathcal{X}, \lambda)$-fibration, a map $g : T \to U$ is an approximate $(\mathcal{X}, \nu)$-fibration, and the class $f \circ \mathcal{A}$ is contained in the class $\mathcal{D}$, then the composition $g \circ f$ is an approximate $(\mathcal{X}, \omega)$-fibration.*

Proof. Let numerable coverings δ of S and α of U be given. Let $\xi \in \alpha^\star$. Let $\pi = g^{-1}(\xi)$. Notice that π is a numerable covering of the space T.

Since f is an approximate $(\mathcal{X}, \lambda)$-fibration, there is a $\varrho \in \mathrm{Cov}(T)$ and an $\eta \in \mathrm{Cov}(S)$ such that for every space $X \in \mathcal{X}$, every $\eta\mathcal{A}$-relation $A : X \to S$, and every $\varrho\mathcal{E}$-homotopy $E : X \times I \to T$ with E_0 and $f \circ A$ being ϱ-close, there is a $\delta\mathcal{C}$-homotopy $C : X \times I \to S$ with $C_0 \overset{\delta}{=} A$ and $E \overset{\pi}{=} f \circ C$.

Now we utilize the assumption that g is an approximate $(\mathcal{X}, \nu)$-fibration to select numerable coverings β of U and σ of T with the property that for every space $X \in \mathcal{X}$, every $\sigma\mathcal{D}$-relation $D : X \to T$, and every $\beta\mathcal{B}$-homotopy $B : X \times I \to U$ with B_0 and $g \circ D$ being β-close, there is a $\xi\mathcal{E}$-homotopy $E : X \times I \to T$ with $E_0 \overset{\xi}{=} D$ and $B \overset{\xi}{=} g \circ E$. Finally, let $\varepsilon \in \mathrm{Cov}(S)$ be a common refinement of η and $f^{-1}(\sigma)$. Then β and ε are numerable coverings that we wanted. ◊

Corollary 9.1. *Let $\mathcal{X}$ be a class of spaces. The composition of approximate $(\mathcal{X}, \varrho)$-fibrations is an approximate $(\mathcal{X}, \varrho)$-fibration.*

The question which we address now can formulated as follows. If the composition $g \circ f$ of maps $f : S \to T$ and $g : T \to U$ is an approximate fibration, under what assumptions on the map f can we conclude that the map g is an approximate fibration? The answer provides the following class of approximate dominations.

Let $\mathcal{G}$ be a class of relations. A map $f : S \to T$ is an *approximate $\mathcal{G}$-domination* provided for every $\varepsilon \in \mathrm{Cov}(S)$ and every $\beta \in \mathrm{Cov}(T)$ there is an $\varepsilon\mathcal{G}$-relation $G : T \to S$ such that $f \circ G \overset{\beta}{=} \mathrm{id}_T$.

Observe that approximate $\mathcal{M}$-dominations between compact metric spaces agree with G. Kozlowski's approximately right invertible maps [9].

Theorem 9.2. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{G}$ be classes of relations. Let $\lambda = (\mathcal{A}, \mathcal{B}, \mathcal{C})$ and $\nu = (\mathcal{D}, \mathcal{B}, \mathcal{E})$. Let $\mathcal{X}$ be a class of spaces. If the composition $g \circ f$ of an approximate $\mathcal{G}$-domination $f : S \to T$ and a map $g : T \to U$ is an approximate $(\mathcal{X}, \lambda)$-fibration, and the classes $\mathcal{G} \circ \mathcal{D}$ and $f \circ \mathcal{C}$ are contained in the classes $\mathcal{A}$ and $\mathcal{E}$, then the map g is an approximate $(\mathcal{X}, \nu)$-fibration.*

Proof. Let an $\alpha \in \mathrm{Cov}(U)$ and a $\delta \in \mathrm{Cov}(T)$ be given. Let $\pi \in \delta^*$ and $\eta = f^{-1}(\pi)$. Since the composition $g \circ f$ is an approximate $(\mathcal{X}, \lambda)$-fibration, there is a $\xi \in \mathrm{Cov}(U)$ and a $\zeta \in \mathrm{Cov}(S)$ such that for every space X in $\mathcal{X}$, every $\zeta\mathcal{A}$-relation $a : X \to S$, and every $\xi\mathcal{B}$-homotopy $b : X \times I \to U$ with $(g \circ f) \circ a \overset{\xi}{=} b_0$, there is an $\eta\mathcal{C}$-homotopy $c : X \times I \to S$ with $c_0 \overset{\eta}{=} a$ and $(g \circ f) \circ c \overset{\alpha}{=} b$. Let $\beta \in \xi^*$. Let $\varrho \in$

$\in \pi^\star \cap g^{-1}(\beta)^\star$. Let $\omega \in \mathrm{Cov}(S)$ be a common refinement of ζ and $f^{-1}(\varrho)$. Next, we utilize the fact that f is an approximate $\mathcal{G}$-domination to select an $\omega\mathcal{G}$-relation $h : T \to S$ such that $f \circ h \overset{\varrho}{=} \mathrm{id}_T$. Pick a numerable covering $\varepsilon \in \mathrm{Cov}(T)$ such that h is an (ε, ξ)-relation and $f \circ h \overset{\varepsilon, st(\varrho)}{=} \mathrm{id}_T$. The numerable coverings β and ε are the ones we were looking for. ◊

Corollary 9.2. *Let $\mathcal{X}$ be a class of topological spaces. If the composition $g \circ f$ of an approximate $\mathcal{R}$-domination $f : S \to T$ and a map $g : T \to$ $\to U$ is an approximate $(\mathcal{X}, \varrho)$-fibration, then g is also an approximate $(\mathcal{X}, \varrho)$-fibration.*

An obvious dual problem is to find conditions on a map $g : T \to$ $\to U$ which are sufficient to guarantee that a map $f : S \to T$ is an approximate fibration whenever the composition $g \circ f$ is an approximate fibration. The solution relies on the following class of approximate injections.

Let $\mathcal{G}$ be a class of relations and let $\mathcal{X}$ be a class of spaces. A map $g : T \to U$ is an *approximate $(\mathcal{X}, \mathcal{G})$-injection* provided for every $\delta \in \mathrm{Cov}(T)$ there is an $\alpha \in \mathrm{Cov}(U)$ and an $\varepsilon \in \mathrm{Cov}(T)$ such that for every space X in $\mathcal{X}$ and every pair $h, k : X \to T$ of $\varepsilon\mathcal{G}$-relations the relation $g \circ h \overset{\alpha}{=} g \circ k$ implies the relation $h \overset{\delta}{=} k$.

Theorem 9.3. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{E}$ be classes of relations. Let $\lambda =$ $= (\mathcal{A}, \mathcal{E}, \mathcal{C})$ and $\nu = (\mathcal{A}, \mathcal{B}, \mathcal{C})$. Let $\mathcal{X}$ be a class of spaces. If the composition $g \circ f$ of an approximate $(\mathcal{X}, \mathcal{E})$-injection $g : T \to U$ and a map $f : S \to T$ is an approximate $(\mathcal{X}, \nu)$-fibration, and the classes $f \circ \mathcal{C}$ and $g \circ \mathcal{E}$ are contained in the classes $\mathcal{E}$ and $\mathcal{B}$, then the map f is an approximate $(\mathcal{X}, \lambda)$-fibration.*

Proof. Let an $\alpha \in \mathrm{Cov}(T)$ and a $\delta \in \mathrm{Cov}(S)$ be given. We first utilize the fact that the map g is an approximate $(\mathcal{X}, \mathcal{E})$-injection to select a $\xi \in \mathrm{Cov}(U)$ and a $\zeta \in \mathrm{Cov}(T)$ such that for every space X in $\mathcal{X}$ and every pair $h, k : X \to T$ of $\zeta\mathcal{E}$-relations the relation $g \circ h \overset{\xi}{=} g \circ k$ implies the relation $h \overset{\alpha}{=} k$. Let $\omega \in \mathrm{Cov}(S)$ be a common refinement of δ and $f^{-1}(\zeta)$. Since the composition $g \circ f$ is an approximate $(\mathcal{X}, \nu)$-fibration, there is a $\pi \in \mathrm{Cov}(U)$ and an $\varepsilon \in \mathrm{Cov}(S)$ such that for every space X in $\mathcal{X}$, every $\varepsilon\mathcal{A}$-relation $a : X \to S$, and every $\pi\mathcal{B}$-homotopy $b : X \times I \to U$ with $(g \circ f) \circ a \overset{\pi}{=} b_0$, there is an $\omega\mathcal{C}$-homotopy $c : X \times$ $\times I \to S$ with $c_0 \overset{\omega}{=} a$ and $(g \circ f) \circ c \overset{\xi}{=} b$. Let $\beta \in \mathrm{Cov}(T)$ be a common refinement of ζ and $g^{-1}(\pi)$. The numerable coverings β and ε are the ones we need. ◊

Corollary 9.3. *Let $\mathcal{X}$ be a class of topological spaces. If the composition $g \circ f$ of an approximate $(\mathcal{X}, \mathcal{R})$-injection $g : T \to U$ and a map $f : S \to$ $\to T$ is an approximate $(\mathcal{X}, \varrho)$-fibration, then f is also an approximate $(\mathcal{X}, \varrho)$-fibration.*

10. Approximate fibrations and products

In this section we shall see that nice behaviour of fibrations with respect to products is also shared by approximate fibrations.

For spaces X and Y, let $p^X_{X\times Y}$ and $p^Y_{X\times Y}$ denote projections of the product $X \times Y$ onto X and Y, respectively. Also, if y is a point of Y, let $j^{X\times Y}_{X,y} : X \to X \times Y$ be the embedding of X into $X \times Y$ defined by $j^{X\times Y}_{X,y}(x) = (x, y)$ for every $x \in X$.

The projection $p^X_{X\times Y}$ is a basic example of a fibration. In order that this map should be an approximate fibration we must assume that numerable coverings of $X \times Y$ can be refined by products of numerable coverings of X and Y. This is made precise in the following definition.

Spaces X and Y are called *entwined* provided for every numerable covering σ of the product $X \times Y$ there are numerable coverings α of X and β of Y such that the numerable covering $\alpha \times \beta$ of $X \times Y$ formed by all products $A \times B$ with $A \in \alpha$ and $B \in \beta$ refines the covering σ.

Theorem 10.1. *The projection $p^X_{X\times Y} : X \times Y \to X$ is an approximate fibration when the spaces X and Y are entwined.*

Proof. Let an $\alpha \in \mathrm{Cov}(X)$ and a $\delta \in \mathrm{Cov}(X \times Y)$ be given. Since spaces X and Y are entwined, there is a $\beta \in \mathrm{Cov}(X)$ and a $\varrho \in \mathrm{Cov}(Y)$ such that $\beta \times \varrho$ refines δ. Let $\varepsilon = (p^Y_{X\times Y})^{-1}(\varrho)$.

Consider a space S, an ε-function $f : S \to X \times Y$, and a βS-homotopy $h : S \times I \to X$ such that $p^X_{X\times Y} \circ f \overset{\beta}{=} h_0$. Define a function $k : S \times I \to X \times Y$ by

$$k(s, t) = (h(s, t), p^Y_{X\times Y}(f(s))),$$

for every $s \in S$ and every $t \in I$. Then k is a δS-homotopy such that $k_0 \overset{\delta}{=} f$ and $p^X_{X\times Y} \circ k = h$. ◊

We shall also need the following construction in order to state a result which describes conditions under which will the product of two approximate fibrations be an approximate fibration.

Let $F : X \to Y$ and $G : X \to Z$ be relations defined on the same space. Then $F \otimes G$ denotes the relation from X into the product $Y \times Z$ which associates to a point x of X the product $F(x) \times G(x)$. This relation will be called the *reduced product* of F and G. It differs from the product $F \times G$ which has the product $X \times X$ as the domain.

For classes $\mathcal{F}$ and $\mathcal{G}$ of relations, $\mathcal{F} \otimes \mathcal{G}$ is the class of all reduced products $F \otimes G$, where $F \in \mathcal{F}$ and $G \in \mathcal{G}$.

We shall state the product theorem for approximate fibrations and its converse only for the product of two maps. It is obvious that similar results can be established for any number (finite or infinite) of factors.

Theorem 10.2. *Let $\lambda = (\mathcal{A}, \mathcal{B}, \mathcal{C})$, $\omega = (\mathcal{D}, \mathcal{E}, \mathcal{F})$, and $\nu = (\mathcal{G}, \mathcal{H}, \mathcal{K})$ be triples of classes of relations. Let $\mathcal{X}$ be a class of spaces. Let $f : S \to T$ be an approximate $(\mathcal{X}, \lambda)$-fibration and let $g : U \to V$ be an approximate $(\mathcal{X}, \omega)$-fibration. If the classes $p^S_{S\times U} \circ \mathcal{G}$, $p^U_{S\times U} \circ \mathcal{G}$, $p^T_{T\times V} \circ \mathcal{H}$, $p^V_{T\times V} \circ \mathcal{H}$, and $\mathcal{C} \otimes \mathcal{F}$ are contained in the classes $\mathcal{A}$, $\mathcal{D}$, $\mathcal{B}$, $\mathcal{E}$, and $\mathcal{K}$, respectively, and spaces S and U and T and V are entwined, then the product map $f \times g$ is an approximate $(\mathcal{X}, \nu)$-fibration.*

Proof. Let numerable coverings α of $T \times V$ and δ of $S \times U$ be given. Since T and V are entwined, there is an $\alpha_1 \in \mathrm{Cov}(T)$ and an $\alpha_2 \in \mathrm{Cov}(U)$ such that the product $\alpha_1 \times \alpha_2$ refines the covering α. Similarly, there are numerable coverings δ_1 of S and δ_2 of U such that $\delta_1 \times \delta_2$ refines δ. Select $\beta_1 \in \mathrm{Cov}(T)$ and $\varepsilon_1 \in \mathrm{Cov}(S)$ with respect to coverings α_1 and δ_1 using the assumption that the map f is an approximate $(\mathcal{X}, \lambda)$-fibration. We now utilize the assumption that the map g is an approximate $(\mathcal{X}, \omega)$-fibration to choose $\beta_2 \in \mathrm{Cov}(V)$ and $\varepsilon_2 \in \mathrm{Cov}(U)$ with respect to coverings α_2 and δ_2. Let $\beta = \beta_1 \times \beta_2$ and $\varepsilon = \varepsilon_1 \times \varepsilon_2$. Then β and ε are the required numerable coverings. $\Diamond$

Corollary 10.1. *Let $\mathcal{X}$ be a class of spaces. The product of two approximate $(\mathcal{X}, \varrho)$-fibrations is an approximate $(\mathcal{X}, \varrho)$-fibration if their domains and codomains are entwined.*

The following result provides a converse to the product theorem for approximate fibrations.

Theorem 10.3. *Let $\lambda = (\mathcal{A}, \mathcal{B}, \mathcal{C})$ and $\nu = (\mathcal{G}, \mathcal{H}, \mathcal{K})$ be triples of classes of relations. Let $\mathcal{X}$ be a class of spaces. If the product map $f \times g$ of maps $f : S \to T$ and $g : U \to V$ is an approximate $(\mathcal{X}, \nu)$-fibration, the class $p^S_{S\times U} \circ \mathcal{K}$ is contained in the class $\mathcal{C}$, there is a point $u \in U$ such that the classes $j^{S\times U}_{S,\,u} \circ \mathcal{A}$ and $j^{T\times V}_{T,\,g(u)} \circ \mathcal{B}$ are contained in the classes $\mathcal{G}$ and $\mathcal{H}$, respectively, and spaces S and U and T and V are* *entwined, then $f : S \to T$ be an approximate $(\mathcal{X}, \lambda)$-fibration.*

Proof. Let an $\alpha \in \mathrm{Cov}(T)$ and a $\delta \in \mathrm{Cov}(S)$ be given. Let $\alpha* = \{A \times V \mid A \in \alpha\}$ and $\delta^* = \{D \times U \mid D \in \delta\}$. Observe that α^* and δ^* are numerable coverings. Since the product map $f \times g$ is an approximate $(\mathcal{X}, \nu)$-fibration, there are numerable coverings $\beta^* \in \mathrm{Cov}(T \times V)$ and $\varepsilon^* \in \mathrm{Cov}(S \times U)$ such that for every space X in $\mathcal{X}$, every $\varepsilon^*\mathcal{G}$-relation $e^* : X \to S \times U$, and every $\beta^*\mathcal{H}$-homotopy $h^* : X \times I \to T \times V$ with $(f \times g) \circ e^* \overset{\beta^*}{=} {h^*}_0$, there is a $\delta^*\mathcal{K}$-homotopy $k^* : X \times I \to S \times U$ with $e^* \overset{\delta^*}{=} {k^*}_0$ and $(f \times g) \circ k^* \overset{\alpha^*}{=} h^*$. Now we use the assumption that the spaces S and U and T and V are entwined to select numerable coverings $\beta \in \mathrm{Cov}(T)$, $\xi \in \mathrm{Cov}(V)$, $\varepsilon \in \mathrm{Cov}(S)$, and $\zeta \in \mathrm{Cov}(U)$ such that $\beta \times \xi$ refines β^* and $\varepsilon \times \zeta$ refines ε^*. Then β and ε are the required numerable coverings. $\lozenge$

Corollary 10.2. *Let $\mathcal{X}$ be a class of spaces. If the product of two maps with entwined domains and codomains is an approximate $(\mathcal{X}, \varrho)$-fibration, then both of these maps are approximate $(\mathcal{X}, \varrho)$-fibrations.*

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