Mathematica Pannonica
9/2 (1998), 195–204

THE SIZE OF AN ANNIHILATOR IN A FACTORIZATION

Keresztély **Corrádi**

Department of Computer Sciences, Eötvös University Budapest, H-1088 Budapest, Hungary

Sándor **Szabó**

Department of General Sciences and Mathematics, College of Health Sciences, P.O.Box 12, Manama, State of Bahrain

Received: December 1997

MSC 1991: 52 C 22

Keywords: Factorization of finite abelian groups, Hajós-Rédei theory.

Abstract: In 1987 K. Corrádi conjectured if a finite abelian group is a direct product of its subsets, then the annihilator of one of the factors must span the whole character group. The paper verifies this conjecture in three special cases.

In this paper we will use multiplicative notation for abelian groups. Let G be a finite abelian group. We denote the identity element of G by e. If B, $A_1, \dots, A_n$ are subsets of G such that each b in B is uniquely expressible in the form

$$b = a_1 \dots a_n, \qquad a_1 \in A_1, \dots, a_n \in A_n,$$

and each product $a_1 \dots a_n$, belongs to B, that is, if the product $A_1 \dots A_n$ is direct and is equal to B, then we say that B is factored by subsets $A_1, \dots, A_n$. The equation $B = A_1 \dots A_n$ is also said to be a *factorization* of B. If $e \in B \cap A_1 \cap \dots \cap A_n$, then the factorization $B = A_1 \dots A_n$ and the subsets $B, A_1, \dots, A_n$ are said to be *normed.*

This work was supported in part by the Hungarian Research Fund Grant number 7441.

There is an equivalent way describing factorizations of B. The product $A_1 \dots A_n$ is a factoring of B if $A_1 \dots A_n = B$ and $|A_1| \dots |A_n| = |B|$.

The subset A of G is called *cyclic* if it is of form $\{e, a, a^2, \dots, a^{r-1}\}$, where a is an element of $G \setminus \{e\}$ and r is a positive integer.

In 1941 G. Hajós [4] proved that if a finite abelian group is a direct product of cyclic subsets, then at least one of the factors must be a subgroup of the group.

By a *character* on a finite abelian group we mean a homomorhism of the group into the multiplicative group of the roots of unity of the field of complex numbers. The characters of a finite abelian group G form a group $\mathcal{G}$ under pointwise multiplication. In addition G and $\mathcal{G}$ are isomorphic. If A is a subset and χ is a character of G, then we will use the notation $\chi(A)$ to denote the sum

$$\sum_{a \in A} \chi(a).$$

The set of all characters for which $\chi(A) = 0$ we will call the *annihilator set* of A and will denote it by $\mathrm{Ann}(A)$.

In 1987 K. Corrádi [1] conjectured that in any factorization of a finite abelian group by its subsets the annihilator of one of the factors spans the whole character group.

In this paper we verify this conjecture in three special cases. In the first case the finite abelian group is a p-group generated by two elements. In the second case the number of the factors in the factorization is not greater than four. In the third case $p \geq n$, where p is the least prime factor of the order of G.

1. The Dirichlet correspondence

Let G be a finite abelian group and let $\mathcal{G}$ be its character group. The identity element of $\mathcal{G}$ is the character which takes the value 1 on each element of G. We call this character the *principal character* of G and we will denote it by ε. For a character χ of G we use the notation $\mathrm{Ker}\,\chi$ to denote the set of elements of G on which χ takes the value 1. $\mathrm{Ker}\,\chi$ is a subgroup of G. It is the largest subgroup of G on which the restriction of χ is principal.

To a subgroup H of G we assign the subgroup $\mathcal{H} = \{\chi : H \subset \mathrm{Ker}\,\chi\}$ of $\mathcal{G}$. The mapping $H \to \mathcal{H}$ is a bijection between the subgroups of G and $\mathcal{G}$. This map has the following properties.

If $H \to \mathcal{H}$, $K \to \mathcal{K}$, then $HK \to \mathcal{H} \cap \mathcal{K}$ and $H \cap K \to \mathcal{HK}$;

$$\{e\} \to \mathcal{G}; \qquad G \to \{e\}.$$

In short, this map is an antiisomorphism between the subgroup lattices of G and $\mathcal{G}$ and it is called the Dirichlet correspondence. To a subgroup $\mathcal{H}$ of $\mathcal{G}$ the inverse map assigns the subgroup

$$H = \bigcap_{\chi \in \mathcal{H}} \operatorname{Ker} \chi$$

of G.

Next we sketch the connection between Hajós' theorem and Corrádi's conjecture. In order to do this we need the following character test for factorization. (See [5])

The product of the subsets $A_1, \dots, A_n$ of a finite abelian group G is direct and gives G if and only if $|A_1| \dots |A_n| = |G|$ and $\operatorname{Ann}(A_1) \cup \cup \dots \cup \operatorname{Ann}(A_n) = \mathcal{G} \setminus \{e\}$.

Next consider a cyclic subset A_i of G. Let $A_i = \{e, a_i, a_i^2, \dots \dots, a_i^{r_i - 1}\}$. Clearly $\chi(A_i) = r_i$ if $\chi(a_i) = 1$ and

$$\chi(A_i) = \frac{1 - (\chi(a_i))^{r_i}}{1 - \chi(a_i)}$$

if $\chi(a_i) \neq 1$. Thus the annihilator of A_i is a difference of two subgroups of $\mathcal{G}$. Namely, $\operatorname{Ann}(A_i) = \mathcal{L}_i \setminus \mathcal{M}_i$, where $\mathcal{L}_i = \{\chi : (\chi(a_i))^{r_i} = 1\}$ and $\mathcal{M}_i = \{\chi : \chi(a_i) = 1\}$.

By Hajós' theorem there is an i, $1 \leq i \leq n$ such that A_i is a subgroup of G. This means that $a_i^{r_i} = e$. Now $(\chi(a_i))^{r_i} = 1$ for each character χ of G and so $\mathcal{L}_i = \mathcal{G}$. Since $a_i \neq e$, $\mathcal{M}_i \neq \mathcal{G}$ and so $|\mathcal{M}_i| \leq |\mathcal{G}|/2$. Hence $\langle \operatorname{Ann}(A_i) \rangle = \mathcal{G}$ as $|\mathcal{G} \setminus \mathcal{M}_i| \geq |\mathcal{G}|/2$. Thus Hajós' theorem implies Corrádi's conjecture in the special case when the factors are cyclic.

Conversely, assume that Corrádi's conjecture holds, that is, $\langle \operatorname{Ann}(A_i) \rangle = \mathcal{G}$ for some i, $1 \leq i \leq n$. From this it follows that $\mathcal{L}_i$ must be $\mathcal{G}$. This gives that $(\chi(a_i))^{r_i} = 1$ for each character χ of G. Therefore $a_i^{r_i} = e$ and so A_i is a subgroup of G. Thus Corrádi's conjecture implies Hajós' theorem.

2. Certain p-groups

In this section we verify the conjecture for finite abelian p-groups generated by two elements. For the proof we need two lemmas.

Lemma 1. *Let G be a finite abelian group and let $\mathcal{G}$ be its character group. If $a \in G$ and $\mathcal{H}$ is a subgroup of $\mathcal{G}$, then*

$$\sum_{\chi \in \mathcal{H}} \chi(a) = \begin{cases} |\mathcal{H}|, & \text{if } a \in K; \\ 0, & \text{if } a \notin K, \end{cases}$$

where

$$K = \bigcap_{\chi \in \mathcal{H}} \operatorname{Ker} \chi.$$

Proof. If $a \in K$, then $\chi(a) = 1$ for each $\chi \in \mathcal{H}$ and so

$$\sum_{\chi \in \mathcal{H}} \chi(a) = |\mathcal{H}|.$$

If $a \notin K$, then there is a $\chi' \in \mathcal{H}$ for which $\chi'(a) \neq 1$. Multiplying the elements of $\mathcal{H}$ by χ' permutes the elements of $\mathcal{H}$. Hence

$$\sum_{\chi \in \mathcal{H}} \chi(a) = \sum_{\chi \in \mathcal{H}} (\chi'\chi)(a) = \sum_{\chi \in \mathcal{H}} \chi'(a)\chi(a).$$

From which we get

$$0 = (1 - \chi'(a)) \sum_{\chi \in \mathcal{H}} \chi(a).$$

As $\chi'(a) \neq 1$ it follows that

$$\sum_{\chi \in \mathcal{H}} \chi(a) = 0. \qquad \diamond$$

The next result is a special case of a theorem of E. Wittmann [7]. For easier reference we state it as a lemma.

Lemma 2. *Let p be a prime and let $\rho_1, \ldots, \rho_n$ be roots of unity of p-power orders. Then $\rho_1 + \cdots + \rho_n = 0$ implies that $n \geq p$.*

We may turn now to the main result of this section.

Theorem 1. *Let p be a prime and let G be a finite abelian p-group generated by two elements. If $G = A_1 \ldots A_n$, is a factorization of G, then there is an i, $1 \leq i \leq n$ such that $\langle \operatorname{Ann}(A_i) \rangle = \mathcal{G}$, where $\mathcal{G}$ is the character group of G.*

Proof. First of all note that in proving the theorem we may restrict our attention to normed factorizations. Indeed, let $a_1 \in A_1, \ldots, a_n \in A_n$. Multiplying the factorization $G = A_1 \ldots A_n$ by $a = a_1^{-1} \ldots a_n^{-1}$ leads to the normed factorization $G = aG = (a_1^{-1}A_1) \ldots (a_n^{-1}A_n)$. In addition, it is clear that $\operatorname{Ann}(A_i) = \operatorname{Ann}(a_i^{-1}A_i)$.

In order to prove the theorem assume the contrary that $\langle \operatorname{Ann}(A_i) \rangle \neq \mathcal{G}$ for each i, $1 \leq i \leq n$. Further assume that n is minimal with this property. As $\langle \mathrm{Ann}(A_i)\rangle \neq \mathcal{G}$ for each i, $1 \leq i \leq n$ there is a maximal subgroup $\mathcal{M}_i$ of $\mathcal{G}$ such that $\mathrm{Ann}(A_i) \subset \mathcal{M}_i$. By the character test for factorization $\mathrm{Ann}(A_1) \cup \ldots \cup \mathrm{Ann}(A_n) = \mathcal{G} \setminus \{\varepsilon\}$ and consequently $\mathcal{M}_1 \cup \ldots \cup \mathcal{M}_n = \mathcal{G}$. The minimality of n in our counterexample implies that the subgroups $\mathcal{M}_1, \ldots, \mathcal{M}_n$ are distinct. To keep the notational difficulties in minimum we prove that $\mathcal{M}_{n-1} \neq \mathcal{M}_n$. From the factorization $G = A_1 \ldots A_{n-2}(A_{n-1}A_n)$ by the minimality of n it follows that $\langle \mathrm{Ann}(A_{n-1}A_n)\rangle = \mathcal{G}$. If we assume that $\mathcal{M}_{n-1} = \mathcal{M}_n$, then we get the contradiction

$$\mathrm{Ann}(A_{n-1}A_n) \subset \mathrm{Ann}(A_{n-1}) \cup \mathrm{Ann}(A_n) \subset \mathcal{M}_{n-1} \cup \mathcal{M}_n = \mathcal{M}_{n-1}.$$

Thus $\mathcal{M}_{n-1} \neq \mathcal{M}_n$ or in general $\mathcal{M}_i \neq \mathcal{M}_j$ if $i \neq j$, $1 \leq i, j \leq n$.

As G and $\mathcal{G}$ are isomorphic, $\mathcal{G}$ is a finite abelian p-group generated by two elements. Let $\Phi(\mathcal{G})$ be the Frattini subgroup of $\mathcal{G}$ and consider the factor group $\overline{\mathcal{G}} = \mathcal{G}/\Phi(\mathcal{G})$. Clearly, $\overline{\mathcal{G}}$ is an elementary abelian group of order p^2. Let $\overline{\mathcal{M}_i}$ denote the image of $\mathcal{M}_i$ in $\overline{\mathcal{G}}$. We know that $\overline{\mathcal{M}_1}, \ldots, \overline{\mathcal{M}_n}$ are distinct subgroups of order p and $\overline{\mathcal{G}} = \overline{\mathcal{M}_1} \cup \ldots \cup \overline{\mathcal{M}_n}$. This gives that $\overline{\mathcal{M}_1}, \ldots, \overline{\mathcal{M}_n}$ are all the subgroups of $\overline{\mathcal{G}}$ of order p and so $n = p + 1$.

We claim that $\mathcal{M}_i \cap \mathcal{M}_j = \Phi(\mathcal{G})$ for each $i \neq j$, $1 \leq i, j \leq n$. By definition $\Phi(\mathcal{G}) = \mathcal{M}_1 \cap \ldots \cap \mathcal{M}_n$. Hence $\Phi(\mathcal{G}) \subset \mathcal{M}_i \cap \mathcal{M}_j$ and so it is enough to establish that $|\mathcal{G} : \Phi(\mathcal{G})| \leq |\mathcal{G} : (\mathcal{M}_i \cap \mathcal{M}_j)|$. As $\mathcal{G}$ is a p-group generated by 2 elements, $|\mathcal{G} : \Phi(\mathcal{G})| = p^2$. As $\mathcal{M}_i$ is a maximal subgroup of $\mathcal{G}$, $|\mathcal{G} : \mathcal{M}_i| = p$. From $\mathcal{M}_i \neq \mathcal{M}_j$ it follows that $\mathcal{M}_i \cap \mathcal{M}_j \neq \mathcal{M}_i$ and so from $\Phi(\mathcal{G}) \subset \mathcal{M}_i \cap \mathcal{M}_j \neq \mathcal{M}_i$ it follows that $|\mathcal{G} : (\mathcal{M}_i \cap \mathcal{M}_j)| \geq p^2$.

Next we claim that $\mathcal{M}_i \backslash \Phi(\mathcal{G}) \subset \mathrm{Ann}(A_i)$ for each i, $1 \leq i \leq n$. Or equivalently, that from $\tau_i \in \mathcal{M}_i \backslash \Phi(\mathcal{G})$ it follows that $\tau_i \Phi(\mathcal{G}) \subset \mathrm{Ann}(A_i)$ for each i, $1 \leq i \leq n$. To prove this claim assume the contrary that there is a $\xi \in \tau_i \Phi(\mathcal{G})$ with $\xi \notin \mathrm{Ann}(A_i)$. Since $\tau_i \Phi(\mathcal{G}) \cap \Phi(\mathcal{G}) = \emptyset$, $\xi \neq \varepsilon$ and so from $\mathrm{Ann}(A_1) \cup \ldots \cup \mathrm{Ann}(A_n) = \mathcal{G} \setminus \{e\}$ it follows that $\xi \in \mathrm{Ann}(A_j)$ for some j, $j \neq i$. But $\mathrm{Ann}(A_j) \subset \mathcal{M}_j$ which leads to the contradiction $\xi \in \mathcal{M}_i \cap \mathcal{M}_j = \Phi(\mathcal{G})$.

Relying on these results we may argue in the following way. For each $\tau_i \in \mathcal{M}_i \setminus \Phi(\mathcal{G})$, $\tau_i \Phi(\mathcal{G}) \subset \mathrm{Ann}(A_i)$, that is, for each $\xi \in \tau_i \Phi(\mathcal{G})$, $\xi(A_i) = 0$ and so

$$0 = \sum_{\xi \in \tau_i \Phi(\mathcal{G})} \xi(A_i) = \sum_{\xi \in \tau_i \Phi(\mathcal{G})} \left(5g \sum_{a \in A_i} \xi(a)\right) = \sum_{a \in A_i} \left(\sum_{\xi \in \tau_i \Phi(\mathcal{G})} \xi(a)\right).$$

Each character ξ in $\tau_i \Phi(\mathcal{G})$ is uniquely expressible in the form $\xi = \tau_i \chi$, where $\chi \in \Phi(\mathcal{G})$. Hence

$$0 = \sum_{a \in A_i} \left(\sum_{\chi \in \Phi(\mathcal{G})} (\tau_i \chi)(a) \right) = \sum_{a \in A_i} \left(\sum_{\chi \in \Phi(\mathcal{G})} \tau_i(a) \chi(a) \right) =$$

$$= \sum_{a \in A_i} \tau_i(a) \left(\sum_{\chi \in \Phi(\mathcal{G})} \chi(a) \right).$$

Consider the Dirichlet correspondence between the subgroup lattices of G and $\mathcal{G}$. Let K be the image of $\Phi(\mathcal{G})$. By Lemma 1

$$\sum_{\chi \in \Phi(\mathcal{G})} \chi(a) = \begin{cases} |\Phi(\mathcal{G})|, & \text{if } a \in K \\ 0, & \text{if } a \notin K, \end{cases}$$

and hence

$$0 = \sum_{a \in K \cap A_i} \tau_i(a).$$

As $e \in K \cap A_i$, the summation is not empty. The terms are roots of unity of p-power orders. Thus by Lemma 2 $|K \cap A_i| \geq p$ for each i, $1 \leq i \leq n$.

Now we can draw two conclusions about $|K|$. On one hand the Dirichlet correspondence gives $|K| = |\mathcal{G} : \Phi(\mathcal{G})| = p^2$. On the other hand note that the product $(K \cap A_1) \dots (K \cap A_n)$ is direct and is part of K. From this and from $|K \cap A_i| \geq p$ it follows that $|K| \geq p^n$. Thus $p^2 = |K| \geq p^n = p^{p+1}$ which lands on the $2 \geq p + 1$ contradiction. $\diamondsuit$

3. The $n \leq 4$ special case and zero divisors

First we reformulate Corrádi's conjecture. Let G be a finite abelian group and let $\mathcal{G}$ be its character group. To a subset A of G we assign the subgroup

$$K = \bigcap_{\chi(A)=0} \operatorname{Ker} \chi$$

of G. Using the Dirichlet correspondence we can verify that if $\langle \operatorname{Ann}(A) \rangle \neq \mathcal{G}$, then $K \neq \{e\}$ and conversely if $K \neq \{e\}$, then $\langle \operatorname{Ann}(A) \rangle \neq$ $\neq G$. The new version of Corrádi's conjecture now reads as follows.

Let $G = A_1 \dots A_n$ be a factorization of the finite abelian group G and let $K_1, \dots, K_n$ be the subgroups assigned to the factors $A_1, \dots, A_n$ respectively. Then there is an i, $1 \leq i \leq n$ such that $K_i = \{e\}$.

We say that a subset A of a finite abelian group G is periodic if there is an element $a \in G$ such that $a \neq e$ and $aA = A$. The element a is called a period of A. The periods of A together with the identity element form a subgroup H of G. Moreover there is a subset B of G such that $A = BH$ is a factorization of A.

Lemma 3. *Let G be a finite abelian group and let $\mathcal{G}$ be its character group. If A is a periodic subset of G, then $\langle \mathrm{Ann}(A) \rangle = \mathcal{G}$.*

Proof. As A is periodic, it admits a factorization of the form $A = BH$, where B is a subset and H is a subgroup of G, where the nonidentity elements of H are the periods of A and so $H \neq \{e\}$. Let $\mathcal{M}$ be the subgroup of $\mathcal{G}$ containing each character χ of G which is principal on H. As $H \neq \{e\}$, $\mathcal{M} \neq \mathcal{G}$. By Th. 1 of [6] $\mathcal{G} \setminus \mathcal{M} \subset \mathrm{Ann}(A)$. Now from $|\mathcal{G} \setminus \mathcal{M}| \geq |\mathcal{G}|/2$ it follows that $\langle \mathrm{Ann}(A) \rangle = \mathcal{G}$. $\Diamond$

Lemma 4. *Let A, B, C be subsets of an abelian group with $e \in B$ and $e \in C$. If the product ABC is direct, then $AB \cap AC = A$.*

Proof. Let $d \in AB \cap AC$. Since $d \in AB$, there are $a \in A$ and $b \in B$ such that $d = ab$. Since $d \in AC$, there are $a' \in A$ and $c \in C$ such that $d = a'c$. Now

$$d = \underbrace{(\,a\,)}_{\in A}\underbrace{(\,b\,)}_{\in B}\underbrace{(\,e\,)}_{\in C} = \underbrace{(\,a'\,)}_{\in A}\underbrace{(\,e\,)}_{\in B}\underbrace{(\,c\,)}_{\in C}.$$

By the directness of the product ABC it follows that $a = a'$, $b = e$, $e = c$, that is, $d \in A$. Hence $A \supset AB \cap AC$. The containment $AB \cap \cap AC \supset A$ is a consequence of $e \in B$ and $e \in C$. $\Diamond$

After these preparations we may turn to the main results of this section.

Theorem 2. *Let $G = A_1 \ldots A_n$ be a factorization of the finite abelian group G and let $K_1, \ldots, K_n$ be the subgroups assigned to the the factors $A_1, \ldots, A_n$, respectively. If $n \leq 4$, then there is an i, $1 \leq i \leq n$ such that $K_i = \{e\}$.*

Proof. We assume in the contrary that $K_i \neq \{e\}$ for each i, $1 \leq i \leq \leq n$. From this assumption we will draw the conclusion that one of the factors $A_1, \ldots, A_n$, is periodic. By Lemma 3 this is a contradiction.

Case $n = 1$. Now $G = A_1$ and so A_1 is clearly periodic.

Case $n = 2$. Now $G = A_1A_2$ and $\chi(A_1) = 0$ for each character χ of G for which $\chi(K_2) = 0$. By Th. 1 of [6] this means that A_1 is periodic.

Case $n = 3$. Now $G = A_1A_2A_3$ and $\chi(A_1) = 0$ for each character χ of G for which $\chi(K_2) = 0$ and $\chi(K_3) = 0$. By Th. 2 of [6] there are subsets X_2, X_3 of G such that $A_1 = X_2K_2 \cup X_3K_3$, where the union is disjoint and the products X_2K_2 and X_3K_3 are direct. If $X_2 = \emptyset$ or $X_3 = \emptyset$, then A_1 is periodic. So we may assume that $X_2 \neq \emptyset$ and $X_3 \neq \emptyset$. Similarly $\chi(A_2) = 0$ for each character χ of G for which $\chi(K_1) = 0$ and $\chi(K_3) = 0$. Thus there are subsets Y_1,Y_3 of G such that $A_2 = Y_1K_1 \cup Y_3K_3$, where the union is disjoint and the products Y_1K_1 and Y_3K_3 are direct. If $Y_1 = \emptyset$ or $Y_3 = \emptyset$, then A_2 is periodic. So we may assume that $Y_1 \neq \emptyset$ and $Y_3 \neq \emptyset$. As $X_3 \neq \emptyset$ and $Y_3 \neq \emptyset$, there are elements $x_3 \in X_3$ and $y_3 \in Y_3$. Multiplying the factorization $G = A_1A_2A_3$ by $g = x_3^{-1}y_3^{-1}$ we get the factorization

$$G = gG = (x_3^{-1}A_1)(y_3^{-1}A_1)A_3 =$$

$$= [x_3^{-1}(X_2K_2 \cup X_3K_3)][y_3^{-1}(Y_1K_1 \cup Y_3K_3)]A_3.$$

Here $K_3 \subset x_3^{-1}A_1$ and $K_3 \subset y_3^{-1}A_2$. This contradicts the definition of factorization as K_3 contains a nonidentity element.

Case $n = 4$. Assume that A_1A_2 is periodic and the periods together with e form the subgroup L of G. By Th. 1 of [6] $\chi(A_1A_2) = 0$ for each χ for which $\chi(L) = 0$. So $\chi(A_1) = 0$ for each χ with $\chi(K_2) = 0$ and $\chi(L) = 0$. There are $U_2, U \subset G$ such that $A_1 = U_2K_2 \cup UL$, where the union is disjoint and the products are direct. Similarly, there are $V_1, V \subset G$ such that $A_2 = V_1K_1 \cup VL$. The directness of the product A_1A_2 implies that $U = \emptyset$ or $V = \emptyset$ and so A_1 or A_2 is periodic. In the remaining part we will show that A_1A_2 is periodic.

First we prove that after a suitable relabeling the factors $K_3 \subset$ $\subset A_4$. From the factorization $G = (A_1A_2)A_3A_4$ it follows that there are $X_3, X_4 \subset G$ such that $A_1A_2 = X_3K_3 \cup X_4K_4$, where the union is disjoint and the products are direct. As $e \in A_1A_2$, one of $K_3 \subset A_1A_2$, $K_4 \subset A_1A_2$ holds. In general, for each $\{k,l\} \subset \{1,2,3,4\}$ there is an $i \notin \{k,l\}$ such that $K_i \subset A_kA_l$. There are 6 choices for $\{k,l\}$ and i ranges over 4 values so by the pigeon hole principle there are i, $\{k,l\}$, $\{k',l'\}$ such that $K_i \subset A_kA_l$ and $K_i \subset A_{k'}A_{l'}$. If $\{k,l\}$ and $\{k',l'\}$ are disjoint, then $i \notin \{k,l\} \cup \{k',l'\} = \{1,2,3,4\}$ is a contradiction. Thus we may assume that $k = k'$. Now, by Lemma 4, $K_i \subset A_kA_l \cap A_kA_{l'} =$ $= A_k$. By relabeling we may assume that $K_3 \subset A_4$.

Let $a \in A_1A_2$ and consider the factorization $G = (a^{-1}A_1A_2)A_3A_4$. From this it follows that there are $X_3, X_4 \subset G$ such that $a^{-1}A_1A_2 =$ $= X_3K_3 \cup X_4K_4$, where the union is disjoint and the product are direct. Consequently

$$K_3 \subset (a^{-1}A_1A_2) \quad \text{or} \quad K_4 \subset (a^{-1}A_1A_2).$$

If $K_3 \subset (a^{-1}A_1A_2)$, then $K_3 \subset (a^{-1}A_1A_2) \cap A_4 = \{e\}$ is a contradiction and so $K_4 \subset (a^{-1}A_1A_2)$ for each $a \in A_1A_2$. By Lemma 1 of [2], A_1A_2 is periodic. $\Diamond$

The group ring $\mathbb{Z}(G)$ consists of the formal linear combinations of elements of G with integer coefficients. We can view elements of G as indeterminates and the elements of $\mathbb{Z}(G)$ as multivariable polynomials. The addition and multiplication in $\mathbb{Z}(G)$ can be defined accordingly using the multiplication in G when we multiply indeterminates. A character of G can be extended linearly to be a character of $\mathbb{Z}(G)$. The following fact is a consequence of the standard orthogonality relations of the characters. If A and B are elements of $\mathbb{Z}(G)$ and $\chi(A) = \chi(B)$ for each character χ of G, then $A = B$.

Theorem 3. *Let $G = A_1 \dots A_n$, be a factorization of the finite abelian group G and let $K_1, \dots, K_n$ be the subgroups assigned to $A_1, \dots, A_n$ respectively. If $p \geq n$, where p is the least prime divisor of $|G|$, then $K_i = \{e\}$ for some i, $1 \leq i \leq n$.*

Proof. Assume the contrary that $K_i \neq \{e\}$ and consequently there is an element $x_i \in K_i \setminus \{e\}$ for each i, $1 \leq i \leq n$. Let us form the product $(e - x_1) \dots (e - x_n)$ which is an element of $\mathbb{Z}(G)$. We claim that $(e - x_1) \dots (e - x_n) = 0$. Since

$$\chi\big((e - x_1) \dots (e - x_n)\big) = \chi(e - x_1) \dots \chi(e - x_n) =$$
$$= \big(1 - \chi(x_1)\big) \dots \big(1 - \chi(x_n)\big)$$

it is enough to show that for each character χ of G there is an x_i with $\chi(x_i) = 1$. This obviously holds for the principal character. If χ is not the principal character of G, then from

$$0 = \chi(G) = \chi(A_1 \dots A_n) = \chi(A_1) \dots \chi(A_n)$$

it follows that $\chi(A_i) = 0$ for some i, $1 \leq i \leq n$. Now $x_i \in K_i \subset \operatorname{Ker} \chi$ gives that $\chi(x_i) = 1$.

By [3] the equation $(e - x_1) \dots (e - x_n) = 0$ implies that $|x_i| < n$ for some i, $1 \leq i \leq n$ and so we get the contradiction $p \leq |x_i| < n$. $\Diamond$

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