Mathematica Pannonica
9/2 (1998), 181–194

SOME REMARKS ON DADE'S CONJECTURE

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Received: Novenber 1997

MSC 1991: 20 C 15; 20 C 20, 20 C 25.

Keywords: Dade's conjecture, characters, blocks, defect, defect group, height, induced block, p-chains, covering, Schur multiplier.

Abstract: In this paper we prove some results concerning Dade's conjecture. First we prove that for every finite group G with $O_p(G) = 1$ for $d = 1$ the ordinary and invariant conjectures are true. Later we consider the connection of the ordinary and the projective conjectures for groups having Schur multiplier of prime order. In the end we show some examples that the analogue of Brauer's first main theorem and that of the Alperin-McKay conjecture is not true in general for chain normalizers.

1. Introduction

We mention a few notations used in this paper: Let G be a finite group, p a prime. R denotes a complete discrete valuation ring with quotient field F of characteristic zero and residue class field $F' = R/J(R)$ of characteristic p. We assume that F, F' are both splitting fields for every subgroup of G. If A is an F-algebra, then $\mathrm{Irr}(A)$ is the set of irreducible F-characters of A. The set of irreducible F-characters

of G are denoted by $\mathrm{Irr}(G)$, if B is a p-block of G, then $\mathrm{Irr}(B)$ stands for the set of all irreducible F-characters of G belonging to the block B. If $N \lhd G$ and $\theta \in \mathrm{Irr}(N)$, then $\mathrm{Irr}(B|\theta)$ denotes the set of all irreducible characters $\chi \in \mathrm{Irr}(B)$ lying over θ and $\mathrm{Bl}(G|\theta)$ is the set of blocks of G, lying over θ. ν_p is the p-adic exponential valuation of R. The p-defect of a character $\chi \in \mathrm{Irr}(G)$ is $d(\chi) = \nu_p(\frac{|G|}{\chi(1)})$ and the p-defect of the block B is $d(B) = \max\{d(\chi)|\chi \in \mathrm{Irr}(B)\}$. The height of the character $\chi \in \mathrm{Irr}(B)$ is $\mathrm{ht}(\chi) = d(B) - d(\chi)$. It is well-known that $d(B)$ is exactly the exponent of p in the order of the defect group of B. With $\mathrm{Bl}(G|D)$ we denote the set of blocks of G with defect group D. $O_p(G)$ is the largest normal subgroup of G of p-power order. For further definitions and notations of modular representation theory, see [14].

Let $\mathcal{R}(G), \mathcal{U}(G), \mathcal{E}(G|P), \mathcal{N}(G|P)$ and $\mathcal{P}(G|P)$ denote the set of radical chains, $\mathcal{U}$-chains, elementary, normal and p-chains, the latter three beginning with the subgroup P, as it is defined in [11]. We mention that subgroups belonging to a chain in $\mathcal{E}(G|P)$ need not be elementary abelian, only their factors by the initial subgroup P, which is normal by definition, see [7].

We recall the conjectures of Dade, about more details see [6], [7] and [8].

Dade's ordinary conjecture 1.1. *Let G be a finite group with $O_p(G) = 1$, and let B be a p-block of G with defect $d(B) > 0$. Let d be a non-negative integer. Then*

$$\sum_{C \in \mathcal{F}/G} (-1)^{|C|} k(N_G(C), B, d) = 0,$$

where $k(N_G(C), B, d)$ is the number of characters in the set

$$\{\chi \in \mathrm{Irr}(N_G(C)) | b(\chi)^G = B, d(\chi) = d\}.$$

Here $b(\chi)$ denotes the block of $N_G(C)$ containing χ and $\mathcal{F}$ is any of the families $\mathcal{R}(G)$, $\mathcal{U}(G)$, $\mathcal{E}(G|1)$, $\mathcal{N}(G|1)$ or $\mathcal{P}(G|1)$.

Let $\alpha : G \times G \to F$ be a factor set. The projective representations of G with factor set equivalent to α correspond bijectively to representations of the twisted group ring $F^{(\alpha)}G$ with factor set α, where $F^{(\alpha)}G = \oplus_{x \in G} u_x F$ and $u_x u_y = u_{xy}\alpha(x, y)$. In this correspondence, irreducible projective representations of G correspond to irreducible representations of $F^{(\alpha)}G$. In [7] Dade developed a theory of blocks for twisted group algebras and proved that most theorems of the block theory of group algebras have analogues in this theory. For $H \leq G$ let

$F^{(\alpha)}[H]$ denote the twisted group ring over H with factor set α. If B is a block of $F^{(\alpha)}G$, then an analogue of the above function k can be defined, see [7], namely

$$k(F^{(\alpha)}[H], B, d) = |\{\psi \in \mathrm{Irr}(F^{(\alpha)}[H]), d(\psi) = d, b(\psi)^G = B\}|.$$

If C is running on representatives of G-conjugacy classes of chains of p-subgroups of G in $\mathcal{P}(G|1)$, then $f(C) = k(F^{(\alpha)}[N_G(C)], B, d)$ satisfies the conditions of (3.2 a) and (3.2 b) in [6] for $G = H = E$. With this terminology, the projective form of Dade's conjecture can be formulated in the following way:

Dade's projective conjecture 1.2. (first form) *Let $O_p(G) = 1$ and let $F^{(\alpha)}G$ be the twisted group ring with factor set $\alpha : G \times G \to F$. Let B be a block of $F^{(\alpha)}G$ with defect $d(B) > 0$ and d be any non-negative integer. Then*

$$\sum_{C \in \mathcal{F}/G} (-1)^{|C|} k(F^{(\alpha)}[N_G(C)], B, d) = 0,$$

where $\mathcal{F}$ is any one of the families $\mathcal{R}(G)$,$\mathcal{U}(G)$, $\mathcal{E}(G|1)$, $\mathcal{N}(G|1)$ or $\mathcal{P}(G|1)$.

The projective conjecture holds for G if the above sum is zero for each factor set $\alpha : G \times G \to F$, each block B of $F^{(\alpha)}G$ of positive defect and each non-negative integer d.

It is well-known, see e.g. [10], that projective representations with factor set α are equivalent to ones that can be lifted to ordinary representations of a covering group G^* of G got as a central extension of G by $Z^* = \langle \alpha \rangle$, where α can be chosen up to equivalence so that its order is equal to the order of its image in the Schur multiplier of G. In this correspondence, irreducible projective representations of G lift to irreducible representations of G^* and if we fix a faithful irreducible character ζ of Z^*, then those irreducible characters of G^*, that come from lifting projective characters of G, are just those, which lie above ζ, while the factor set belonging to the central extension G^* of G, is just $\zeta^{-1}(\alpha)$.

Remark 1.3. According to Prop. 2.2 of [7], if $K = K_p \times Z$, where $K \lhd G$, K_p is a p-subgroup and Z is a central p'-subgroup of G, then if $\overline{G} = G/K$ then there is an inclusion preserving bijection between p-subgroups of $\overline{G}$ and p-subgroups of G containing K_p, which induces a length preserving bijection on p-chains C of $\overline{G}$ and those p-chains C^* of G that $K_p \leq P_0$, where P_0 is the initial subgroup of the chain C. In this correspondence normalizers of p-subgroups and p-chains corre-

spond to similar normalizers in the other group, radical p-subgroups correspond to radical p-subgroups etc. Applying this to $K := Z^*$, $G := G^*$ and $\overline{G} := G$, we get that $\mathcal{R}(G^*/Z^*)$ and $\mathcal{R}(G^*)$, $\mathcal{U}(G^*/Z^*)$ and $\mathcal{U}(G^*)$, $\mathcal{P}(G^*/Z^*|PZ^*/Z^*)$ and $\mathcal{P}(G^*|P)$, etc. correspond to each other.

Remark 1.4. If $\chi \in \mathrm{Irr}(F^{(\alpha)}G)$, and $\chi^* \in \mathrm{Irr}(G^*)$ is the corresponding character, then $d(\chi^*) = d(\chi)+\nu_p(|Z^*|)$, a block B of $F^{(\alpha)}G$ corresponds to $B^* \in \mathrm{Bl}(G^*|\zeta)$ and $d(B^*) = d(B) + \nu_p(|Z^*|)$. In this way we get a bijection of blocks of $F^{(\alpha)}G$ and $\mathrm{Bl}(G^*|\zeta)$ and between $\mathrm{Irr}(F^{(\alpha)}G)$ and $\mathrm{Irr}(G^*|\zeta)$.

It was proved in [7], that if $O_p(G) = 1$ and the chains $C \in \mathcal{P}(G|1)$ and $C^* \in \mathcal{P}(G^*|O_p(G^*))$ correspond to each other, then

$$k(F^{(\alpha)}[N_G(C)], B, d) = k(N_{G^*}(C^*), B^*, d^*, \zeta) =$$
$$= |\{\psi \in \mathrm{Irr}(N_{G^*}(C^*))|d(\psi) = d^*, b(\psi)^{G^*} = B^*,$$
$$(\psi_{Z^*}, \zeta) \neq 0\},$$

where ζ is a faithful irreducible character of Z^* and $d^* = d + \nu_p(|Z^*|)$.

Thus we can state Dade's projective conjecture in another form, just using ordinary characters of the covering group G^* above:

Dade's projective conjecture 1.5. (second form) *Let G be a finite group with $O_p(G) = 1$. Let Z^* be a cyclic subgroup of the Schur multiplier of the group G and let G^* be a central extension of G with Z^*. Let ζ be a faithful irreducible character of Z^*. Then*

$$\sum_{C^* \in \mathcal{F}(G^*|O_p(G^*))/G^*} (-1)^{|C^*|} k(N_{G^*}(C^*), B^*, d^*, \zeta) = 0$$

for each block $B^* \in \mathrm{Bl}(G^*|\zeta)$ with $d(B^*) > \nu_p(|Z^*|)$, where $\mathcal{F}$ is any of $\mathcal{R}, \mathcal{U}, \mathcal{E}, \mathcal{N}, \mathcal{P}$.

For the invariant conjecture, one has to embed G as a normal subgroup into an extension group E. If the centre of G is trivial, then we can identify G with its inner automorphism group, and so $G \lhd \mathrm{Aut}(G)$. In this case, we may assume that $E = \mathrm{Aut}(G)$. Then E acts on the chains in $\mathcal{R}(G)$, $\mathcal{U}(G)$, $\mathcal{E}(G|1)$, $\mathcal{N}(G|1)$ or $\mathcal{P}(G|1)$. For each chain C, $N_G(C) \lhd N_E(C)$ and so for each $\psi \in \mathrm{Irr}(N_G(C))$ there exists an inertia subgroup $T(\psi)$ in $N_E(C)$. Of course $N_G(C) \leq T(\psi)$. For each chain C and for each subgroup H of E containing G, let $k(C, B, d, H)$ denote the number of irreducible characters ψ of $N_G(C)$, with $d(\psi) = d$ and $GT(\psi) = H$, which belong to a p-block $b = b(\psi)$ of $N_G(C)$ with $b^G = B$. Now we formulate

Dade's invariant conjecture 1.6. *Let $O_p(G) = 1$ and $d(B) > 0$. Then*

$$\sum_{C \in \mathcal{F}/G} (-1)^{|C|} k(C, B, d, H) = 0,$$

where $\mathcal{F}$ is any of $\mathcal{R}(G)$, $\mathcal{U}(G)$, $\mathcal{E}(G|1)$, $\mathcal{N}(G|1)$, $\mathcal{P}(G|1)$, d is any nonnegative integer, H is any subgroup of E containing G.

2. The defect one case

It is easy to see that if $C \in \mathcal{P}(G|1)$ is of positive length then $N_G(C)$ has no blocks of defect zero. In this section we investigate the case of defect one blocks of $N_G(C)$, and Dade's conjecture for $d = 1$.

Proposition 2.1.

(i) *If C is a p-chain $1 = P_0 < P_1 < \cdots < P_n$ with $|P_n| > p$ then its normalizer, $N_G(C)$ has no defect one blocks (and hence no defect one characters, either). Thus $k(N_G(C), B, 1) = 0$ for these chains and for each block B of G.*

(ii) *If the defect of the block B of G is greater than one, then blocks of $N_G(C)$ which induce B are also of defect greater than one, for all p-chains C. Hence $k(N_G(C), B, 1) = 0$ for these blocks B, and for all p-chains C.*

(iii) *If the defect group D of a block B of G is abelian, then each block b of a chain normalizer $N_G(C)$ that induces B has defect group G-conjugate to D. Especially, if B is of defect one, then blocks of $N_G(C)$ which induce B are also of defect one. So it can only be induced from blocks of chain normalizers $N_G(C)$ of chains of length zero and one. In the latter case $|P_1| = p$ also holds.*

(iv) *If G is an arbitrary finite group with $O_p(G) = 1$, then the ordinary and invariant Dade's conjectures are true for G for the prime p and $d = 1$.*

Proof. (i) It is easy to see, that $P_1 \lhd N_G(C)$ so it is contained in the defect group of each block of $N_G(C)$. If $|P_1| > p$, then we get that $N_G(C)$ has no blocks of defect one. If $|P_1| = p$ then by our assumption, the length of the chain is at least two, and hence $|P_2| \geq \geq p^2$. Let $G_2 = N_G(C^2)$ be the second final subchain normalizer, where $C^2 : P_2 < \cdots < P_n$. Then $P_2 \leq G_2 \leq N_G(P_2)$, thus $P_2 \lhd G_2$. But $N_G(C) = N_{G_2}(P_1) \geq N_{G_2}(P_1) \cap P_2 = N_{P_2}(P_1)$, hence $N_{P_2}(P_1) \lhd N_G(C)$ of order at least p^2. So in this case G has no blocks of defect one, either.

By Th. 4.5 of [15], each irreducible character of defect one belongs to a block of defect one, hence $N_G(C)$ has no defect one characters.

(ii) Let us suppose that the defect of B is greater than 1. By (i) if $|P_n| > p$ then $N_G(C)$ has no blocks of defect one and $k(N_G(C), B, 1) = 0$. If $n = 0$ then $N_G(C)$ has no blocks of defect one which induces B. If $|P_n| = p$ then $n = 1$. Let us suppose in this latter case that some block b of $N_G(C)$ is of defect one. Then b has defect group P_1. By Brauer's first main theorem, blocks of $N_G(P_1)$ with defect group P_1 induce blocks with defect group P_1, hence $b^G \neq B$ and $k(N_G(C), B, 1) = 0$.

(iii) Let us suppose that B has abelian defect group D. Let b be a block of $N_G(C)$ with $b^G = B$ and defect group $\delta(b)$. Let us denote by G_i $i = 1, \ldots, n$ the normalizer of the ith final subchain $C^i : P_i < \cdots < P_n$, and let $G_{n+1} = G$. Then $N_G(C) = N_{G_2}(P_1) \leq G_2$, $P_1 \leq \delta(b)$ and hence $C_{G_2}(\delta(b)) \leq C_{G_2}(P_1)$. Thus by Th.5.21 of Chapter 5 in [14], the defect group $\delta(b^{G_2})$ is conjugate in G_2 to $\delta(b)$. Let us suppose that we have already proved that the defect group of b^{G_i}, $\delta(b^{G_i})$, is conjugate to $\delta(b)$ in G_i. As $G_i = N_G(P_i) \cap G_{i+1} \leq G_{i+1}$, $P_i \leq \delta(b^{G_i})$ and hence $C_{G_{i+1}}(\delta(b^{G_i})) \leq C_{G_{i+1}}(P_i)$. Thus by the same Th. 5.21, the defect group $\delta(b^{G_{i+1}})$ of the block $b^{G_{i+1}}$ is conjugate in G_{i+1} to $\delta(b)$. And so the defect group D of the block $B = b^{G_{n+1}}$ is conjugate in $G = G_{n+1}$ to $\delta(b)$. Especially if $d(B) = 1$, then it can only be induced from blocks of $N_G(C)$ of defect 1. As $N_G(C)$ does not have blocks of defect one if $|P_n| > p$ so B can be induced from itself and from a block of defect one of the normalizer of the length one chain C with $|P_1| = p$.

(iv) For blocks B of defect greater than 1 the ordinary and invariant Dade's conjectures hold trivially for $d = 1$. If $d(B) = 1$, then the defect group of B is cyclic, so by [9] we know that Dade's conjectures are true. $\Diamond$

Remark 2.2. Similarly as in Prop. 2.1(iii), one can also prove the generalization of Th. 5.21 of [14] for chain normalizers, namely: Let C be a p-chain of G, let b be a p-block of $N_G(C)$ with defect group Q. Then for a suitable defect group D of b^G, $D \cap N_G(C) = Q$. Moreover,

$$Z(D) \leq C_D(Q) = Z(Q) \leq Q \leq D.$$

3. Connections of the ordinary and the projective conjectures

From now on let G be a finite group with Schur multiplier of order

q, where q is a prime. Let us suppose further that $O_p(G) = 1$. Let G^* be a nonsplit central extension of G with a cyclic subgroup Z^* of order q. Let $\zeta_2, \ldots, \zeta_q$ be the faithful irreducible characters of Z^*. Let C be a p-chain from $\mathcal{P}(G|1)$ and C^* its image under the bijection from Remark 1.3. There are two cases, $p = q$ and $p \neq q$.

Remark 3.1. By Th. 8.11 of Chapter 5 in [14], if $p = q$, then domination of blocks gives a one-to-one correspondence between p-blocks B^* of G^* and blocks B of G, where the image of the defect group is the defect group of the image, thus $d(B^*) = d(B)+1$. By Lemma 8.6 of Chapter 5 in [14], $\mathrm{Irr}(B) \subseteq \mathrm{Irr}(B^*)$, so $\mathrm{Irr}(B^*)\setminus \mathrm{Irr}(B) = \cup_{j=2}^{q} \mathrm{Irr}(B^*|\zeta_j)$. Similar statements hold for p-blocks of chain normalizers $N_{G^*}(C^*)$ and $N_G(C)$, as well.

Remark 3.2. According to Lemma 8.6 and Th. 8.9 in Chapter 5 of [14], if B^* is a p-block of G^* for $p \neq q$, then it either dominates no block of G, and in this case does not contain any characters of G, or it dominates exactly one block B of G, containing the same irreducible characters. Thus $\mathrm{Irr}(B^*) = \cup_{j=2}^{q} \mathrm{Irr}(B^*|\zeta_j)$ in the first case and $\mathrm{Irr}(B^*|\zeta_j) = \emptyset$, for $j \in \{2, \ldots, q\}$ in the second case. In this case $d(B^*) = = d(B)$. Similar statements hold for p-blocks of chain normalizers $N_{G^*}(C^*)$ and $N_G(C)$, as well.

Remark 3.3. Let ${\mu_{Z^*}}^*$ be the algebra homomorphism $F'G^* \to F'G$, the so called domination map, given by $\sum \alpha_g g \to \sum \alpha_{\overline{g}} \overline{g}$, where $\overline{g}$ is the image of g under the natural homomorphism $G^* \to G$. Let $H^* = = N_{G^*}(C^*)$ be a chain normalizer. Let ${s_{H^*}}^*$ be the F'-homomorphism $Z(F'G^*) \to Z(F'H^*)$ given by the projection on the H^*-components, while ${s_H}^*$ denotes the similar F'-homomorphism $Z(F'G) \to Z(F'H)$ where $H = N_G(C)$ is the image of H^*. Then the domination map and the projection maps commute, namely

$$ {\mu_{Z^*}}^* \circ {s_{H^*}}^* = {s_H}^* \circ {\mu_{Z^*}}^*. $$

Proposition 3.4. *Let $b \in \mathrm{Bl}(N_{G^*}(C^*))$ and $B^* \in \mathrm{Bl}(G^*)$. Let $\overline{b} \in \in \mathrm{Bl}(N_G(C))$ and $B \in \mathrm{Bl}(G)$.*

(i) *If B is dominated by B^* and $\overline{b}$ is dominated by b then $B^* = b^{G^*}$ if and only if $\overline{b}^G = B$.*

(ii) *If $\overline{b}^G = B$ and $b^{G^*} = B^*$ then B is dominated by B^* if and only if $\overline{b}$ is dominated by b.*

Proof. (i) Let ${\omega_{B^*}}^*$, ${\omega_b}^*$, ${\omega_B}^*$ and ${\omega_{\overline{b}}}^*$ be the central F'-characters of the above mentioned blocks. Let us suppose that B is dominated by B^*, $\overline{b}$ is dominated by b and $b^{G^*} = B^*$. Then ${\omega^*}_b \circ {s_{H^*}}^* = {\omega_{B^*}}^*$. By

Lemma 8.5 in Chapter 5 of [14], ${\omega_{\overline{b}}}^* \circ {\mu_{Z^*}}^* = {\omega_b}$. Thus

$$ {\omega_{B^*}}^* = {\omega_{\overline{b}}}^* \circ {\mu_{Z^*}}^* \circ {s_{H^*}}^*. $$

Thus by Remark 3.3, ${\omega_{B^*}}^* = {\omega_{\overline{b}}}^* \circ s_H^* \circ {\mu_{Z^*}}^*$. As $\overline{b}^G$ is defined, we get that ${\omega_{\overline{b}^G}}^* \circ {\mu_{Z^*}}^* = {\omega_{B^*}}^*$. If e_{B^*} is the central primitive idempotent of $F'G^*$ belonging to B^*, then ${\omega_{\overline{b}^G}}^* \circ {\mu_{Z^*}}^*(e_{B^*}) = 1$. Thus B^* dominates $\overline{b}^G$. As B^* dominates exactly one block of G, we get that $B = \overline{b}^G$. The converse statement follows similarly, by taking the above equations in the reverse order.

(ii) Let us suppose that $\overline{b}^G = B$ and $b^{G^*} = B^*$. If $\overline{b}$ is dominated by b then by Lemma 8.5 in Chapter 5 of [14] ${\omega_b}^* = {\omega_{\overline{b}}}^* \circ {\mu_{Z^*}}^*$. Since ${\omega_{B^*}}^* = {\omega_b}^* \circ {s_{H^*}}^*$ and ${\omega_{\overline{b}}}^* \circ {s_H}^* = {\omega_B}^*$, we get that ${\omega_B}^* \circ {\mu_{Z^*}}^* = {\omega_{\overline{b}}}^* * \circ {s_H}^* \circ {\mu_{Z^*}}^* = {\omega_{\overline{b}}}^* \circ {\mu_{Z^*}}^* \circ {s_{H^*}}^* = {\omega_b}^* \circ {s_{H^*}}^* = {\omega_{B^*}}^*$. If we put in e_{B^*} into both sides of this identity, we get that B^* is dominating B. If B^* dominates B, then ${\omega_b}^* \circ {s_{H^*}}^* = {\omega_{B^*}}^* = {\omega_B}^* \circ {\mu_{Z^*}}^* = {\omega_{\overline{b}}}^* \circ {s_H}^* * \circ {\mu_{Z^*}}^* = {\omega_{\overline{b}}}^* \circ {\mu_{Z^*}}^* \circ {s_{H^*}}^* *$. Thus, if we apply both sides of this to e_b, the central primitive idempotent of $F'H^*$ belonging to the block b, then we get that $1 = {\omega_{\overline{b}}}^* \circ {\mu_{Z^*}}^*(e_b)$. So we get that b dominates $\overline{b}$. $\Diamond$

3.1. $p = q$

Proposition 3.5. *Let B and B^* be corresponding p-blocks of G and G^* as above, and C and C^* corresponding p-chains in G and in G^*, respectively, $d^* = d + 1$. Let $k(H, B, d)$ be the function from the ordinary Dade's conjecture, while $k(H^*, B^*, d^*, \zeta)$ is the function from the projective Dade's conjecture. Then*

$$ k(N_{G^*}(C^*), B^*, d^*) - k(N_G(C), B, d) = $$

$$ = \sum_{j=2}^{p} k(N_{G^*}(C^*), B^*, d^*, \zeta_j) = (p-1)k(N_{G^*}(C^*), B^*, d^*, \zeta_2) $$

Proof. We have to prove that $\{\psi \in \mathrm{Irr}(N_{G^*}(C^*)),\ d(\psi) = d^*,\ b(\psi)^{G^*} = B^*\} \backslash \{\psi \in \mathrm{Irr}(N_G(C)),\ d(\psi) = d,\ b(\psi)^G = B\} = \cup_{j=2}^{p} \{\psi \in \in \mathrm{Irr}(N_{G^*}(C^*)),\ d(\psi) = d^*,\ b(\psi)^{G^*} = B^*,\ (\psi_{Z^*}, \zeta_j) \neq 0\}$. If ψ belongs to the left-hand side of this equality, then $\mathrm{Ker}(\psi)$ cannot contain Z^*, as then $\psi \in \mathrm{Irr}(N_G(C))$ and if $\overline{b}$ is the block of $N_G(C)$ dominated by $b(\psi)$ then by Prop. 3.4, $\overline{b}^G = B$, contradicting the assumption. Thus it belongs to the right-hand side, too. If ψ belongs to the right-hand side, then it obviously belongs to the left-hand side, too.

By Lemma 5.8 (ii) in Chapter 5 of [14] $\mathrm{Irr}(B^*) = \cup_{j=1}^{p} \mathrm{Irr}(B^*|\zeta_j)$, and this is a disjoint union. Let Q_m denote an extension field of the rationals Q with a primitive m^{th} root of unity, let $n = |G|$ and n_p, $n_{p'}$ its p-part and p'-part respectively. Let $\mathcal{G} = \mathrm{Gal}(Q_n/Q_{n_{p'}})$ and $\sigma \in \mathcal{G}$. Then for every $\chi \in \mathrm{Irr}(B^*)$ and every p'-element $x \in G$ $\chi(x) = \chi^\sigma(x)$, so χ^σ belongs to $\mathrm{Irr}(B^*)$, too. As $Q_n = Q_{n_{p'}} Q_{n_p}$, $\mathcal{G}$ is acting on $\mathrm{Irr}(Z^*)$. We state that this action is transitive on the set of nontrivial characters of Z^*, $\{\zeta_2, \ldots, \zeta_p\}$. If there would occur a proper $\mathcal{G}$-orbit $\{\zeta_{i_1}, \ldots, \zeta_{i_k}\}$, then for every $x \in Z^*$ the sum $\sum_{j=1}^{k} \zeta_{i_j}(x)$ were fixed by $\mathcal{G}$, hence this sum belongs to $Q_{n_{p'}} \cap Q_{n_p} = Q$. If we choose $1 \neq x \in Z^*$, then $\zeta_{i_1}(x) = \epsilon$ is a primitive p^{th} root of unity and $\zeta_{i_j}(x)$ is a power of it for every j of degree at most $p-1$. Thus ϵ satisfies an equation of degree at most $p-1$ with leading coefficient 1, so it has to be equal to its minimal polinomial $x^{p-1} + \cdots + 1$. Hence the action of $\mathcal{G}$ has to be transitive on nontrivial irreducible characters of Z^*. This action defines bijections between the sets $\mathrm{Irr}(B^*|\zeta_j)$ for $j \in \{2, \ldots, p\}$. Hence these sets have the same number of elements. If instead of G^* one takes $N_{G^*}(C^*)$, then with similar argument we get that the numbers $k(N_{G^*}(C^*), B^*, d^*, \zeta_j)$ are independent of $j \in \{2, \ldots, p\}$. Hence the second equation is also true. $\Diamond$

3.2. $p \neq q$

Proposition 3.6. *Let p be a prime different from q. Let B^* and B be corresponding p-blocks under domination as above, and C and C^* corresponding p-chains of G and G^*, $d^* = d$. Let $k(H, B, d)$ be the function from ordinary Dade's conjecture, while $k(H^*, B^*, d^*, \zeta)$ is the function from projective Dade's conjecture. Then $k(N_{G^*}(C^*), B^*, d^*, \zeta_j) = 0$ for $j = 2, \ldots, q$. If B^* does not dominate any blocks of G then*

$$k(N_{G^*}(C^*), B^*, d^*) =$$

$$= \sum_{j=2}^{q} k(N_{G^*}(C^*), B^*, d^*, \zeta_j) = k(N_{G^*}(C^*), B^*, d^*, \zeta_i)$$

for suitable $i \in \{2, \ldots, q\}$.

Proof. We get similarly as in the proof of Prop. 3.5 that

$$\{\psi \in \mathrm{Irr}(N_{G^*}(C^*)), d(\psi) = d^*, b(\psi)^{G^*} = B^*\}\backslash$$
$$\backslash\{\psi \in \mathrm{Irr}(N_G(C)), d(\psi) = d, b(\psi)^G = B\} =$$
$$= \cup_{j=2}^{q}\{\psi \in \mathrm{Irr}(N_{G^*}(C^*)), d(\psi) = d^*, b(\psi)^{G^*} = B^*, (\psi_{Z^*}, \zeta_j) \neq 0\}$$

if B^* dominates B. If ψ belongs to the first set of the left-hand side, then by Prop. 3.4 $b(\psi)$ dominates $\overline{b}(\psi)$, so by Remark 3.2, they contain the same characters, so the set on the right-hand side is empty. Thus we get $k(N_{G^*}(C^*), B^*, d^*, \zeta_j) = 0$ for $j \in \{2, \dots q\}$. If B^* does not dominate any blocks of G then $\{\psi \in \mathrm{Irr}(N_{G^*}(C^*)), d(\psi) = d^*, b(\psi)^{G^*} = = B^*\} = \cup_{j=2}^{q}\{\psi \in \mathrm{Irr}(N_{G^*}(C^*)), d(\psi) = d^*, b(\psi)^{G^*} = B^*, (\psi_{Z^*}, \zeta_j) \neq \neq 0\}$ since if $(\psi_{Z^*}, \zeta_j) = 0$ for some $j > 1$, then $b(\psi)$ dominates the block $\overline{b}(\psi)$ of $N_G(C)$, and thus by Prop. 3.4 B^* dominates $\overline{b}(\psi)^G$.

Let $G_1^*, \dots G_n^*$ be the final subchain normalizers $G_i^* = N_{G^*}(C^{*^i})$ for $i = 1, \dots n$, where $C^{*^i} : P_i < \dots < P_n$, together with $G_{n+1}^* = G^*$. Then we have that $G_1^* = N_{G^*}(P_1) \cap N_{G^*}(C^{*^2}) = N_{G_2^*}(P_1) \leq G_2^* \dots \leq \leq G_n^* = N_{G^*}(P_n) \leq G^*$. Thus we can define a Brauer homomorphism

$$\mathrm{Br}_{N_{G^*}(C^*)} = \mathrm{Br}_{P_1} \circ \dots \circ \mathrm{Br}_{P_n} : Z(F'G^*) \to Z(F'(N_{G^*}(C^*)),$$

where

$$\mathrm{Br}_{P_i} : Z(F'G_{i+1}^*) \to Z(F'N_{G_{i+1}^*}(P_i)),$$

given by $\hat{K} \mapsto \hat{K}^0$, where $\hat{K}$ denotes the class sum of the conjugacy class K of G_{i+1}^* and $K^0 = K \cap C_{G_{i+1}^*}(P_i)$.

Let b be a p-block of $N_{G^*}(C^*)$ such that $b^{G^*} = B^*$ and b_i the p-block of Z^* containing the faithful irreducible character ζ_i. Let us suppose that b covers b_i. We claim that B^* covers b_i. If B^* would cover b_j containing the faithful irreducible character $\zeta_j \neq \zeta_i$ then since $e_{B^*}e_{b_j} = = e_{B^*}$ and $e_b e_{b_i} = e_b$, we have $\mathrm{Br}_{N_{G^*}(C^*)}(e_{B^*}) = \mathrm{Br}_{N_{G^*}(C^*)}(e_{B^*}e_{b_j}) = = e_b e_{b_j} = e_b$. But $e_b = e_b e_{b_i}$ which is a contradiction. In this way we get a faithful irreducible character ζ_i of Z^* such that

$$\sum_{j=1}^{q} k(N_{G^*}(C^*), B^*, d^*, \zeta_j) = k(N_{G^*}(C^*), B^*, d^*, \zeta_i). \qquad \Diamond$$

3.3. For $p = q$ the connection of the ordinary and projective conjectures

In [6] it is shown, see Ex. 7.3, that for $O_p(G) > 1$ the alternating sum of 1.1 may also be not zero. We show that for the above group G^* it cannot happen if for G the projective Dade's conjecture is true for

the prime p.

Theorem 3.7. *Let G be a group with Schur multipier of prime order p and with $O_p(G) = 1$. Let G^* be a nonsplit central extension of G by a cyclic group of order p. Then the following are equivalent:*

(i) *For G the ordinary Dade's conjecture is true for the prime p and*

$$(1) \qquad \sum_{C^* \in \mathcal{P}(G^*|O_p(G^*))/G^*} (-1)^{|C^*|} k(N_{G^*}(C^*), B^*, d^*) = 0,$$

where k is the function from 1.1

(ii) *For G the projective conjecture is true for the prime p.*

Proof. By Lemma 5.8 (ii) in Chapter 5 of [14], every p-block B^* of G^* belongs to $\mathrm{Bl}(G^*|\zeta)$ for every irreducible character $\zeta \in \mathrm{Irr}(Z^*)$. By summing up the two sides of the equation in Prop. 3.5, we get the desired result. $\Diamond$

Corollary 3.8.

a) *If G is a finite group with Schur multiplier Z^* of order $p = 2$, $O_2(G) = 1$ and G^* is a nonsplit central extension of Z^* with G, $B^* \in \mathrm{Bl}(G^*)$ with $d(B^*) > 1$, then "on the average" on dimensions d^* the sum of 1.1 for G^* is zero, namely:*

$$(2) \qquad \sum_{d^* \geq 1} \sum_{C^* \in \mathcal{P}(G^*|O_p(G^*))/G^*} (-1)^{|C^*|} k(N_{G^*}(C^*), B^*, d^*) = 0.$$

b) *If G is a finite group with Schur multiplier Z^* of prime order $p > 2$, $O_p(G) = 1$ and G^* is a nonsplit central extension of Z^* with G, $B^* \in \mathrm{Bl}(G^*)$ with $d(B^*) > 1$, then if Dade's ordinary conjecture is true for the block $B \in \mathrm{Bl}(G)$ dominated by B^* and for the prime p then the sum (2) is zero. Especially if the Knörr-Robinson formulation of Alperin's weight conjecture* [13] *is true for G for the prime p and the block B, then (2) is zero.*

Proof. a) We sum up the two sides of the equality in 3.5 for $d \geq 0$, where $d^* = d + 1$ and for C, where C^* is the corresponding chain in G^*, with coefficients $(-1)^{|C|}$, then if we use Th. 2 of [16] then we get that if $p = 2$ then the sum of (2) is zero.

b) If the Knörr-Robinson formulation of Alperin's weight conjecture is true for G, e.g. if Dade's ordinary conjecture is true, then if we sum up the two sides of the equality in 3.5 for $d \geq 0$, where $d^* = d + 1$ and for C, where C^* is the corresponding chain in G^*, with coefficients $(-1)^{|C|}$, and if we use Th. 2 of [16], then we get that "on the average"

of dimensions the projective conjecture holds for the extension G^*, so the sum of (2) is zero. $\Diamond$

Remark 3.9. We mention that in Th. 3.7 (i) and in the Cor. 3.8 in the equation (2) we could have replaced $\mathcal{P}(G^*|O_p(G^*))$ by $\mathcal{R}(G^*)$, $\mathcal{U}(G^*)$, $\mathcal{E}(G^*|O_p(G^*))$ or $\mathcal{N}(G^*|O_p(G^*))$. This follows from results of [13], [6], [7] and [11].

3.4. For $p \neq q$ the connection of the ordinary and projective conjecture

Theorem 3.10. *Let G be a finite group with Schur multiplier of prime order q. Let G^* be a nonsplit central extension of G by a cyclic group of order q. Then the following are equivalent:*

(i) *For G and G^* the ordinary Dade's conjecture is true for the prime p, where $p \neq q$.*

(ii) *For G the projective Dade's conjecture is true for the prime p, where $p \neq q$.*

Proof. (i) $\rightarrow$ (ii): If the ordinary conjecture is true for G then the projective conjecture is true for the extension of G with the trivial group. For the central extension G^*, if B^* dominates a block B of G, then the projective conjecture holds trivially, as $k(N_{G^*}(C^*), B^*, d^*, \zeta_j) = 0$ for $j \in \{2, \dots q\}$. If B^* dominates no blocks of G then $k(N_{G^*}(C^*), B^*, d^*) = k(N_{G^*}(C^*), B^*, d^*, \zeta_i)$, for some $i \in \{2, \dots, q\}$. Hence the projective conjecture holds, as for G^* the ordinary conjecture holds.

(ii) $\rightarrow$ (i): If the projective conjecture holds for G, then the ordinary conjecture also holds for G. Each block B of G is dominated by exactly one block B^* of G^*. Thus $k(N_{G^*}(C^*), B^*, d^*) = k(N_G(C), B, d)$, and hence the ordinary conjecture holds for these blocks B^* of G^*. If B^* does not dominate any blocks of G then $k(N_{G^*}(C^*), B^*, d^*) = = k(N_{G^*}(C^*), B^*, d^*, \zeta_i)$, for some $i \in \{2, \dots, q\}$. Hence the ordinary conjecture holds for these blocks of G^*, too. $\Diamond$

4. Examples for block correspondence between $N_G(C)$ and G

In [11] we proved the generalization of Brauer's third main theorem for chain normalizers. The first main theorem of Brauer tells

that block induction gives a one-to-one correspondence between $\mathrm{Bl}(N_G(D)|D)$ and $\mathrm{Bl}(G|D)$. The first example shows that the analogue of the first main theorem is not true for chain normalizers.

Let G be a finite group, let D be a defect group of G for the prime p. Let $k_0(B)$ denote the number of height zero ordinary irreducible characters of a given p-block B. The Alperin-McKay conjecture tells, see e.g. [1] that $k_0(b) = k_0(b^G)$ if $b \in \mathrm{Bl}(N_G(D)|D)$. The second example shows that the analogue of this for chain normalizers is not true. In an earliaer version of this paper we had examples form [11], got from calculations with GAP [17]. The first the example was $2HS$ for $p = 3$ and C of length one, with $|P_1| = 3$ and $|S| = 3$. The second example was HS for $p = 2$ and C of length 1 and $|P_1| = 2$ for the principal block.

The following much simpler examples are due to Prof. Külshammer.

Example 4.1. There is a group G and a p-chain C in G such that for $S \in \mathrm{Syl}_p(G)$, $S \leq N_G(C)$ and there is no bijection between $\mathrm{Bl}(N_G(C)|S)$ and $\mathrm{Bl}(G|S)$:

Let A be the Klein four group, B cyclic of order 7, H cyclic of order 3 acting on $A \times B$ faithfully. Let G be the semidirect product of $A \times B$ with H. Let $p = 2$. Then $S = A$ and if C is a chain of length 1 with $|P_1| = 2$, then $N_G(C) = A \times B$, $|\mathrm{Bl}(N_G(C)|S)| = 7$, while $|\mathrm{Bl}(G|S)| = 3$.

Example 4.2. There is a group G, a p-chain C of G and a block b of $N_G(C)$ such that $k_o(b) \neq k_0(b^G)$:

Let G be the extraspecial group of order 32 which is the central product of two Dihedral groups of order 8. Then there are 16 height zero characters in the unique block of G. Hoewever if we take a noncentral involution in G and define C to be the length one chain where P_1 is generated by this involution, then $N_G(C)$ is a nonabelian group of order 16, so its unique block has less than 16 height zero irreducible characters.

Acknowledgements

Thanks are due to Professor Külshammer for letting us know the above examples and also for his comments and suggestions concerning a previous version of this paper.

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